

THE 1997 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours

NO calculators are to be used.

Each question is worth seven points.

Question 1

Given

$$S = 1 + \frac{1}{1 + \frac{1}{3}} + \frac{1}{1 + \frac{1}{3} + \frac{1}{6}} + \cdots + \frac{1}{1 + \frac{1}{3} + \frac{1}{6} + \cdots + \frac{1}{1993006}} ,$$

where the denominators contain partial sums of the sequence of reciprocals of triangular numbers (i.e. $k = n(n+1)/2$ for $n = 1, 2, \dots, 1996$). Prove that $S > 1001$.

Question 2

Find an integer n , where $100 \leq n \leq 1997$, such that

$$\frac{2^n + 2}{n}$$

is also an integer.

Question 3

Let ABC be a triangle inscribed in a circle and let

$$l_a = \frac{m_a}{M_a} , \quad l_b = \frac{m_b}{M_b} , \quad l_c = \frac{m_c}{M_c} ,$$

where m_a, m_b, m_c are the lengths of the angle bisectors (internal to the triangle) and M_a, M_b, M_c are the lengths of the angle bisectors extended until they meet the circle. Prove that

$$\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} \geq 3,$$

and that equality holds iff ABC is an equilateral triangle.

Question 4

Triangle $A_1A_2A_3$ has a right angle at A_3 . A sequence of points is now defined by the following iterative process, where n is a positive integer. From A_n ($n \geq 3$), a perpendicular line is drawn to meet $A_{n-2}A_{n-1}$ at A_{n+1} .

(a) Prove that if this process is continued indefinitely, then one and only one point P is interior to every triangle $A_{n-2}A_{n-1}A_n$, $n \geq 3$.

(b) Let A_1 and A_3 be fixed points. By considering all possible locations of A_2 on the plane, find the locus of P .

Question 5

Suppose that n people A_1, A_2, \dots, A_n , ($n \geq 3$) are seated in a circle and that A_i has a_i

objects such that

$$a_1 + a_2 + \cdots + a_n = nN,$$

where N is a positive integer. In order that each person has the same number of objects, each person A_i is to give or to receive a certain number of objects to or from its two neighbours A_{i-1} and A_{i+1} . (Here A_{n+1} means A_1 and A_n means A_0 .) How should this redistribution be performed so that the total number of objects transferred is minimum?

Solutions

Note: The points to be awarded for each part of the solution are indicated on the right side.

Problem 1.

$$1 = \frac{1 \times 2}{2}$$

$$1 + \frac{1}{3} = \frac{2 \times 2}{3}$$

$$1 + \frac{1}{3} + \frac{1}{6} + \dots + \frac{1}{n} = \frac{n \times 2}{n+1}$$

which is easily shown by induction.

(up to 3 points)

Now S is the sum of the reciprocals of these numbers where the last, $1993006 = \frac{1996 \times 1997}{2} = 1996$. Thus we have

$$\begin{aligned} S &= \frac{1}{2} \left(\frac{2}{1} + \frac{3}{2} + \dots + \frac{1997}{1996} \right) \\ &= \frac{1}{2} \left(1996 + \left(1 + \frac{1}{2} + \dots + \frac{1}{1996} \right) \right) \end{aligned}$$

(up to 3 points)

$$> \frac{1}{2} (1996 + 6)$$

(1 point)

$$= 1001$$

Problem 2. Note that $2^n + 2 = 2(2^{n-1} + 1)$ so that n is of the form $2r$ with r odd. We will consider two cases.

i) $n = 2p$ with p prime. $2p \mid 2^{2p} + 2$, implies that $p \mid 2^{2p-1} + 1$ and hence, hence $p \mid 2^{4p-2} - 1$. On the other hand Fermat's little theorem guarantees that $p \mid 2^{p-1} - 1$. Let $d = \text{g.c.d.}(p-1, 4p-2)$. It follows that $p \mid 2^d - 1$. But $d \mid p-1$ and $d \mid 4p-2 = 4(p-1) + 2$. Hence $d \mid 2$ and since $p-1, 4p-2$ are even $d = 2$. Then $p = 3$ and $n = 6 < 100$.

(up to 2 points)

ii) $n = 2pq$ where p, q are odd primes. $p < q$ and $pq < \frac{1997}{2}$. Now $n \mid 2^n + 2$ implies that $p \mid 2^{n-1} + 1$ and therefore that $p \mid 2^{2q-2} - 1 = 2^{4pq-2} - 1$. Once again by Fermat's theorem we have $p \mid 2^{p-1} - 1$ which implies that $p-1 \mid 4pq-2$. The same holds true for q so that

$$q-1 \mid 4pq-2 \quad (1)$$

Both $p-1$ and $q-1$ are thus multiples of 2 but not of 4 so that $p \equiv q \equiv 3 \pmod{4}$.

(2 points)

Taking $p = 3$, we have $4pq - 2 = 12q - 2$. Now from (1) we have

$$12 \leq \frac{12q-2}{q-1} < \frac{12q-2}{q-1} = \frac{12(q-1)+10}{q-1} = 12 + \frac{10}{q-1} \leq 1$$

if $q \geq 11$, and clearly $\frac{12q-2}{q-1} = 13$ if $q = 11$. But this gives $n = 2(3)(11) = 66 < 100$. Furthermore $(p, q) = (3, 7)$ does not satisfy (1).

Taking $p = 7$ we observe that $4pq - 2 = 28q - 2$, and from (1) we have

$$28 < \frac{28q-2}{q-1} = \frac{28(q-1)+26}{q-1} = 28 + \frac{26}{q-1} \leq 2$$

if $q \geq 27$ and clearly $\frac{28q-2}{q-1} = 29$ if $q = 27$. But 27 is not prime and the cases $(p, q) = (7, 11)$, $(7, 19)$ and $(7, 23)$ do not satisfy (1).

Taking $p = 11$, then $4pq - 2 = 44q - 2$, and

$$44 < \frac{44q-2}{q-1} \text{ and } \frac{44q-2}{q-1} \leq 45 \text{ if } q \geq 43.$$

Now clearly $\frac{44q-2}{q-1} = 45$ when $q = 43$. In this case we have $n = 2pq = 2(11)(43) = 946$.

Furthermore, $\frac{2^{946}+2}{946}$ is indeed an integer. The cases $(p, q) = (11, 19)$, $(11, 23)$ and $(11, 31)$ do not satisfy (1).

(2 points)

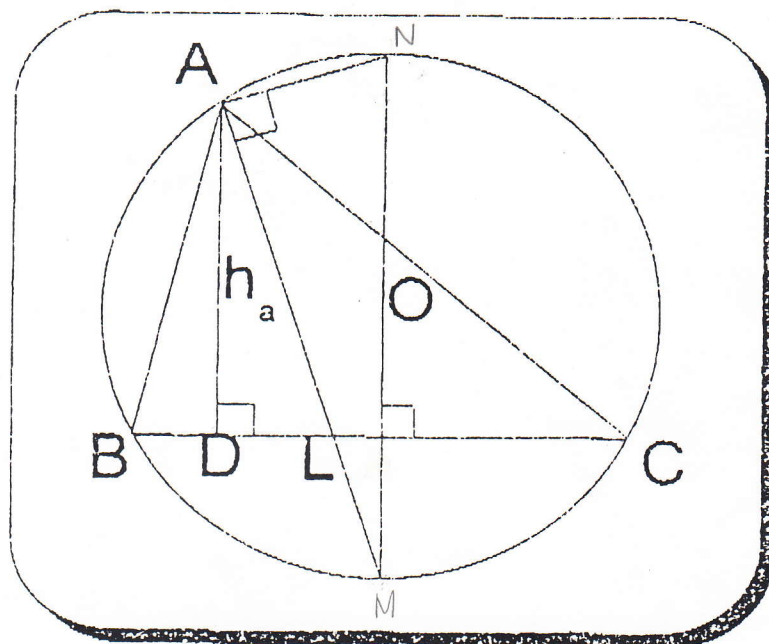
[Additionally for completeness, if $p = 19$ then $4pq - 2 = 76q - 2$ and $76 < \frac{76q-2}{q-1} \leq 77$ if $q \geq 75$. Now 75 is not prime and for the cases $(p, q) = (19, 23)$, $(19, 31)$, $(19, 43)$ and $(19, 47)$, $q-1$ is not a divisor of $74 = 2 \times 37$.

Similarly, if $p = 23$ then $4pq - 2 = 92q - 2$ and $92 < \frac{92q - 2}{q - 1} \leq 93$ if $q \geq 91$ and $\frac{92q - 2}{q - 1} = 93$ if $q = 91$. But 91 is not prime and of the cases $(p, q) = (23, 31), (23, 43)$, when $q = 31$ all of the conditions are satisfied. But, $n = 2pq = 1426$ is not a solution because $\frac{2 \cdot 1426 + 2}{1426}$ is not an integer.

No other pairs of p, q yield numbers within the required range.]

(1 point)

Problem 3.



$$\begin{aligned}\angle ALD &= \frac{1}{2} (\widehat{MC} + \widehat{AB}) \\ &= \frac{1}{2} (\widehat{BM} + \widehat{AB}) \\ &= \angle ANM\end{aligned}$$

It is known (see Geometry Revisited) or easily derivable that

$$m_a^2 = (AL)^2 = bc \left(1 - \left(\frac{a}{b+c} \right)^2 \right)$$

From $\triangle ADL \sim \triangle MAN$ we have

$$\frac{AD}{AL} = \frac{AM}{MN} \Rightarrow \underline{AD \cdot MN} = \underline{AL \cdot AM}$$

$$h_a \cdot 2R = AL \cdot AM = m_a \cdot M_a$$

$$h_a = \frac{2(ABC)}{a}$$

Using Stewart's Theorem:

Let AX be a cevian of length p , dividing BC into segments $BX = m$ and $XC = n$. Then

$$a(p^2 + mn) = b^2m + c^2n.$$

(1 point)

Here, we use $m = kc, n = kb$
 $k = \frac{a}{b+c}$

$$\frac{2(ABC)}{a} \cdot 2R = m_a M_a$$

$$(ABC) = \frac{abc}{4R}$$

$$\frac{\frac{abc}{4R} \cdot 4R}{a} = m_a M_a$$

$$bc = m_a M_a$$

So that

$$l_a = \frac{m_a^2}{m_a M_a} = 1 - \left(\frac{a}{b+c} \right)^2$$

with similar expressions for l_b and l_c .

(2 points)

Given that $\sin A = \frac{a}{2R}$, etc. the expression we are working with becomes

$$\begin{aligned} \frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} &= \frac{4R^2}{a^2} \left(1 - \left(\frac{a}{b+c} \right)^2 \right) + \frac{4R^2}{b^2} \left(1 - \left(\frac{b}{a+c} \right)^2 \right) + \frac{4R^2}{c^2} \left(1 - \left(\frac{c}{a+b} \right)^2 \right) \\ &= 4R^2 \left[\left(\frac{1}{a^2} - \frac{1}{(b+c)^2} \right) + \left(\frac{1}{b^2} - \frac{1}{(a+c)^2} \right) + \left(\frac{1}{c^2} - \frac{1}{(a+b)^2} \right) \right] \\ &\geq 4R^2 \left[\frac{1}{a^2} - \frac{1}{4bc} + \frac{1}{b^2} - \frac{1}{4ac} + \frac{1}{c^2} - \frac{1}{4ab} \right] \\ &= 2R^2 \left[\left(\frac{1}{a^2} + \frac{1}{b^2} \right) + \left(\frac{1}{a^2} + \frac{1}{c^2} \right) + \left(\frac{1}{b^2} + \frac{1}{c^2} \right) - \frac{1}{2ab} - \frac{1}{2ac} - \frac{1}{2bc} \right] \\ &\stackrel{2R^2}{\geq} \left[\frac{2}{ab} + \frac{2}{ac} + \frac{2}{bc} - \frac{1}{2ab} - \frac{1}{2ac} - \frac{1}{2bc} \right] \\ &= 2R^2 \left[\frac{3}{2ab} + \frac{3}{2ac} + \frac{3}{2bc} \right] \\ &= 3R^2 \left[\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} \right] = 3R^2 \left[\frac{a+b+c}{abc} \right] \end{aligned}$$

But $abc = 4R(ABC)$ so that this last expression becomes

$$\frac{3R(a+b+c)}{4(ABC)} = \frac{3R \cdot 2s}{4sr} = 3 \cdot \frac{R}{2r} \geq 3$$

(3 points)

since $R \geq 2r$. All of the inequalities are equalities iff $a = b = c$.

(1 point)

Problem 4.

(a) Consider the sequence of triangles on the plane $A_1A_2A_3, A_2A_3A_4, A_3A_4A_5, \dots$

It is easy to see that any pair of them are similar.

Let's prove that triangles $A_1A_2A_3$ and $A_3A_4A_5$ are similar.

Triangles $A_2A_3A_4$ and $A_4A_5A_6$ are similar and their altitudes are A_2A_4 and A_4A_6 , then

$$\frac{A_2A_3}{A_4A_5} = \frac{A_4A_5}{A_6A_7}$$

Triangles $A_3A_4A_5$ and $A_5A_6A_7$ are similar, then

$$\frac{A_4A_5}{A_6A_7} = \frac{A_3A_4}{A_5A_6}$$

Now we can conclude that

$$\frac{A_1A_2}{A_3A_4} = \frac{A_3A_4}{A_5A_6}$$

and triangles $A_1A_2A_3$ and $A_3A_4A_5$ are similar.

Hence, if P is the point where line A_1A_3 meets A_3A_5 , $\angle A_1A_3A_5 = \angle P A_3A_5 = \angle A_1A_2A_3$ and $\angle A_1A_2A_3 = \angle A_3A_4A_5 = \angle A_3A_5P = \angle A_3A_4A_5$, so triangle A_1A_3P has a right angle at P and lines A_1A_3 and A_3A_5 are perpendicular. In the same way lines A_3A_5 and A_5A_7 are perpendicular and lines A_5A_7 and A_7A_9 are perpendicular, hence A_1, A_3, A_5 are collinear and A_3, A_5, A_7 are collinear. It follows that triangle $A_1A_2A_3$ and $A_3A_4A_5$ are homothetic and the center of homothety is P . Moreover, all triangles from the family $A_1A_2A_3, A_3A_4A_5, A_5A_6A_7, A_7A_8A_9, \dots$ are homothetic. Of course the point P is an interior point to any of these triangles and there is no other point distinct from P that is interior to any of these triangles. So this is the point we are looking for.

(up to 4 points)

(b) Since $\angle A_1PA_3 = 90^\circ$ then P lies on the circle with diameter A_1A_3 . Let $A_1A_2 = 1, A_2A_3 = s, A_1A_3 = r$, and let $A_1A_2A_3$ be clockwise. Triangles $A_1A_2A_3$ and $A_3A_4A_5$ are similar, thus $A_3A_4 : r = s : 1$, and so $A_3A_4 = rs$. Besides $A_3A_4 = r\sqrt{1+s^2}$ (Pythagoras), and area of triangle $A_1A_2A_3 = \frac{1}{2}r\sqrt{1+s^2} \cdot \sqrt{1+s^2} = \frac{1}{2}s \cdot 1$. Thus $r = \frac{s}{1+s^2}$. By the arithmetic-geometric mean $\frac{s}{1+s^2} \leq \frac{1}{2}$, thus $r \leq \frac{1}{2}$ and the set of all possible values of r consists of two real intervals $[-\frac{1}{2}, 0)$ and $(0, \frac{1}{2}]$. $\angle A_3A_1P$ takes the maximum value when $r = \frac{1}{2}$ thus the locus of P

Problem 5.

A redistribution can be written as (x_1, x_2, \dots, x_n) where x_i denotes the number of objects transferred from A_i to A_{i+1} . Our objective is to minimize the function

$$F(x_1, x_2, \dots, x_n) = \sum_{i=1}^n |x_i|$$

After redistribution we should have at each A_i , $a_i - x_i + x_{i-1} = N$ for $i \in \{1, 2, \dots, n\}$ where x_0 means x_n . (1 point)

Solving this system of linear equations we obtain:

$$x_i = x_1 - [(i-1)N - a_2 - a_3 - \dots - a_i]$$

for $i \in \{1, 2, \dots, n\}$.

Hence

$$F(x_1, x_2, \dots, x_n) = |x_1| + |x_1 - (N - a_2)| + |x_1 - 2N - a_2 - a_3| + \dots + |x_1 - [(n-1)N - a_2 - a_3 - \dots - a_n]|$$

Basically the problem reduces to find the minimum of $F(x) = \sum_{i=1}^n |x - \alpha_i|$

where $\alpha_i = (i-1)N - \sum_{j=2}^i a_j$. (up to 3 points)

First rearrange $\alpha_1, \alpha_2, \dots, \alpha_n$ in non decreasing order. Collecting terms which are equal to one another we write the ordered sequence $\beta_1 < \beta_2 < \dots < \beta_m$, each β_i occurs k_i times in the family $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$.

Thus $k_1 + k_2 + \dots + k_m = n$.

Consider the intervals $(-\infty, \beta_1], [\beta_1, \beta_2], \dots, [\beta_{m-1}, \beta_m], [\beta_m, \infty)$

the graph of $F(x) = \sum_{i=1}^n |x - \alpha_i| = \sum_{i=1}^m k_i |x - \beta_i|$ is a continuous piece wise linear graph define in the following way:

$$F(x) = \begin{cases} k_1(\beta_1 - x) + k_2(\beta_2 - x) + \dots + k_m(\beta_m - x) & \text{if } x \in (-\infty, \beta_1] \\ k_1(x - \beta_1) + k_2(\beta_2 - x) + \dots + k_m(\beta_m - x) & \text{if } x \in [\beta_1, \beta_2] \\ \vdots \\ k_1(x - \beta_1) + k_2(x - \beta_2) + \dots + k_m(x - \beta_m) & \text{if } x \in [\beta_m, \infty) \end{cases}$$

(up to 4 points)

The slopes of each line segment on each interval are respectively:

$$S_0 = -k_1 - k_2 - k_3 - \dots - k_m$$

$$S_1 = k_1 - k_2 - k_3 - \dots - k_m$$

$$S_2 = k_1 + k_2 - k_3 - \dots - k_m$$

$$S_m = k_1 + k_2 + k_3 + \dots + k_m$$

Note that this sequence of increasing numbers goes from a negative to a positive number, hence for some $t \geq 1$ there is an

$$S_t = 0 \text{ or } S_{t-1} < 0 < S_t$$

In the first case the minimum occurs at $x = \beta_t$ or β_{t+1} and in the second case the minimum occurs at $x = \beta_t$

(Up to 7 points)

We can rephrase the computations above in terms of $\alpha_1, \alpha_2, \dots, \alpha_n$ rather than $\beta_1, \beta_2, \dots, \beta_m$. After rearranging the α 's in non decreasing order,

pick $x = \alpha_{\frac{n+1}{2}}$ if n is odd and take $x = \alpha_{\frac{n}{2}}$ or $\alpha_{\frac{n}{2}+1}$ if n is even.

If no justification is given for the choice of x , give up to 4 points.