SEEMOUS 2008

South Eastern European Mathematical Olympiad
for University Students

Athens, March 2008

Analysis

Problem AN1

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous convex function. Show that the inequality

$$\int_{-a}^{a} [f(x+t) - f(x)]dt \geq 0$$

holds for any $a > 0$, $x \in \mathbb{R}$.

Remark. A function f is convex, if for any $x_0 \in \mathbb{R}$ there exists a linear function $l(x, x_0)$ such that $l(x_0, x_0) = f(x_0)$ and $l(x, x_0) \leq f(x)$ for any $x \in \mathbb{R}$.

Remark 2. It can be proved that every convex function $f : \mathbb{R} \to \mathbb{R}$ is continuous. Also, using the definition given above one can see that

$$f(x+t) + f(x-t) \geq 2f(x)$$

for every $x, t \in \mathbb{R}$.

First Solution. Fix some $x \in \mathbb{R}$ and $a > 0$. Consider a rectangle P with center at point $A(x, f(x))$ and one side parallel to Ox and equal to $2a$, the other side parallel to Oy and equal to $2b$. Take b so large that each point $(t, f(t))$ lies in P for $t \in [x-a, x+a]$. Since f is convex, there is at least one tangent l to its graph at the point $A(x, f(x))$. Then, l divides the area of P into two equal parts (A is a center of P) and since the graph of f is situated above l, it evidently follows that the set

$$\{(t, y) \mid y \leq f(t)\} \cap P$$

has area $\geq \frac{1}{2}\, area(P) = 2ab$. In other terms,

$$\int_{x-a}^{x+a} [f(t) - (f(x) - b)]dt \geq 2ab\ ,$$

whereby

$$\int_{x-a}^{x+a} [f(t) - f(x)]dt \geq 0.$$

Now, by the change of variable $u = t - x$, one gets

$$\int_{-a}^{a} [f(x+u) - f(x)]du \geq 0.$$

Second Solution. The change of variables $u = -t$ shows that

$$\int_{-a}^{a} [f(x+t) - f(x)]\,dt = \int_{-a}^{a} [f(x-t) - f(x)]\,dt.$$

So, it suffices to prove the inequality

$$\int_{-a}^{a} [f(x+t) + f(x-t) - 2f(x)]\,dt \geq 0.$$

This is clear because the integrand is non-negative by Remark 2.

Problem AN2

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the equation

$$(*) \qquad af(x+1) + cf(x-1) = 2bf(x).$$

a.) Show that $(*)$ has a non-constant continuous periodic solution if and only if $a = c \neq 0$ and $\frac{b}{a} \in [-1, 1]$.

b.) Does the equation have a periodic solution if the condition for continuity is omitted? Justify your answer.

Solution. a.) Suppose that $(*)$ has a continuous periodic solution f. We distinguish three cases:

(i) $b^2 - ac > 0$. Then, for some x_0 we have $f(x_0) \neq 0$, and hence, one can find constants C_1 and C_2 such that $f(x) = C_1\lambda_1^x + C_2\lambda_2^x$ for $x \in x_0 + \mathbb{Z}$ where $\lambda_{1,2}$ are the roots of the characteristic equation

$$(\mathbf{he}) \qquad a\lambda^2 - 2b\lambda + c = 0.$$

Therefore f is unbounded on the set $x_0 + \mathbb{Z}$ which contradicts the hypothesis that f is continuous and periodic.

(ii) $b^2 = ac$. Then, the function f has the form $(C_1 + C_2x)\left(\frac{c}{a}\right)^{\frac{x}{2}}$ on the set $x_0 + \mathbb{Z}$.

(iii) $b^2 - ac < 0$. Then, the equation (**he**) has conjugate complex roots $\lambda_{1,2} = r(\cos\varphi \pm i\sin\varphi)$. It is easy to see that in this case the restriction of the function f over the set $x_0 + \mathbb{Z}$ has the form $f(x) = r^x(C_1\cos\varphi x + C_2\sin\varphi x)$. Thus we obtain again that if $r \neq 1$ the function f is unbounded on $x_0 + \mathbb{Z}$ (observe that for $r > 0$; $r \neq 1$ and $n \in \mathbb{Z}$ the sequences $\{r^n\cos n\varphi\}$ and $\{r^n\sin n\varphi\}$ are unbounded).

From the above we see that necessary conditions for the existence of a continuous periodic solution are: $b^2 - ac < 0$ and $r = 1$. It is easy to check that the condition $r = 1$ implies $a = c$. Indeed, if $b^2 - ac < 0$ then $\lambda_{1,2} = \frac{b \pm i\sqrt{ac - b^2}}{a}$ and $r^2 = \frac{b^2}{a^2} + \frac{ac - b^2}{a^2} = \frac{c}{a}$. Note that from $b^2 - ac < 0$ we must have $\frac{b^2}{ac} = \frac{b^2}{a^2} \in [0, 1]$.

And so, the equation (**he**) takes the form $f(x+1) + f(x-1) = 2\frac{b}{a}f(x)$. Now, we have for example the solution $f(x) = \cos\alpha x$ where $\alpha = \arccos\frac{b}{a}$.

b.) The answer is "yes". One can construct a function f by using the set M of all real numbers of the form $x = m + n\xi$, where $m, n \in \mathbb{Z}$ and ξ is irrational - for example, $\xi = \sqrt{2}$. Let furthermore $\lambda \neq 0$ be a root of the equation (**he**). If $\lambda \in \mathbb{R}$ we define the function f by $f(x) = 0$ if $x \notin M$ and $f(x) = \lambda^m$ if $x = m + n\sqrt{2}$ for some $m, n \in \mathbb{Z}$.

In the case $\lambda = r(\cos\varphi + i\sin\varphi)$ is complex one may put as above $f(x) = 0$ for $x \notin M$ and $f(x) = r^m\cos m\varphi$ otherwise.

Problem AN3

$g(x) = x$

Let $f : [1, \infty) \to (0, \infty)$ and $g : (0, \infty) \to (0, \infty)$ be continuous functions such that g is strictly increasing and $g(\infty) = \infty$. Assume that for every $a > 0$ the equation $f(x) = ag(x)$ has a solution in the interval $[1, \infty)$.

a.) Prove that for every $a > 0$ the equation $f(x) = ag(x)$ has infinitely many solutions.

b.) Is it possible to choose such a function f to be strictly increasing? Justify your answer.

Solution. a.) Suppose that one can find constants $a > 0$ and $b > 0$ such that $f(x) \neq ag(x)$ for all $x \in [b, \infty)$. Since f and g are continuous we obtain two possible cases:

1.) $f(x) > ag(x)$ for $x \in [b, \infty)$. Define

$$c = \min_{x \in [1,b]} \frac{f(x)}{g(x)} = \frac{f(x_0)}{g(x_0)}.$$

Then, for every $x \in [1, \infty)$ one should have

$$f(x) > \frac{\min(a, c)}{2} g(x),$$

a contradiction.

2.) $f(x) < ag(x)$ for $x \in [b, \infty)$. Define

$$C = \max_{x \in [1,b]} \frac{f(x)}{g(x)} = \frac{f(x_0)}{g(x_0)}.$$

Then,

$$f(x) < 2 \max(a, C) g(x)$$

for every $x \in [1, \infty)$ and this is again a contradiction.

b.) The answer is yes. In view of the fact that g is increasing and $\lim_{x \to \infty} g(x) = \infty$ one can choose a sequence $1 = x_1 < x_2 < \cdots < x_k < \cdots$ such that the sequence $y_k = 2^{k \cos k\pi} g(x_k)$ is also increasing. Next define $f(x_k) = y_k$ and extend f linearly on each interval $[x_{k-1}, x_k]$: $f(x) = a_k x + b_k$ for suitable a_k, b_k. In this way we obtain an increasing continuous function f, for which $\lim_{n \to \infty} \frac{f(x_{2n})}{g(x_{2n})} = \infty$ and $\lim_{n \to \infty} \frac{f(x_{2n-1})}{g(x_{2n-1})} = 0$. It now follows that the continuous function $\frac{f(x)}{g(x)}$ takes every positive value on $[1, \infty)$.

Problem AN4

Let A be a subset of $\mathbb{R}$ and let $f : A \to (0, +\infty)$ be a function such that

$$\min\{f(x_1), f(x_2)\} \leq |x_1 - x_2|$$

for all $x_1, x_2 \in A$. Prove that A is countable.

Solution. Assume that A is uncountable. We consider the following decomposition of $\mathbb{R} \times (0, +\infty)$:

$$\mathbb{R} \times (0, +\infty) = \bigcup_{n=-\infty}^{\infty} \bigcup_{m=0}^{\infty} D_{n,m},$$

where

$$D_{n,m} = [n, n+1) \times \left[\frac{1}{m+1}, \frac{1}{m}\right) \text{ and } D_{n,0} = [n, n+1) \times [1, +\infty), \qquad n \in \mathbb{Z}, m \in \mathbb{N}.$$

Consider the graph $G = \{(x, f(x)) \mid x \in A\}$ of f. Since A is uncountable, the set G is also uncountable. If all the sets $G \cap D_{n,m}$ were finite, then the set G would be countable. So, there exist $n_0 \in \mathbb{Z}$, $m_0 \geq 0$ such that $G \cap D_{n_0,m_0}$ is infinite. This means that there exists an infinite set $X \subset [n_0, n_0 + 1) \cap A$: for all $x \in X$, $f(x) > \frac{1}{m_0+1}$. Then, by Bolzano-Weierstrass theorem, there exists a sequence of distinct elements $(x_n) \subset X$ which converges to some $x_0 \in [n_0, n_0 + 1]$. But then, (x_n) is a Cauchy sequence and for all $k \in N$ we have $f(x_k) > \frac{1}{m_0+1}$. This leads to a contradiction:

$$\lim_{k\to\infty} |x_{k+1} - x_k| = 0 \quad \text{and} \quad \min\{f(x_k), f(x_{k+1})\} \geq \frac{1}{m_0 + 1}.$$

Problem AN5

Let a_n $(n = 1, 2, \dots)$ be a non-increasing sequence of non-negative numbers such that $\sum_{n=1}^{\infty} a_n < +\infty$. Prove that $na_n \to 0$ as $n \to \infty$.

First Solution. Assume that na_n does not converge to 0. Then, we can find $c > 0$ and a subsequence a_{i_k} $(k = 1, 2, \dots)$ such that $a_{i_k} \geq c/k$ for all $k \in \mathbb{N}$.

We define a new sequence b_n $(n = 1, 2, \dots)$ as follows: for all $n \in \mathbb{N}$ we set $b_n = c/i_{k(n)}$, where $k(n) = \min\{k : i_k \geq n\}$. Since a_n is non-increasing, one can check that for all $n \in \mathbb{N}$,

$$b_n = c/i_{k(n)} \leq a_{i_{k(n)}} \leq a_n.$$

Hence,

$$\sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} b_n.$$

We shall prove that $\sum_{n=1}^{\infty} b_n = +\infty$, which is a contradiction. Suppose that $\lim_{n\to\infty} b_n = 0$, otherwise the statement is clearly true. Consider $\sum_{n=1}^{\infty} b_n$ as the area of a histogram, where the n-th column has width 1 and height b_n. Set $i_0 = 0$. Summing by columns we get

$$\sum_{n=1}^{\infty} b_n = \sum_{k=1}^{\infty} (i_k - i_{k-1}) b_{i_k} = c \sum_{k=1}^{\infty} (i_k - i_{k-1})/i_k = c \sum_{k=1}^{\infty} (1 - i_{k-1}/i_k).$$

Summing by layers, we get

$$\sum_{n=1}^{\infty} b_n = c \sum_{k=1}^{\infty} (b_{i_k} - b_{i_{k+1}}) i_k = c \sum_{k=1}^{\infty} (1/i_k - 1/i_{k+1}) i_k = c \sum_{k=1}^{\infty} (1 - i_k/i_{k+1}).$$

The last sum can be rewritten as $c \sum_{k=2}^{\infty} (1 - i_{k-1}/i_k)$. So, the two sums differ by $c(1 - i_0/i_1) = c \neq 0$. This can be possible only if $\sum_{n=1}^{\infty} b_n = +\infty$.

Second Solution. Let $\varepsilon > 0$. Since $\sum\limits_{n=1}^{\infty} a_n < +\infty$, the sequence $s_n = a_1 + \cdots + a_n$ of partial sums is a Cauchy sequence. So, there exists $n_0 \in \mathbb{N}$ with the following property: if $n > m \geq n_0$ then

$$a_{m+1} + \cdots + a_n = |s_n - s_m| < \frac{\varepsilon}{2}.$$

In particular, if $n \geq 2n_0$, choosing $m = n_0$ and using the fact that (a_n) is non-increasing, we get

$$\frac{\varepsilon}{2} > a_{n_0+1} + \cdots + a_n \geq (n - n_0) a_n \geq \frac{n a_n}{2},$$

because $n - n_0 \geq \frac{n}{2}$. In other words, for all $n \geq 2n_0$ we have $na_n < \varepsilon$. It follows that $\lim\limits_{n\to\infty} (na_n) = 0$.

Problem AN6

Let n be a positive integer and $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 x^k f(x)\,dx = 1$$

for every $k \in \{0,1,\dots,n-1\}$. Prove that

$$\int_0^1 f^2(x)\,dx \geq n^2.$$

Solution. There exists a polynomial $p(x) = a_1 + a_2x + \cdots + a_nx^{n-1}$ which satisfies

$$\int_0^1 x^k p(x)\,dx = 1 \quad \text{for all } k = 0,1,\dots,n-1. \tag{1}$$

It follows that, for all $k = 0,1,\dots,n-1$,

$$\int_0^1 x^k(f(x) - p(x))\,dx = 0,$$

and hence

$$\int_0^1 p(x)(f(x) - p(x))\,dx = 0.$$

Then, we can write

$$\begin{aligned}\int_0^1 (f(x) - p(x))^2 dx &= \int_0^1 f(x)(f(x) - p(x))\,dx \\ &= \int_0^1 f^2(x)\,dx - \sum_{k=0}^{n-1} a_{k+1} \int_0^1 x^k f(x)\,dx,\end{aligned}$$

and since the first integral is non-negative we get

$$\int_0^1 f^2(x)\,dx \geq a_1 + a_2 + \cdots + a_n.$$

To complete the proof we show the following:

Claim. For the coefficients $a_1, \dots, a_n$ of p we have

$$a_1 + a_2 + \cdots + a_n = n^2.$$

Proof of the Claim. The defining property of p can be written in the form

$$\frac{a_1}{k+1} + \frac{a_2}{k+2} + \cdots + \frac{a_n}{k+n} = 1, \qquad 0 \leq k \leq n-1.$$

Equivalently, the function

$$r(x) = \frac{a_1}{x+1} + \frac{a_2}{x+2} + \cdots + \frac{a_n}{x+n} - 1$$

has $0, 1, \ldots, n-1$ as zeros. We write r in the form

$$r(x) = \frac{q(x) - (x+1)(x+2)\cdots(x+n)}{(x+1)(x+2)\cdots(x+n)},$$

where q is a polynomial of degree $n-1$. Observe that the coefficient of x^{n-1} in q is equal to $a_1 + a_2 + \cdots + a_n$. Also, the numerator has $0, 1, \ldots, n-1$ as zeros, which shows that

$$q(x) = (x+1)(x+2)\cdots(x+n) + x(x-1)\cdots(x-(n-1)).$$

This expression for q shows that the coefficient of x^{n-1} in q is $\frac{n(n+1)}{2} + \frac{(n-1)n}{2}$. It follows that

$$a_1 + a_2 + \cdots + a_n = n^2.$$

Problem AN7

Prove that every positive rational number can be written as the sum of finitely many fractions of the form $\frac{1}{n}$ with $n \geq 2$ and distinct denominators.

Solution. Let x be a positive rational number. We distinguish three cases:

(i) $x = 1$. Then, we can write $x = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$.

(ii) $0 < x < 1$. We write $x = \frac{a}{b}$, where $a, b \in \mathbb{N}$ with $0 < a < b$. We write n_1 for the least positive integer such that $n_1 \geq \frac{b}{a}$ and set $a_1 = n_1 a - b$, $b_1 = n_1 b$. Then, we can easily check that $0 \leq a_1 < a$, $n_1 > 1$ and $x = \frac{a}{b} = \frac{1}{n_1} + \frac{a_1}{b_1}$. If $a_1 = 0$ then $x = \frac{a}{b} = \frac{1}{n_1}$ and this proves the assertion. Assume that $a_1 > 0$. Repeating the same argument for $x_1 = \frac{a_1}{b_1} \in (0,1)$ we find n_1, a_2, b_2 such that n_2 is the least positive integer satisfying $n_2 \geq \frac{b_2}{a_2}$, $a_2 = n_2 a_1 - b_1$ and $b_2 = n_2 b_1$. Then, $0 \leq a_2 < a_1$ and $\frac{a_1}{b_1} = \frac{1}{n_2} + \frac{a_2}{b_2}$. Moreover, $n_2 a > n_2 a_1 \geq b_1 = n_1 b$ which implies $\frac{n_2}{n_1} > \frac{b}{a} > 1$, and hence, $n_2 > n_1$. As before, if $a_2 = 0$ the proof is complete. If $a_2 > 0$, we proceed in a similar way to obtain a sequence of fractions

$$\frac{a}{b}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_k}{b_k}, \ldots.$$

Our construction shows that the sequence of numerators is strictly decreasing. Therefore, for some k we will have $a_k = 1$. This means that there exists k such that

$$x = \frac{a}{b} = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k}$$

and all the denominators are distinct integers greater than 1.

(iii) $x > 1$. Since the series $\sum_{k=2}^{\infty} \frac{1}{k}$ is divergent and the sequence of its partial sums is strictly increasing, there exists $n \geq 2$ such that

$$\sum_{k=2}^{n} \frac{1}{k} \leq x < \sum_{k=2}^{n+1} \frac{1}{k}.$$

If $x = \sum_{k=2}^{n} \frac{1}{k}$, then the proof is complete. If $x > \sum_{k=2}^{n} \frac{1}{k}$, we define $y = x - \sum_{k=2}^{n} \frac{1}{k} \in (0,1)$. As in the first part of the proof, we can write y in the form

$$y = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_s}, \tag{1}$$

where $2 \leq n_1 < n_2 < \cdots < n_s$. If we prove that $n_1 > n$ then the proof is complete: we have

$$x = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_s}.$$

For the inequality $n_1 > n$, observe that $y = x - \sum_{k=2}^{n} \frac{1}{k} < \frac{1}{n+1}$ and, by (1), $y \geq \frac{1}{n_1}$. It follows that $\frac{1}{n_1} < \frac{1}{n+1}$, and hence, $n_1 > n + 1 > n$.

Let $f : [0, a] \to \mathbb{R}$ be twice differentiable. Assume that f, f' and f'' are continuous on $[0, a]$ and $f(0) = f'(0) = 0$. Prove that

$$\int_0^a \sqrt{|f(x)f'(x)|}\,|f''(x)|\,dx \leq \frac{a\sqrt{a}}{2}\int_0^a (f''(x))^2 dx.$$

Solution. An application of the Cauchy-Schwarz inequality shows that

$$\left(\int_0^a \sqrt{|f(x)f'(x)|}\,|f''(x)|\,dx\right)^2 \leq \left(\int_0^a |f(x)f'(x)|\,dx\right)\left(\int_0^a (f''(x))^2 dx\right).$$

So, it is enough to prove that

$$(1) \qquad \int_0^a |f(x)f'(x)|\,dx \leq \frac{a^3}{4}\int_0^a (f''(x))^2 dx.$$

Since $f(0) = 0$, we can write

$$|f(x)| = |f(x) - f(0)| = \left|\int_0^a f'(t)\,dt\right| \leq \int_0^a |f'(t)|\,dt.$$

Define

$$G(x) = \left(\int_0^x |f'(t)|\,dt\right)^2.$$

Then,

$$G'(x) = 2|f'(x)|\int_0^x |f'(t)|\,dt \geq 2|f(x)f'(x)|.$$

It follows that

$$(2) \qquad \int_0^a |f(x)f'(x)|\,dx \leq \frac{1}{2}\int_0^x G'(x)\,dx = \frac{G(a)}{2} = \frac{1}{2}\left(\int_0^a |f'(t)|\,dt\right)^2.$$

One more application of the Cauchy-Schwarz inequality shows that

$$(3) \quad \left(\int_0^a |f'(t)|\,dt\right)^2 \leq \left(\int_0^a dx\right)\left(\int_0^a (f'(x))^2 dx\right) = a\left(\int_0^a (f'(x))^2 dx\right).$$

Since $f'(0) = 0$, we can write $f'(x) = \int_0^x f''(t)\,dt$, and Cauchy-Schwarz inequality shows that

$$(f'(x))^2 = \left(\int_0^x f''(t)\,dt\right)^2 \leq x\int_0^x (f''(t))^2\,dt \leq x\int_0^a (f''(t))^2\,dt.$$

It follows that

$$(4) \qquad \int_0^a (f'(x))^2 dx \leq \int_0^a x\int_0^a \left((f''(t))^2\,dt\right)\,dx = \frac{a^2}{2}\int_0^a (f''(t))^2\,dt.$$

Combining (2), (3) and (4) we get

$$\int_0^a |f(x)f'(x)|\,dx \leq \frac{1}{2}\left(\int_0^a |f'(t)|\,dt\right)^2 \leq \frac{a}{2}\left(\int_0^a (f'(x))^2 dx\right) \leq \frac{a^3}{4}\int_0^a (f''(t))^2\,dt.$$

This proves (1).

Problem AN9

Let $f:(0,\infty)\to(0,\infty)$ *be a continuous and strictly increasing function. Assume that the equation* $f(x)=x$ *has a unique solution* $c>0$. *Let* $u_0>0$ *and define* $u_{n+1}=f(u_n)$, $n\geq 0$. *Prove that the sequence* $(u_n)_{n\geq 0}$ *is convergent and find its limit.*

Solution. Using the intermediate value theorem one can check that $f(x)>x$ for $x\in[0,c)$ and $f(x)<x$ for $x\in(c,\infty)$.

We distinguish three cases:

(i) If $u_0=c$ then $u_1=f(u_0)=f(c)=c=u_0$. Inductively, we see that $u_n=c$ for all $n\geq 0$. It follows that $u_n\to c$.

(ii) If $0<u_0<c$ we have: $u_1=f(u_0)<f(c)=c$ (because f is strictly increasing) and $u_1=f(u_0)>u_0$ (because $u_0<c$). Therefore, $0<u_0<u_1<c$. Inductively, we see that $(u_n)_{n\geq 0}$ is increasing and bounded from above by c. It follows that $u_n\to y$ for some $y\leq c$. By the continuity of f at y we get $u_{n+1}=f(u_n)\to f(y)$, and hence, $f(y)=y$. It follows that $y=c$ and $u_n\to c$.

(iii) If $u_0>c$ we have: $u_1=f(u_0)>f(c)=c$ (because f is increasing) and $u_1=f(u_0)<u_0$ (because $u_0>c$). Therefore, $u_0>u_1>c$. Inductively, we see that $(u_n)_{n\geq 0}$ is decreasing and bounded from below by c. It follows that $u_n\to y$ for some $y\geq c$. By the continuity of f at y we get $u_{n+1}=f(u_n)\to f(y)$, and hence, $f(y)=y$. It follows that $y=c$ and $u_n\to c$.

Problem AN10

Let $f : [0, \infty) \to [0, \infty)$ be a continuous function and let $p \geq 1$ be a real number. Define $I_p = \int_0^\infty f^p(x)\,dx$ and assume that $I_1 < \infty$. Prove the following:

(i) If f is uniformly continuous on $[0, \infty)$ then it is bounded on $[0, \infty)$.

(ii) If f is bounded on $[0, \infty)$ then $I_p < \infty$ for all $p \geq 1$.

(iii) The converse of the above statements are false.

Solution. (i) One can actually show that $\lim_{x\to\infty} f(x) = 0$. If not, we can find $M > 0$ and a sequence (x_n) in $[0, \infty)$ such that $x_n \to \infty$ and $f(x_n) \geq M$ for all $n \in \mathbb{N}$. We can also assume that $x_{n+1} > x_n + 1$ for all $n \in \mathbb{N}$ (define the x_n's inductively).

Since f is uniformly continuous, we can find $0 < \delta < 1$ such that $|x - y| \leq \delta$ implies $|f(x) - f(y)| \leq \frac{M}{2}$. It follows that $f(x) \geq \frac{M}{2}$ for all $n \in \mathbb{N}$ and all $x \in J_n := [x_n - \delta, x_n + \delta]$. Since $0 < \delta < 1$, the intervals J_n are disjoint. Now, f is non-negative, and hence,

$$I_1 = \int_0^\infty f(x)\,dx \geq \sum_{n=1}^{N} \int_{J_n} f(x)\,dx \geq NM\delta$$

for all $N \in \mathbb{N}$. This shows that $I_1 = \infty$, a contradiction.

(ii) Assume that there exists $M > 0$ such that $f(x) \leq M$ for every $x \in [0, \infty)$. Then, for every $p \geq 1$ we have

$$I_p = \int_0^\infty f^{p-1}(x) f(x)\,dx \leq M^{p-1} \int_0^\infty f(x)\,dx = M^{p-1} I_1 < \infty.$$

(iii) A counterexample for the converse of (i). We set $I_n = \left[n - \frac{1}{n^2}, n\right]$, $J_n = \left[n, n + \frac{1}{n^2}\right]$ for $n \geq 2$, and define f as follows:

$$f(x) = 1 + n^2(x - n) \text{ on } I_n, \quad f(x) = 1 - n^2(x - n) \text{ on } J_n, \quad f(x) = 0 \text{ otherwise.}$$

Check that f is continuous on $[0, \infty)$ and

$$I_1(f) = \int_0^\infty f(x)\,dx = \sum_{n=2}^{\infty} \int_{J_n \cup I_n} f(x)\,dx = \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty.$$

On the other hand, f is not uniformly continuous: consider the sequences $a_n = n$ and $b_n = n + \frac{1}{n^2}$, $n \geq 2$. We have $b_n - a_n = \frac{1}{n^2} \to 0$ but $|f(b_n) - f(a_n)| = 1$ for all $n \geq 2$.

A counterexample for the converse of (ii). Fix $p \geq 1$ and consider any $q > p$. We set $I_n = \left[n - \frac{1}{n^{q+1}}, n\right]$, $J_n = \left[n, n + \frac{1}{n^{q+1}}\right]$ for $n \geq 2$, and define f as follows:

$$f(x) = n^{q+2}(x - n) + n \text{ on } I_n, \; f(x) = n - n^{q+2}(x - n) \text{ on } J_n, \; f(x) = 0 \text{ otherwise.}$$

Check that f is continuous on $[0, \infty)$ and

$$\begin{aligned} I_p(f) &= \int_0^\infty f(x)\,dx = 2\sum_{n=2}^{\infty}\int_n^{n+\frac{1}{n^{q+1}}} \left(n - n^{q+2}(x-n)\right)^p\,dx \\ &= 2\sum_{n=2}^{\infty}\int_0^{\frac{1}{n^{q+1}}} \left(n - n^{q+2}y\right)^p\,dy = 2\sum_{n=2}^{\infty}\frac{1}{n^{q-p+1}}\int_0^1 (1-z)^p\,dz \\ &= \frac{2}{p+1}\sum_{n=2}^{\infty}\frac{1}{n^{q-p+1}} < \infty. \end{aligned}$$

On the other hand, f is not bounded: observe that $f(n) = n$ for all $n \geq 2$.

Problem AN11

Let $f : [0,\infty) \to [0,\infty)$, $g : [0,\infty) \to (0,\infty)$ be two continuous functions. Define $I_g = \int_0^\infty g(f(x))f(x)\,dx$ and assume that $I_1 = \int_0^\infty f(x)\,dx < \infty$. Prove the following:

(i) If f is uniformly continuous on $[0,\infty)$ then it is bounded on $[0,\infty)$.

(ii) If f is bounded on $[0,\infty)$ then $I_g < \infty$.

(iii) The converse of the above statements are false.

Solution. (i) One can actually show that $\lim_{x\to\infty} f(x) = 0$. If not, we can find $M > 0$ and a sequence (x_n) in $[0,\infty)$ such that $x_n \to \infty$ and $f(x_n) \ge M$ for all $n \in \mathbb{N}$. We can also assume that $x_{n+1} > x_n + 1$ for all $n \in \mathbb{N}$ (define the x_n's inductively).

Since f is uniformly continuous, we can find $0 < \delta < 1$ such that $|x-y| \le \delta$ implies $|f(x) - f(y)| \le \frac{M}{2}$. It follows that $f(x) \ge \frac{M}{2}$ for all $n \in \mathbb{N}$ and all $x \in J_n := [x_n - \delta, x_n + \delta]$. Since $0 < \delta < 1$, the intervals J_n are disjoint. Now, f is non-negative, and hence,

$$I_1 = \int_0^\infty f(x)\,dx \ge \sum_{n=1}^{N} \int_{J_n} f(x)\,dx \ge NM\delta$$

for all $N \in \mathbb{N}$. This shows that $I_1 = \infty$, a contradiction.

(ii) Assume that there exists $M > 0$ such that $f(x) \le M$ for every $x \in [0,\infty)$. Since g is continuous on $[0,M]$, there exists $D > 0$ such that $g(y) \le D$ for all $0 \le y \le M$. It follows that $g(f(x)) \le D$ for all $x \ge 0$. Then,

$$I_g = \int_0^\infty g(f(x))f(x)\,dx \le D\int_0^\infty f(x)\,dx = DI_1 < \infty.$$

(iii) A counterexample for the converse of (i). We set $I_n = \left[n - \frac{1}{n^2}, n\right]$, $J_n = \left[n, n + \frac{1}{n^2}\right]$ for $n \ge 2$, and define f as follows:

$f(x) = 1 + n^2(x-n)$ on I_n, $\quad f(x) = 1 - n^2(x-n)$ on J_n, $\quad f(x) = 0$ otherwise.

Check that f is continuous on $[0,\infty)$ and

$$I_1(f) = \int_0^\infty f(x)\,dx = \sum_{n=2}^{\infty} \int_{J_n \cup I_n} f(x)\,dx = \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty.$$

On the other hand, f is not uniformly continuous: consider the sequences $a_n = n$ and $b_n = n + \frac{1}{n^2}$, $n \ge 2$. We have $b_n - a_n = \frac{1}{n^2} \to 0$ but $|f(b_n) - f(a_n)| = 1$ for all $n \ge 2$.

A counterexample for the converse of (ii). For $n \ge 2$ we define

$$a_n = \int_0^n xg(x)\,dx \ge \int_0^1 xg(x)\,dx =: r > 0.$$

There exists $k \in \mathbb{N}$ such that $na_n \geq nr > 2$ for all $n \geq k$ We set $I_n = \left[n - \frac{1}{na_n}, n\right]$, $J_n = \left[n, n + \frac{1}{na_n}\right]$ for $n \geq k$, and define f as follows:

$f(x) = a_n n^2(x-n)+n$ on I_n, $f(x) = n-a_n n^2(x-n)$ on J_n, $f(x) = 0$ otherwise.

Check that the intervals $I_n \cup J_n$, $n \geq k$ are disjoint, f is continuous on $[0, \infty)$ and

$$
\begin{aligned}
I_g(f) &= \int_0^\infty g(f(x))f(x)\,dx \\
&= 2\sum_{n=k}^{\infty} \int_n^{n+\frac{1}{na_n}} (n - a_n n^2(x-n))g(n - a_n n^2(x-n))\,dx \\
&= 2\sum_{n=k}^{\infty} \int_0^{\frac{1}{na_n}} (n - a_n n^2 y)g(n - a_n n^2 y)\,dy \\
&= 2\sum_{n=k}^{\infty} \frac{1}{a_n n^2} \int_0^n (n-z)g(n-z)\,dz \\
&= 2\sum_{n=k}^{\infty} \frac{1}{a_n n^2} a_n = 2\sum_{n=k}^{\infty} \frac{1}{n^2} < \infty.
\end{aligned}
$$

On the other hand, f is not bounded: observe that $f(n) = n$ for all $n \geq k$.

Problem AN12

Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that

$$\int_a^b x^k f(x)\, dx = 0$$

for $k = 0, 1, 2, \ldots, n-1$.

Prove that the equation $f(x) = 0$ has at least n roots.

Sketch of a Solution. Suppose the contrary. Then there are $k \leq n-1$ points $a < x_1 < x_2 < \ldots < x_k < b$ where the sign of $f(x)$ changes. Consider the polynomial

$$g(x) = (x - x_1)(x - x_2) \cdots (x - x_k) = x^k + \alpha_{k-1} x^{k-1} + \cdots + \alpha_1 x + \alpha_0.$$

The sign of the product $f(x)g(x)$ is constant (with the exception of the finite set of points x which satisfy $f(x)g(x) = 0$), hence

$$\int_a^b g(x) f(x)\, dx \neq 0.$$

On the other hand,

$$\int_a^b g(x) f(x)\, dx = \sum_{i=0}^{k} \alpha_i \int_a^b x^i f(x)\, dx = \sum 0 = 0.$$

This leads to a contradiction.

Chapter 2

Algebra

Problem AL1

For any square matrix $A = (a_{ij})$, denote by $A^{\mathsf{H}} = (\widetilde{a}_{ij})$ the matrix with elements

$$\widetilde{a}_{ij} = \begin{cases} a_{ij}, & \text{if } i+j \text{ is even} \\ a_{ji}, & \text{if } i+j \text{ is odd.} \end{cases}$$

For each positive integer n, find all $n \times n$ matrices A such that the equality $(AB)^{\mathsf{H}} = B^{\mathsf{H}} A^{\mathsf{H}}$ holds for all $n \times n$ matrices B.

Solution. If $n \leq 2$ then any matrix A satisfies the condition, since in this case the operation $A \mapsto A^{\mathsf{H}}$ is the usual transposition of a matrix.

Suppose $n \geq 3$ and let B be the matrix with $b_{ij} = 1$ and all other entires equal to zero. It follows from $(AB)^{\mathsf{H}} = B^{\mathsf{H}} A^{\mathsf{H}}$ that $a_{jk} = 0$ for $k \neq j$ and $a_{jj} = a_{ii}$ if $i + j$ is even. Hence the matrix A should be diagonal with elements $a, b, a, b, \ldots$ in the diagonal. It is easy to check that such matrices A satisfy the condition.

Problem AL2

Let $m \leq n$ be positive integers and $P(x) = x^n + p_{n-1}x^{n-1} + \cdots + p_0$ be a polynomial with complex coefficients and no more than m distinct complex roots. It is given that $P(x) \not\equiv x^n$. Prove that at least one coefficient among $p_{n-1}, \cdots, p_{n-m}$ is not equal to 0.

Solution. Let $P(x) = (x - x_1)^{s_1} \cdots (x - x_k)^{s_k}$, where $x_1, x_2, \ldots, x_k$ are the distinct roots of $P(x)$, so that $k \leq m$ by our hypothesis. Then $P'(x)$ is divisible by $Q(x) = (x - x_1)^{s_1-1} \cdots (x - x_k)^{s_k-1}$, which has degree $n - k \geq n - m$. The polynomial $nP(x) - xP'(x)$ is divisible by $Q(x)$ too, so either it is identically zero or its degree is at least $n - m$. But

$$nP(x) - xP'(x) = p_{n-1}x^{n-1} + 2p_{n-2}x^{n-2} + \cdots + np_0$$

is identically zero only if $P(x) \equiv x^n$, which is forbidden. Therefore we must have $\deg(nP(x) - xP'(x)) \geq n - m$ and hence some coefficient among $p_{n-1}, \cdots, p_{n-m}$ is not equal to 0.

Problem AL3

Let $P(x)$ and $Q(x)$ be two real polynomials (possibly of different degrees). If the set of points where $P(x)$ attains integer values is equal to the set of points where $Q(x)$ attains integer values, prove that $P(x) - Q(x) \equiv c$ or $P(x) + Q(x) \equiv c$.

Solution. By continuity considerations, one may assume that neither $P(x)$ nor $Q(x)$ is a constant polynomial. One may assume further that $P(x)$ and $Q(x)$ have leading coefficients with the same sign, for example positive (otherwise one may replace $Q(x)$ by $-Q(x)$).

So let the leading coefficients of $P(x)$ and $Q(x)$ be positive. Then there exists $a > 0$ such that $P(x)$ and $Q(x)$ increase when $x > a$. Let $t_1 < t_2 < \cdots$ be those points $t > a$ where $P(x)$ attains integer values. By continuity, these values must be successive integers $f, f+1, f+2 \ldots$. By our hypothesis, $Q(x)$ also attains integer values at these points, which must also be successive integers. So $P(x) - Q(x)$ attains the same value at infinitely many points $t_1 < t_2 < \cdots$ and hence must be a constant polynomial.

Problem AL4

Let $P(x), Q(x), R(x)$ be nonzero polynomials with complex coefficients. Let a, b, c be distinct nonzero complex numbers and set

$$z_n = P(n)\, a^n + Q(n)\, b^n + R(n)\, c^n.$$

Prove that if the set $\{z_n : n \in \mathbb{N}\}$ is finite, then there exists $p \in \mathbb{N} \setminus \{0\}$ such that $z_{n+p} = z_n$ for every $n \in \mathbb{N}$.

Solution. Let $z'_n = z_{n+1} - az_n$, so that

$$z'_n = P_1(n)\, a^n + Q_1(n)\, b^n + R_1(n) c^n$$

where

$$\begin{aligned} P_1(n) &= a(P(n+1) - P(n)), \\ Q_1(n) &= bQ(n+1) - aQ(n), \\ R_1(n) &= cR(n+1) - aR(n) \end{aligned}$$

have degrees $\deg(P) - 1$, $\deg(Q)$ and $\deg(R)$, respectively. Since $\{z_n : n \in \mathbb{N}\}$ is a finite set, so is $\{z'_n : n \in \mathbb{N}\}$. Repeating the argument finitely many times we can transform any two of the polynomials $P(x), Q(x), R(x)$ into zero and the third into a nonzero constant polynomial. We conclude that all three sets $\{a^n : n \in \mathbb{N}\}$, $\{b^n : n \in \mathbb{N}\}$ and $\{c^n : n \in \mathbb{N}\}$ are finite and hence that $a^p = b^q = c^r = 1$ for some positive integers p, q, r. Replacing these integers by their least common multiple we may assume that $p = q = r$, so that $a^p = b^p = c^p = 1$ for some positive integer p. Setting $n = kp$ in the formula for z_n, it follows that the set $\{P(kp) + Q(kp) + R(kp) : k \in \mathbb{N}\}$ is finite and hence that the polynomial $P + Q + R$ is constant. Similarly, setting $n = kp + 1$ and $n = kp + 2$ we find that the polynomials $aP + bQ + cR$ and $a^2P + b^2Q + c^2R$ are constants. Hence P, Q and R are constants and the result follows.

Remark. Using generating functions instead, we have

$$\sum_{n \geq 0} z_n t^n = \frac{f(n)}{(1 - at)^m} + \frac{g(n)}{(1 - bt)^r} + \frac{h(n)}{(1 - ct)^s}$$

for some polynomials f, g and h (of degrees less than m, r and s, respecively). The operation of replacing z_n by z'_n in the preceeding proof corresponds (essentially) to multiplying this generating function by $1 - at$.

Problem AL5

Let $\mathcal{M}_n(\mathbb{R})$ denote the set of all real $n \times n$ matrices. Find all surjective functions $f : \mathcal{M}_n(\mathbb{R}) \to \{0, 1, \dots, n\}$ which satisfy

$$f(X \cdot Y) \leq \min\{f(X), f(Y)\}$$

for all $X, Y \in \mathcal{M}_n(\mathbb{R})$.

Solution. We will show that the only such function is $f(X) = \operatorname{rank}(X)$. Setting $Y = I_n$ we find that $f(X) \leq f(I_n)$ for all $X \in \mathcal{M}_n(\mathbb{R})$. Setting $Y = X^{-1}$ we find that $f(I_n) \leq f(X)$ for all invertible $X \in \mathcal{M}_n(\mathbb{R})$. From these facts we conclude that $f(X) = f(I_n)$ for all $X \in GL_n(\mathbb{R})$.

For $X \in GL_n(\mathbb{R})$ and $Y \in \mathcal{M}_n(\mathbb{R})$ we have

$$f(Y) = f(X^{-1}XY) \leq f(XY) \leq f(Y),$$
$$f(Y) = f(YXX^{-1}) \leq f(YX) \leq f(Y).$$

Hence we have $f(XY) = f(YX) = f(Y)$ for all $X \in GL_n(\mathbb{R})$ and $Y \in \mathcal{M}_n(\mathbb{R})$. For $k = 0, 1, \dots, n$, let

$$J_k = \begin{pmatrix} I_k & O \\ O & O \end{pmatrix}.$$

It is well known that every matrix $Y \in \mathcal{M}_n(\mathbb{R})$ is equivalent to J_k for $k = \operatorname{rank}(Y)$. This means that there exist matrices $X, Z \in GL_n(\mathbb{R})$ such that $Y = XJ_kZ$. From the discussion above it follows that $f(Y) = f(J_k)$. Thus it suffices to determine the values of the function f on the matrices $J_0, J_1, \dots, J_n$. Since $J_k = J_k \cdot J_{k+1}$ we have $f(J_k) \leq f(J_{k+1})$ for $0 \leq k \leq n-1$. Surjectivity of f imples that $f(J_k) = k$ for $k = 0, 1, \dots, n$ and hence $f(Y) = \operatorname{rank}(Y)$ for all $Y \in \mathcal{M}_n(\mathbb{R})$.

Problem AL6

Let G be a finite group and let p be the smallest prime number which divides the order of G. If H is a subgroup of G of index p in G, prove that H is a normal subgroup of G.

Solution. Let S be the set of left cosets of H in G and let $\text{Perm}(S)$ denote the group of permutations of S. For $x \in G$ consider the permutation $\phi_x : S \to S$ which associates the coset xyH to a coset yH. For $x, y \in G$ one checks that $\phi_x \circ \phi_y = \phi_{xy}$ and therefore the map $\phi : G \to \text{Perm}(S)$, defined by $\phi(x) = \phi_x$, is a group homomorphism (known as the left coset representation of G with respect to H).

Let $K = \ker(\phi)$. Since $\text{Perm}(S)$ has order $p!$ and G/K is isomorphic to a subgroup of $\text{Perm}(S)$, it follows that $(G : K)$ divides $p!$. We have $K \subseteq H$ (since $x \in K$ implies $xH = \phi_x(H) = H$, so that $x \in H$) and hence

$$(G : K) = (G : H)(H : K),$$

where $(G : H) = p$. It follows that $(H : K)$ divides $(p-1)!$. Finally, noting that $(H : K)$ also divides the order of G and that p is the smallest prime number which divides the order of G, we conclude that $(H : K) = 1$, so that $H = K$ is normal in G.

Remark. This problem generalizes the well known fact that any subgroup of index 2 in a finite group G is a normal subgroup of G.

Chapter 3

Combinatorics

Problem C1

Some m horizontal and n vertical lines break the square $D = [0,1] \times [0,1]$ into $(m+1)(n+1)$ rectangles (not necessarily congruent). Denote the left upper rectangle by A and the lower right one by B. A trace *from A to B is a set Π of $m+n+1$ such rectangles which includes A and B and whenever rectangle P belongs to Π, either the rectangle under P or the one right to P (but not both) belong to Π. Let $l(\Pi)$ be the sum of the areas of the rectangles in Π and denote by $\pi(D; m, n)$ the set of all traces from A to B. Prove that*

$$\max_{\Pi \in \pi(D;\, m,\, n)} l(\Pi) \;\geq\; \frac{m+n+1}{(m+1)(n+1)}.$$

Solution. Let us consider a little more general situation with rectangle $D_{a,b} = [0,a] \times [0,b]$ instead of the unit square. We will show that

$$\max_{\Pi \in \pi(D_{a,b};\, m,n)} l(\Pi) \;\geq\; \frac{m+n+1}{(m+1)(n+1)}\, ab.$$

Let us prove this statement by induction on the number $N = m+n+1$ of rectangles in a trace. For $N = 3$ our statement is trivial. Assume it has been proved for $N \geq 3$ and let $k = N+1$. It is clear that any trace Π from A to B includes either the rectangle B_1 over B or the rectangle B_2 to the left of B. Let rectangle B have dimensions $x \times y$, where $(x,y) \in (0,a) \times (0,b)$. Let us consider the rectangles P_1 and P_2 with dimensions $a \times (b-y)$ and $(a-x) \times b$, respectively, which are included in $D_{a,b}$ and include A. Clearly P_1 contains exactly $m(n+1)$ of the subrectangles of $D_{a,b}$ and P_2 contains exactly $n(m+1)$ of them. The induction hypothesis for rectangles P_1 and P_2 gives:

$$\max_{\Pi \in \pi(D_{a,b-y};\, m-1,n)} l(\Pi) \;\geq\; \frac{m+n}{m(n+1)} \cdot a(b-y)$$

and

$$\max_{\Pi \in \pi(D_{a-x,b};\, m,n-1)} l(\Pi) \;\geq\; \frac{m+n}{(m+1)n} \cdot (a-x)b.$$

Let $\widetilde{\Pi}_1$ be a trace in $\pi(D_{a,b-y};\, m-1, n)$ with length at least $\frac{m+n}{m(n+1)} \cdot a(b-y)$ and $\widetilde{\Pi}_2$ be a trace in $\pi(D_{a-x,b};\, m, n-1)$ with length at least $\frac{m+n}{(m+1)n} \cdot (a-x)b$. Then it is clear that $\Pi_1 = \widetilde{\Pi}_1 \cup B$ and $\Pi_2 = \widetilde{\Pi}_2 \cup B$ are traces in $\pi(D_{a,b};\, m, n)$. Consequently

$$\max_{\Pi \in \pi(D_{a,b};\, m,n)} l(\Pi) \;\geq\; \max\{l(\Pi_1),\, l(\Pi_2)\}$$
$$\geq \max\left\{\frac{m+n}{m(n+1)} \cdot a(b-y),\, \frac{m+n}{(m+1)n} \cdot (a-x)b\right\} + xy = f(x,y).$$

The minimum of the function $f(x,y)$ for $(x,y) \in D_{a,b}$ can be found to equal

$$\frac{m+n+1}{(m+1)(n+1)} \cdot ab$$

(for instance with Lagrange's method). This completes the inductive step.

Problem C2

What is the largest positive integer N for which we can draw a complete graph with N vertices on the plane, so that each arc is intersected by at most one other arc, no three arcs have a common inner point and no two arcs are tangent? (Of course, no vertex can be an inner point of any arc.)

Solution. We will show that the maximum is 6. For $N = 6$ an explicit drawing of the graph can be given, for instance so that the six vertices are those of two homothetic equilateral triangles with common centroid. Suppose, by the way of contradiction, that a drawing is also possible for $N = 7$. Consider two arcs AC and BD of the graph which cross each other at point X. A configuration such as this will be called a *cross*. Clearly, arcs AX, BX, CX and DX are not crossed by other arcs. Consider the (curvilinear) triangle ABX and suppose it contains a vertex E of the graph other than A, B. This vertex must be connected to C and D. Since arcs CE and DE cannot cross AX or BX, they must escape the triangle ABX through AB. Hence AB is crossed twice, which is forbidden. We conclude that triangle ABX does not contain vertices of the graph other than A, B. We will also show that no arc crosses edge AB. Indeed, if an arc enters triangle ABX, it must also escape it. Since both can be done only through the arc AB, this arc would be intersected twice, which is forbidden. Similar reasoning applies to triangles BCX, CDX and DAX. Thus each cross is bounded from all four sides by arcs which are not intersected. So, if we have n crosses, these correspond to $4n$ arcs with no intersections. Those arcs might be counted twice (since each arc can be approached from two sides) and hence there are at least $2n$ arcs which are not intersected. Also, each intersection is formed by two arcs, so n intersections are formed by $2n$ intersecting arcs and none is counted twice. So n crosses require at least $4n$ arcs ($2n$ intersecting ones and $2n$ non-intersecting). In a complete graph of 7 vertices we have only $7 \cdot 6/2 = 21$ arcs, so no more than 5 crosses. Consider the graph whose vertices are the 7 original points and all n intersection points, formed by the crosses, and arcs among them. Since this graph is planar, Euler formula $v - e + f = 2$ applies to it. Clearly $v = 7 + n$ and $e = 21 + 2n$, since for each of the n crosses we have split two arcs of the original graph in two. Hence the number of faces satisfies $(7 + n) - (21 + 2n) + f = 2$ and hence $f = n + 16$. Each face of the new graph has at least three edges and each edge has exactly two incident faces. Hence, by counting face-edge incidencies, we get $3f \leq 2e$. This gives $3(n + 16) \leq 2(21 + 2n)$, so $n \geq 6$. However we have already shown that $n \leq 5$. This contradiction shows that such a drawing of a complete graph with 7 vertices is impossible.

Chapter 4

Analytic Geometry

Problem G1

Let A, B, C, D be four distinct spheres in space. Suppose spheres A and B intersect along a circle which is contained in a plane P, spheres B and C intersect along a circle which is contained in a plane Q, spheres C and D intersect along a circle which is contained in a plane S, and spheres D and A intersect along a circle which is contained in a plane T. Show that planes P, Q, S, T are either parallel to the same line or have a common point.

Solution. Let $a(x, y, z)$ be a polynomial of the form $x^2 + y^2 + z^2 +$ linear part which defines sphere A. Similarly, let b, c, d, be the polynomials defining spheres B, C, D. The difference of two such polynomials is linear, since the quadratic part is cancelled out, and gives the equation of the plane which contains the circle of their intersection. Hence the equation of the four planes, mentioned in the statement of the problem, are $a - b = 0, b - c = 0, c - d = 0, d - a = 0$. Therefore the sum of the four equations is zero and hence the sum of normals to the planes (whose coordinates are the coefficients of x, y, z in the equations of these planes) is also zero. If three normals are linearly dependent, then they lie in a plane and are orthogonal to the same nonzero vector. The fourth normal, which is the negative of their sum, is also orthogonal to the same vector. Hence, in this case all normals are orthogonal to the same nonzero vector and the line parallel to it will be parallel to all four planes.

If thee of the normals are linearly independent, then three of the planes have a common point (x, y, z). The equations of these planes are valid at this point and so does that of the fourth plane, since the sum of the four equations is zero. Hence all four planes have a common point.

Problem G2

Find the curve traced out by the center of an ellipse that rolls along two perpendicular lines.

Solution. Suppose our ellipse is congruent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Choosing the two orthogonal tangents as the axes of a coordinate system and assuming the ellipse lies in the first quadrant, the center of the ellipse will trace out the arc of the circle $x^2 + y^2 = a^2 + b^2$ in the first quadrant between points (a, b) and (b, a). This follows easily from the next lemma and its proof.

Lemma. The set of all points of intersection of two orthogonal tangents to an ellipse is a circle.

Proof. Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Suppose that $t_1 : y = y_0 + k_1(x - x_0)$ and $t_2 : y = y_0 + k_2(x - x_0)$ are two orthogonal tangents, intersecting at (x_0, y_0). For $k = k_1, k_2$, the system

$$\begin{cases} b^2x^2 + a^2y^2 = a^2b^2 \\ y = y_0 + k(x - x_0) \end{cases}$$

has a unique solution. Equivalently, the quadratic equation

$$b^2x^2 + a^2(y_0 + kx - kx_0)^2 = a^2b^2$$

has a unique solution. This equation can be written as $Ax^2 + Bx + C = 0$, where $A = b^2 + a^2k^2$, $B = 2a^2k(y_0 - kx_0)$ and $C = a^2[(y_0 - kx_0)^2 - b^2]$, and has a unique solution if and only if $B^2 - 4AC = 0$. It follows that k_1 and k_2 are the two solutions of the equation

$$(a^2 - x_0^2)k^2 + 2x_0y_0k + b^2 - y_0^2 = 0.$$

As a result, we have

$$k_1k_2 = \frac{b^2 - y_0^2}{a^2 - x_0^2}.$$

On the other hand, the lines t_1 and t_2 are orthogonal if and only if $k_1k_2 = -1$. From the last two conditions we gather that $x_0^2 + y_0^2 = a^2 + b^2$.

Problem G3

Let $0 < \alpha < 1$. Let P_k be a convex n-gon $A_1A_2\cdots A_n$. Let $B_1, B_2, \ldots, B_n$ be the points of the sides $A_1A_2, A_2A_3, \ldots, A_nA_1$, respectively, for which

$$A_1B_1/A_1A_2 = A_2B_2/A_2A_3 = \cdots = A_nB_n/A_nA_1 = \alpha.$$

We denote by P_{k+1} the n-gon $B_1B_2\cdots B_n$. Given a convex n-gon P_0, define the sequence of n-gons $P_0, P_1, P_2, \ldots$. Prove that there exists a unique point lying inside P_k for all $k = 0, 1, 2, \ldots$.

Solution. Let O be the center of gravity of the points $A_1, A_2, \ldots, A_n$. Since $\overrightarrow{OB_1} + \overrightarrow{OB_2} + \cdots + \overrightarrow{OB_n} = \overrightarrow{OA_1} + \overrightarrow{OA_2} + \cdots + \overrightarrow{OA_n} = \overrightarrow{0}$, O belongs to each one of the polygons $P_0, P_1, P_2, \ldots$.

Let $R = \max\{OA_1, \ldots, OA_n\}$, so that P_k lies in the ball of radius R centered at O. Let $C_1, \ldots, C_n$ be the vertices of the polygon P_{k+n}. It is easy to check that $\overrightarrow{OC_1} = \sum\limits_{i=1}^{n} \beta_i \overrightarrow{OA_i}$ where $\beta_i > 0$ and $\sum\limits_{i=1}^{n} = 1$. Let $\lambda = \min\limits_{i=1,\ldots,n} \beta_i$. Since $\overrightarrow{OA_1} + \overrightarrow{OA_2} + \cdots + \overrightarrow{OA_n} = \overrightarrow{0}$, we have the following:

$$|\overrightarrow{OC_1}| = \left|\sum_{i=1}^{n}(\beta_i - \lambda)\overrightarrow{OA_{i+1}}\right| \leq \sum_{i=0}^{n-1}(\beta_i - \lambda)|\overrightarrow{OA_{i+1}}| \leq R\sum_{i=0}^{n-1}(\beta_i - \lambda) = R(1 - n\lambda).$$

This means that P_{k+n} lies in the ball of radius $R(1 - n\lambda)$ centered at O.

Continuing in the same way we see that the polygon P_{k+mn} lies in the ball of radius $R(1 - n\lambda)^m$ centered at O, therefore O is the unique common point for $P_0, P_1, P_2, \ldots$.

Note. The problem could be proposed for $\alpha = 1/2$.

Chapter 5

Number Theory

Problem NT1

Find all positive integers n such that there exist infinitely many pairs of positive integers (x, y) satisfying the equation

$$(*) \qquad 1^n + 2^n + \cdots + x^n = (x+1)^y.$$

Sketch of a Solution We claim that the number of pairs is finite for each n.

If $y \geq n+1$, then $x^{n+1} = x \cdot x^n \geq 1^n + 2^n + \cdots + x^n = (x+1)^y \geq (x+1)^{n+1}$ – a contradiction.

So, it is sufficient to show that for each $y = 1, 2, \ldots, n$ the set of x satisfying $(*)$ is finite. This can be seen as follows:

$$\lim_{x \to \infty} \frac{1}{x^{n+1}}(1^n + 2^n + \cdots + x^n) = \frac{1}{n+1},$$

but

$$\lim_{x \to \infty} \frac{1}{x^{n+1}}(x+1)^y = 0.$$