OJBM - 1

Algebra

A1. Let a, b, c, d, e be real numbers such that $a+b+c+d+e=0$. Let, also

$$A=ab+bc+cd+de+ea \text{ and } B=ac+ce+eb+bd+da.$$

Show that

$$2005A + B \leq 0 \quad \text{or} \quad A + 2005B \leq 0.$$

Solution

We have

$$0=(a+b+c+d+e)^2 = a^2 + b^2 + c^2 + d^2 + e^2 + 2A + 2B.$$

This implies that

$$A + B \leq 0 \text{ or } 2006(A + B) = (2005A + B) + (A + 2005B) \leq 0.$$

This implies the conclusion.

Alternative solution

We have

$$2A+2B = a(b+c+d+e)+b(c+d+e+a)+c(d+e+a+b)$$
$$+d(e+a+b+c)+e(a+b+c+d)$$
$$=-a^2-b^2-c^2-d^2-e^2 \leq 0$$

Therefore we have $A+B \leq 0$, etc.

A2. Find all positive integers x, y satisfying the equation

$$9\left(x^2+y^2+1\right)+2(3xy+2)=2005.$$

Solution

The given equation can be written into the form

$$2(x+y)^2+(x-y)^2=664 \qquad (1).$$

Therefore, both numbers $x+y$ and $x-y$ are even.

Let $x+y=2m$ and $x-y=2t,\ t\in\mathbb{Z}$.

Now from (1) we have that t and t^2 are even and m is odd.

So, if $t=2k, k\in\mathbb{Z}$ and $m=2n+1, n\in\mathbb{N}$, then from (1) we get

$$k^2=41-2n(n+1) \qquad (2).$$

Thus $41-2n(n+1)\geq 0$ or $2n^2+2n-41\leq 0$. The last inequality is satisfied for the positive integers $n=1,2,3,4$ and for $n=0$.

However, only for $n=4$, equation (2) gives a perfect square $k^2=1 \Leftrightarrow k=\pm 1$.

Therefore the solutions are $(x,y)=(11,7)$ or $(x,y)=(7,11)$.

A3. Find the maximum value of the area of a triangle having side lengths a, b, c with

$$a^2+b^2+c^2=a^3+b^3+c^3.$$

Solution

Without any loss of generality, we may assume that $a\le b\le c$.
On the one hand, Tchebyshev's inequality gives

$$(a+b+c)(a^2+b^2+c^2)\le 3(a^3+b^3+c^3).$$

Therefore using the given equation we get

$$a+b+c\le 3 \text{ or } p\le\frac{3}{2},$$

where p denotes the semi perimeter of the triangle.
On the other hand,

$$p=(p-a)+(p-b)+(p-c)\ge 3\sqrt[3]{(p-a)(p-b)(p-c)}.$$

Hence

$$p^3\ge 27(p-a)(p-b)(p-c)\Leftrightarrow p^4\ge 27p(p-a)(p-b)(p-c)$$
$$\Leftrightarrow p^2\ge 3\sqrt{3}\cdot S,$$

where S is the area of the triangle.

Thus $S\le\frac{\sqrt{3}}{4}$ and equality holds whenever when $a=b=c=1$.

Comment

Cauchy's inequality implies the following two inequalities are true:

$$\frac{a+b+c}{3}\le\frac{a^2+b^2+c^2}{a+b+c}\le\frac{a^3+b^3+c^3}{a^2+b^2+c^2}$$

Now note that

$$\frac{a+b+c}{3}\le\frac{a^2+b^2+c^2}{a+b+c}$$

gives

$$(a+b+c)^2\le 3(a^2+b^2+c^2), \qquad (1)$$

whereas $\frac{a^2+b^2+c^2}{a+b+c}\le\frac{a^3+b^3+c^3}{a^2+b^2+c^2}$, because of our assumptions,

becomes $\frac{a^2+b^2+c^2}{a+b+c}\le 1$, and so,

$$a^2+b^2+c^2\le a+b+c \qquad (2)$$

Combining (1) and (2) we get
$(a+b+c)^2\le 3(a+b+c)$ and then $a+b+c\le 3$.

A4. Find all the integer solutions of the equation

$$9x^2y^2+9xy^2+6x^2y+18xy+x^2+2y^2+5x+7y+6=0.$$

Solution

The equation is equivalent to the following one

$$(9y^2+6y+1)x^2+(9y^2+18y+5)x+2y^2+7y++6=0$$
$$\Leftrightarrow (3y+1)^2(x^2+x)+4(3y+1)x+2y^2+7y+6=0.$$

Therefore *3y+1* must divide $2y^2+7y+6$ and so it must also divide

$$9(2y^2+7y+6)=18y^2+63y+54=2(3y+1)^2+17(3y+1)+35$$

from which it follows that it must divide 35 as well. Since $3y+1\in\mathbb{Z}$ we conclude that $y\in\{0,-2,2,-12\}$ and it is easy now to get all the solutions (-2, 0), (-3,0), (0,-2), (-1,2).

A5. Solve the equation

$$8x^3+8x^2y+8xy^2+8y^3=15(x^2+y^2+xy+1)$$

in the set of integers.

Solution

We transform the equation to the following one

$$\left(x^2+y^2\right)(8x+8y-15)=15(xy+1).$$

Since the right side is divisible by 3, then$3/\left(x^2+y^2\right)(8x+8y-15)$. But if$3/\left(x^2+y^2\right)$, then $3/x$ and $3/y$, so 9 will divide $15(xy+1)$ and $3/(xy+1)$, which is impossible. Hence $3/(x+y)$ and 3 does not divide x or y. Without loss of generality we can assume that $x=3a+1$ and $y=3b+2$. Substituting in the equation, we obtain

$$\left(x^2+y^2\right)(8(a+b)+3)=5(xy+1).$$

Since $xy+1\equiv 0(\text{mod}3)$, we conclude that $3/(a+b)$.

Now we distinguish the following cases:

- If $a+b=0$, then $x=3a+1$ and $y=-3a+2$ from which we get

$$\left(9a^2+6a+1+9a^2-12a+4\right).3=5(-9a^2+3a+3) \text{ or } 3a^2-a=0.$$

But $a=\frac{1}{3}$ is not an integer, so $a=0$ and $x=1$, $y=2$. Thus, by symmetry, we have two solutions $(x,y)=(1,2)$ and $(x,y)=(2,1)$.

- If $a+b\neq 0$, then $|8(a+b)+3|\geq 21$. So we obtain

$$\left|(x^2+y^2)(8(a+b)+3)\right|\geq 21x^2+21y^2\geq|5xy+5|,$$

which means that the equation has no other solutions.

Geometry

G1. Let $ABCD$ be an isosceles trapezoid with $AB = AD = BC$, $AB // DC$, $AB > DC$. Let E be the point of intersection of the diagonals AC and BD and N be the symmetric point of B with respect to the line AC. Prove that quadrilateral $ANDE$ is cyclic.

Solution

Let ω be a circle passing through the points A, N, D and let M the point where ω intersects BD for the second time. The quadrilateral $ANDM$ is cyclic and it follows that

$$\angle NDM + \angle NAM = \angle NDM + \angle BDC = 180^0$$

and

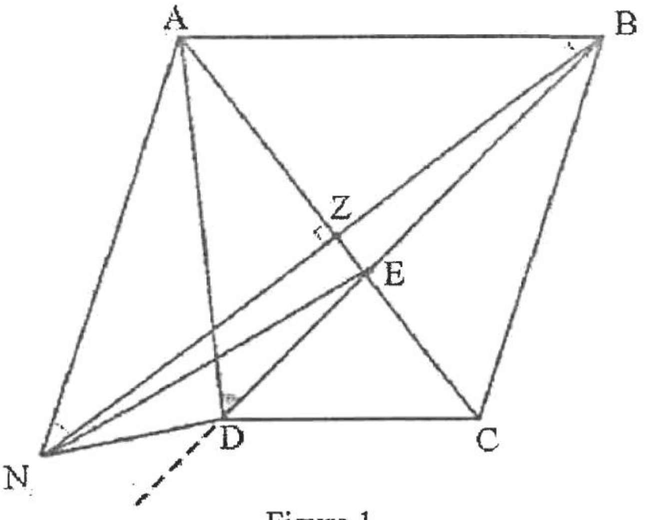

Figure 1

$$\angle NAM = \angle BDC$$

Now we have

$$\angle BDC = \angle ACD = \angle NAC$$

and

$$\angle NAM = \angle NAC.$$

So the points A, M, C are collinear and $M \equiv E$.

Alternative solution

In this solution we do not need the circle passing through the points A, N and D. Because of the given symmetry we have

$$\angle ANE = \angle ABD \qquad (1)$$

and from the equality AD=AB the triangle ABD is isosceles with

$$\angle ABD = \angle ADE \qquad (2)$$

From (1) and (2) we get that $\angle ANE = \angle ADE$, which means that the quadrilateral $ANDE$ is cyclic.

G2. Let ABC be a triangle inscribed in a circle K. The tangent from A to the circle meets the line BC at point P. Let M be the midpoint of the line segment AP and let R be the intersection point of the circle K with the line BM. The line PR meets again the circle K at the point S. Prove that the lines AP and CS are parallel.

Solution

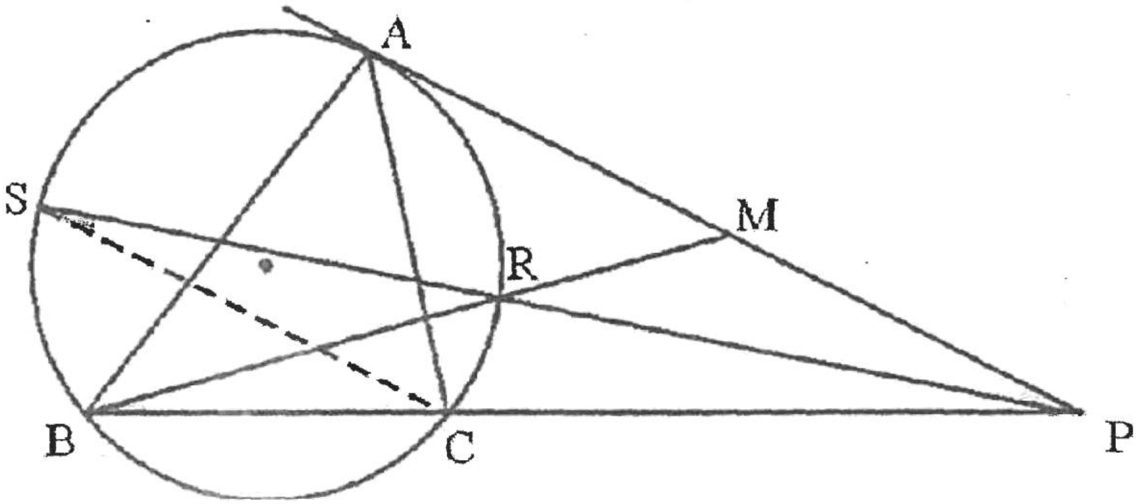

Figure 2

Assume that point C lies on the line segment BP. By the Power of Point theorem we have $MA^2 = MR \cdot MB$ and so $MP^2 = MR \cdot MB$. The last equality implies that the triangles MRP and MPB are similar. Hence $\angle MPR = \angle MBP$ and since $\angle PSC = \angle MBP$, the claim is proved.

Slight changes are to be made if the point B lies on the line segment PC.

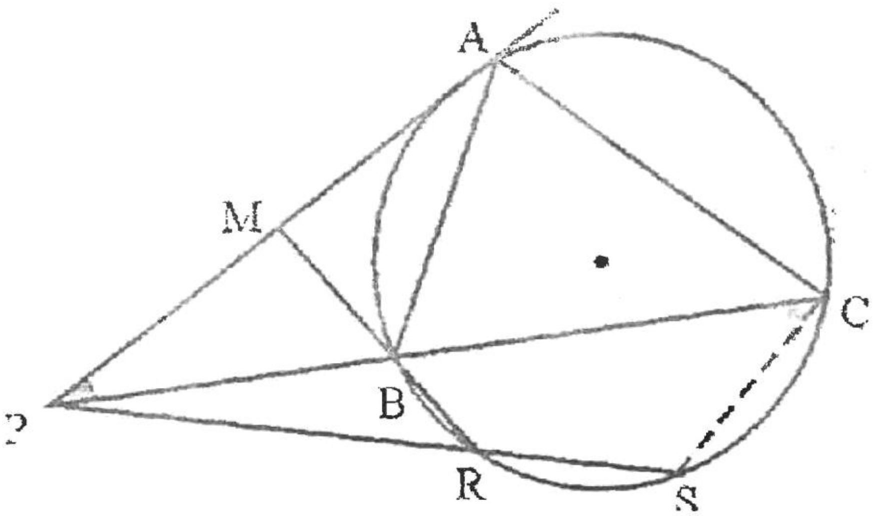

Figure 3

G3. Let $ABCDEF$ be a regular hexagon. The points M and N are internal points of the sides DE and DC respectively, such that $\angle AMN = 90^\circ$ and $AN = \sqrt{2} \cdot CM$. Find the measure of the angle $\angle BAM$.

Solution

Since $AC \perp CD$ and $AM \perp MN$ the quadrilateral $AMNC$ is inscribed. So, we have

$$\angle MAN = \angle MCN .$$

Let P be the projection of the point M on the line CD. The triangles AMN and CPM are similar implying

$$\frac{AM}{CP} = \frac{MN}{PM} = \frac{AN}{CM} = \sqrt{2} .$$

So, we have

$$\frac{MP}{MN} = \frac{1}{\sqrt{2}} \Rightarrow \angle MNP = 45^0 .$$

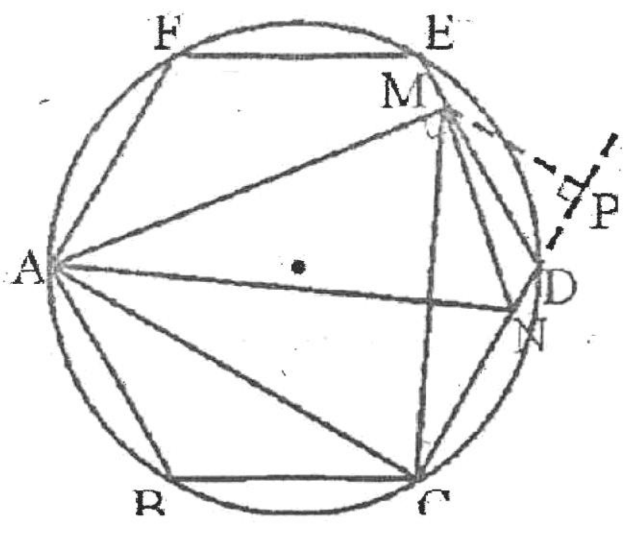

Figure 4

Hence we have

$$\angle CAM = \angle MNP = 45^0$$

and finally, we obtain

$$\angle BAM = \angle BAC + \angle CAM = 75^0 .$$

G4. Let ABC be an isosceles triangle such that $AB = AC$ and $\angle\frac{A}{2} < \angle B$. On the extension of the altitude AM we get the points D and Z such that $\angle CBD = \angle A$ and $\angle ZBA = 90^o$. E is the foot of the perpendicular from M to the altitude BF and K is the foot of the perpendicular from Z to AE. Prove that $\angle KDZ = \angle KBD = \angle KZB$.

Solution

The points *A, B, K, Z* and *C* are co-cyclic.
Because ME//AC so we have

$$\angle KEM = \angle EAC = \angle MBK.$$

Therefore the points *B, K, M* and *E* are co-cyclic.
Now, we have

$$\begin{aligned}\angle ABF &= \angle ABC - \angle FBC \\ &= \angle AKC - \angle EKM = \angle MKC\end{aligned} \quad (1)$$

Also, we have

$$\begin{aligned}\angle ABF &= 90^o - \angle BAF = 90^o - \angle MBD \\ &= \angle BDM = \angle MDC\end{aligned} \quad (2)$$

From (1) and (2) we get $\angle MKC = \angle MDC$ and so the points *M, K, D* and *C* are co-cyclic.
Consequently,

$$\angle KDM = \angle KCM = \angle BAK = \angle BZK,$$

and because the line BD is tangent to the circumcircle of triangle ABC, we have

$$\angle KBD = \angle BAK.$$

Finally, we have

$$\angle KDZ = \angle KBD = \angle KZB.$$

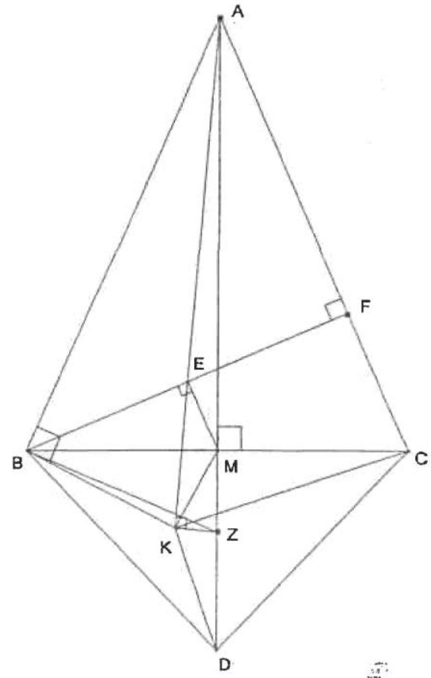

Figure 5

G5. Let A and P are the points of intersection of the circles k_1 and k_2 with centers O and K, respectively. Let also B and C be the symmetric points of A with respect to O and K, respectively. A line through A intersects the circles k_1 and k_2 at the points D and E, respectively. Prove that the centre of the circumcircle of the triangle DEP lies on the circumcircle OKP.

Solution

The points B, P, C are collinear, and

$$\angle APC = \angle APB = 90^o$$

Let N be the midpoint of DP. So we have:

$$\angle NOP = \angle DAP = \angle ECP = \angle ECA + \angle ACP \quad (1)$$

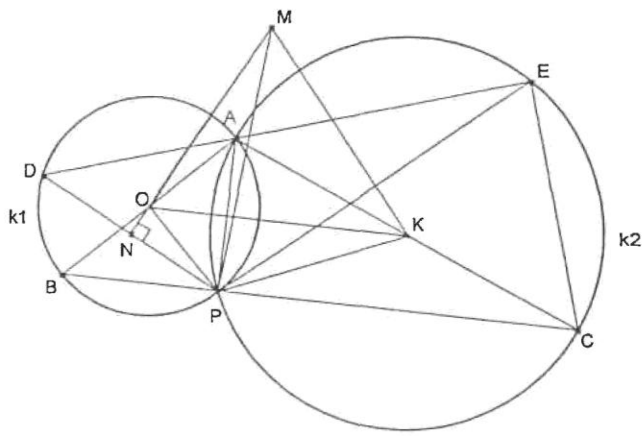

Figure 6

Since $OK//BC$ and OK is the bisector of $\angle AKP$ we get

$$\angle ACP = OKP \quad (2)$$

Also, since $AP \perp OK$ and $MK \perp PE$ we have that

$$\angle APE = \angle MKO \quad (3).$$

The points A, E, C, P are co-cyclic, and so $\angle ECA = \angle APE$.
Therefore, from (1), (2) and (3) we have that $\angle NOP = \angle MKP$.
Thus O, M, K and P are co-cyclic.

Comment

Points B and C may not be included in the statement of the problem

Alternative solution

It is sufficient to prove that the quadrilateral *MOPK* is circumscrible.
Since *MO* and *MK* are perpendicular bisectors of the line segments *PD* and *PE*, respectively, we have

$$\angle OMK = 180^0 - \angle DPE = \alpha + \beta = \alpha' + \beta' = 180^0 - \angle OPK .$$

Therefore the quadrilateral *MOPK* is circumscrible.

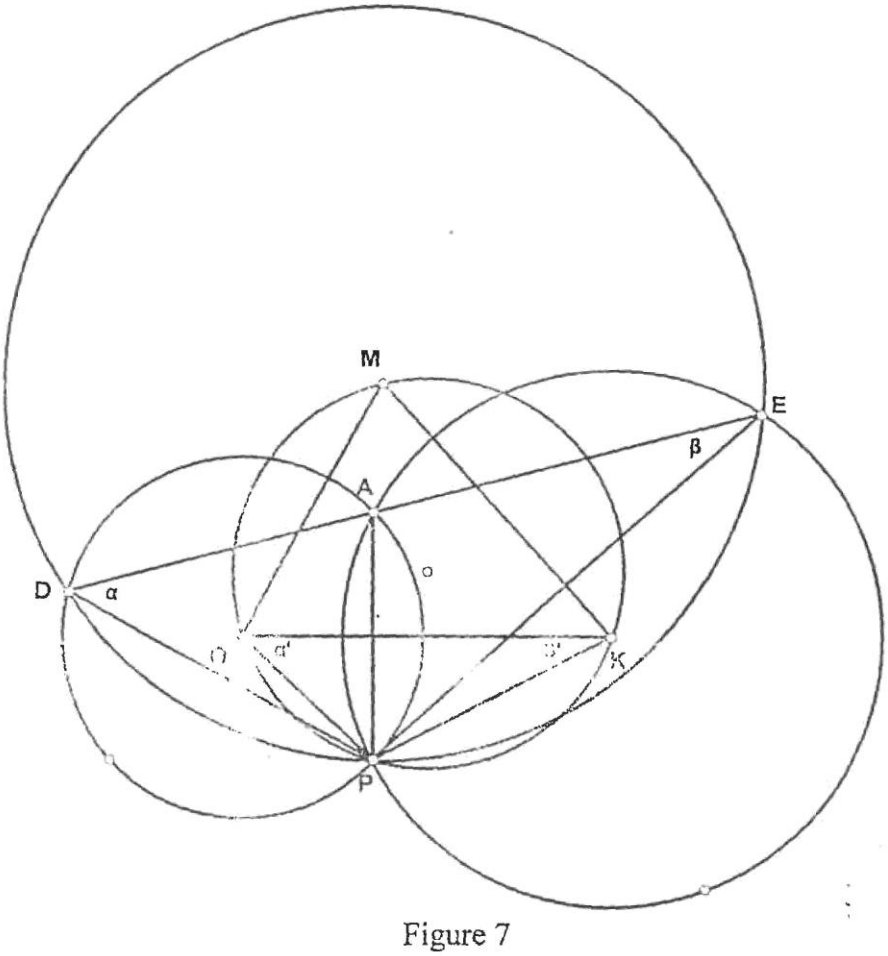

Figure 7

G6. A point O and the circles k_1 with center O and radius 3, k_2 with center O and radius 5, are given. Let A be a point on k_1 and B be a point on k_2. If ABC is equilateral triangle, find the maximum value of the distance OC.

Solution

It is easy to see that the points O and C must be in different semi-planes with respect to the line AB.

Let OPB be an equilateral triangle (P and C on the same side of OB). Since $\angle PBC = 60^0 - \angle ABP$ and $\angle OBA = 60^0 - \angle ABP$, then $\angle PBC = \angle OBA$. Hence the triangles AOB and CPB are equal and $PC = OA$. From the triangle OPC we have

$$OC \leq OP + PC = OB + OA = 8.$$

Hence, the maximum value of the distance OC is 8 (when the point P lies on OC)

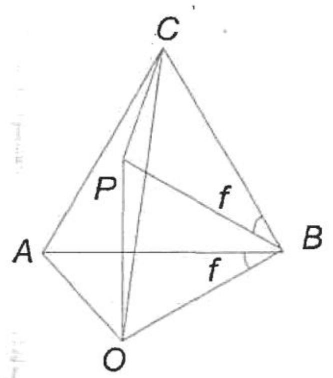

Figure 8

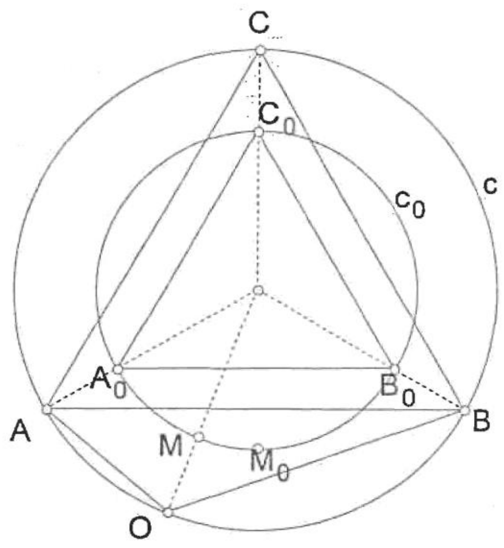

Figure 9

Complement to the solution

Indeed there exists a triangle OAB with $OA = 3, OB = 5$ and $OC = 8$.

To construct such a triangle, let's first consider a point M on the minor arc $\overset{\frown}{A_0B_0}$ of the circumference (c_0) of an arbitrary equilateral triangle $A_0B_0C_0$. As M moves along $\overset{\frown}{A_0B_0}$ from the midpoint position M_0 towards A_0, the ratio $\frac{MA_0}{MB_0}$ takes on all the decreasing values from 1 to 0. Thus there exists a position of M such that $\frac{MA_0}{MB_0} = \frac{3}{5}$. Now a homothesy centered at the center of (c_0) can take A_0, B_0, C_0, M to the new positions A, B, C, O so that $OA = 3$ and $OB = 5$. Then, since C lies on the minor arc $\overset{\frown}{AB}$ of the circumference (c) of the equilateral triangle ABC we get $OC = OA + OB = 3 + 5 = 8$ as wanted, (figure 9).

Alternative solution 1 (*by the proposer*)

Let ϕ be a 60° rotation with center at B. Then $\phi(A) = C$, $\phi(O) = P$ and $PC = OA$, $OP = O$, etc.

Alternative solution 2

Let $OA = x, AB = BC = CA = a$. From the second theorem of Ptolemy we get

$$OA \cdot BC + OB \cdot AC \geq AB \cdot OC \Leftrightarrow 3a + 5a = ax \Leftrightarrow x \leq 8.$$

The value x=8 is attained when the quadrilateral OACB is circumscrible, i.e. when $\angle AOB = 120^0$.

The point B can be constructed as follows:

It is the point of intersection of the circle (O,5) with the ray coming from the rotation of the ray OA with center O by an angle $\theta = -120^0$, (figure 10).

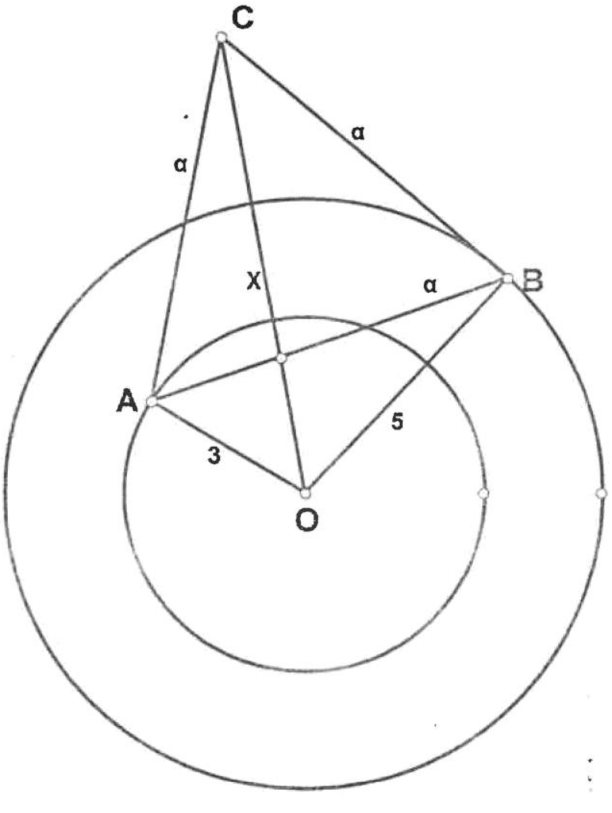

Figure 10

G7. Let $ABCD$ be a parallelogram, **P** a point on CD, *and* Q a point on AB. Let also $M = AP \cap DQ$, $N = BP \cap CQ$, $K = MN \cap AD$, and $L = MN \cap BC$. Show that $BL = DK$.

Solution

Let O be the intersection of the diagonals. Let P_1 be on AB such that $PP_1 \, // AD$, and let Q_1 be on CD such that $QQ_1 \, // AD$. Let σ be the central symmetry with center O.

Let $P' = \sigma(P)$, $Q' = \sigma(Q)$, $P_1' = \sigma(P_1))$ and, (figure 1).

Let $M_1 = AQ_1 \cap DP_1$, $N_1 = BQ_1 \cap CP_1, N' = AQ' \cap DP'$ and $M' = BQ' \cap CP'$.

Then: $M' = \sigma(M)$, $N' = \sigma(N)$, $M_1' = \sigma(M_1)$ and $N_1' = \sigma(N_1)$.

Since AP and DP_1 are the diagonals of the parallelogram AP_1PD, CP_1 and BP are the diagonals of the parallelogram P_1BCP, and AQ_1 and DO are the diagonals of the parallelogram AQQ_1D, it follows that the points U, V, W (figure 2) are collinear and they lie on the line passing through the midpoints R of AD and Z of BC. The diagonals AM and DM_1 the quadrilateral AM_1MD intersect at U and the diagonals AM_1 and DM intersect at W. Since the midpoint of AD is on the line UW, it follows that the quadrilateral AM_1MD is a trapezoid. Hence, MM_1 is parallel to AD and the midpoint S of MM_1 lies on the line UW, (figure 2).

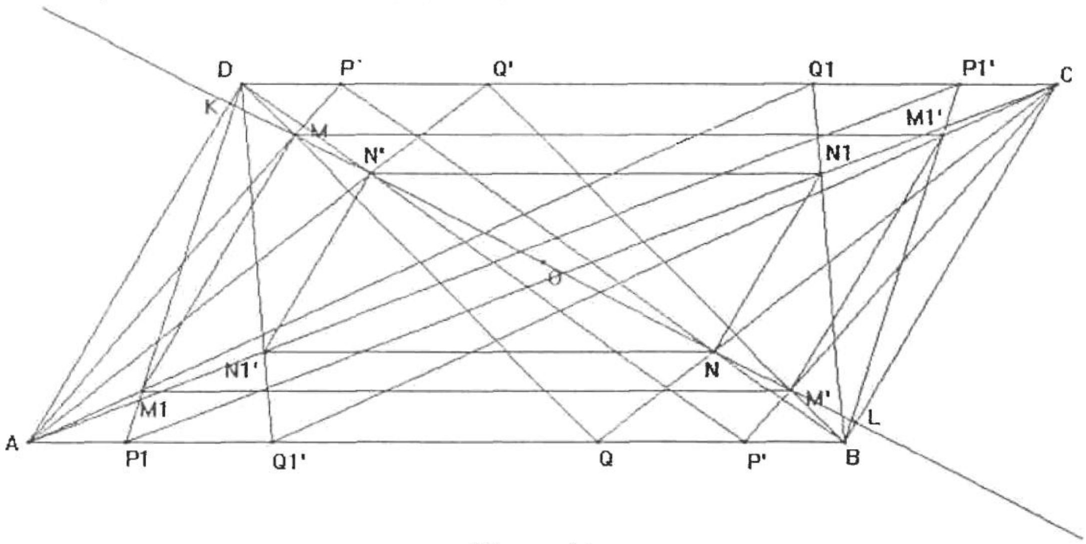

Figure 11

Similarly $M'M_1'$ is parallel to AD and its midpoint lies on UW. So $M_1M'M_1'M$ is a parallelogram whose diagonals intersect at O.

Similarly, $N_1'NN_1N'$ is a parallelogram whose diagonals intersect at O.

All these imply that M, N, M', N' and O are collinear, i.e. O lies on the line KL.

This implies that $K = \sigma(L)$, and since $D = \sigma(B)$, the conclusion follows.

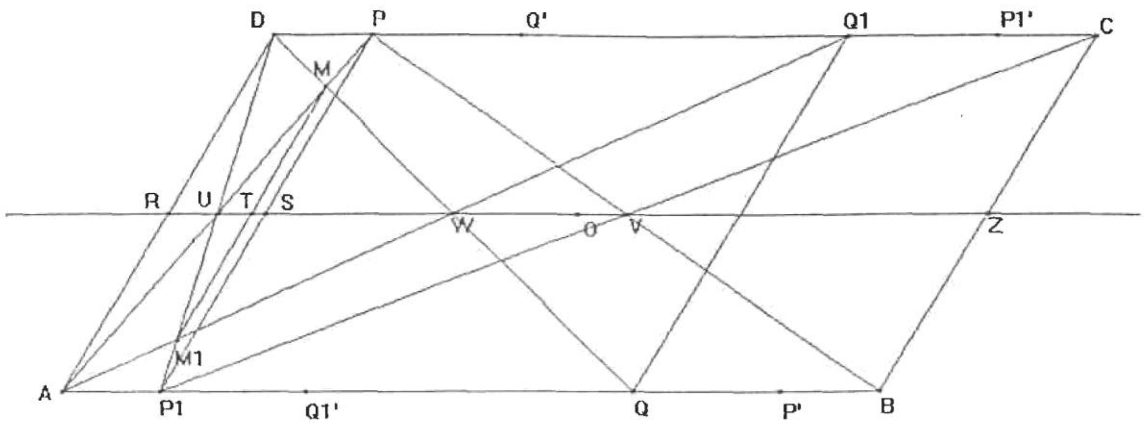

Figure 12

Alternative solution

Let the line (ε) through the points M, N intersect the lines DC, AB at points T_1, T_2 respectively. Applying Menelaus Theorem to the triangles DQC, APB with intersecting line (ε) in both cases we get:

$$\frac{MD}{MQ}\frac{NQ}{NC}\frac{T_1C}{T_1D}=1 \text{ and } \frac{MP}{MA}\frac{NB}{NP}\frac{T_2A}{T_2B}=1.$$

But it is true that $\frac{MD}{MQ}=\frac{MP}{MA}$ and $\frac{NQ}{NC}=\frac{NB}{NP}$.

It follows that $\frac{T_1C}{T_1D}=\frac{T_2A}{T_2B}$ i.e. $\frac{T_1D+DC}{T_1D}=\frac{T_2B+BA}{T_2B}$ hence $T_1D=T_2B$ (since $DC=BA$).

Then of course by the similarity of the triangles T_1DK and T_2BL we get the desired equality $DK=BL$.

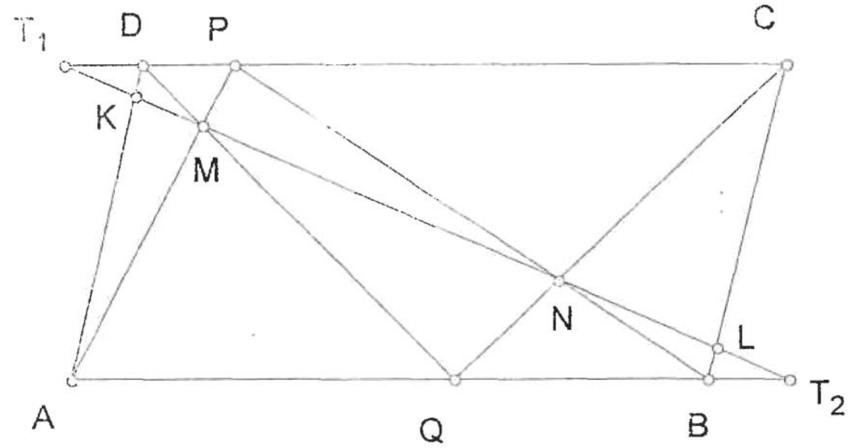

Figure 13

Number Theory

NT1.Find all the natural numbers m and n, such that the square of m minus the product of n with k, is 2, where the number k is obtained from n by writing 1 on the left of the decimal notation of n.

Solution

Let t be the number of digits of n. Then $k = 10^t + n$. So

$$m^2 = n(10^t + n) + 2, \text{ i.e. } m^2 - n^2 = 10^t n + 2.$$

This implies that m, n are even and both m, n are odd.

If t=1, then, 4 is divisor of 10n + 2, so, n is odd. We check that the only solution in this case is m=11 and n=7.

If t>1, then 4 is divisor of $m^2 - n^2$, but 4 is not divisor of $10^t + 2$.

Hence the only solution is m=11 and n=7.

NT2. Find all natural numbers n such that $5^n + 12^n$ is perfect square.

Solution

By checking the cases *n=1,2,3* we get the solution *n=2* and $13^2 = 5^2 + 12^2$.

If *n=2k+1* is odd, we consider the equation modulo 5 and we obtain

$$x^2 \equiv 5^{2k+1} + 12^{2k+1} (\bmod 5) \equiv 2^{2k} \cdot 2 (\bmod 5)$$

$$\equiv (-1)^k \cdot 2 (\bmod 5) \equiv \pm 2 (\bmod 5)$$

This is not possible, because the square residue of any natural number module 5 is 0,1 or 4. Therefore n is even and $x^2 = 5^{2k} + 12^{2k}$. Rearrange this equation in the form

$$5^{2k} = (x - 12^k)(x + 12^k).$$

If 5 divides both factors on the right, it must also divide their difference, that is

$$5 \mid (x + 12^k) - (x - 12^k) = 2 \cdot 12^k,$$

which is not possible . Therefore we must have

$$x - 12^k = 1 \text{ and } x + 12^k = 5^{2k}$$

By adding the above equalities we get

$$5^{2k} - 1 = 2 \cdot 12^k.$$

For $k \geq 2$, we have the inequality

$$25^k - 1 > 24^k = 2^k \cdot 12^k > 2 \cdot 12^k.$$

Thus we conclude that there exists a unique solution to our problem, namely *n=2.*

NT3. Let p be an odd prime. Prove that p divides the integer

$$\frac{2^{p!}-1}{2^k-1},$$

for all integers $k=1,2,...,p$.

Solution

At first, note that $\frac{2^{p!}-1}{2^k-1}$ is indeed an integer.

We start with the case k=p. Since $p|2^p-2$, then $p\nmid 2^p-1$ and so it suffices to prove that $p|2^{(p)!}-1$. This is obvious as $p|2^{p-1}-1$ and $(2^{p-1}-1)|2^{(p)!}-1$.

If k=1,2,…,p-1, let $m=\frac{(p-1)!}{k}\in\mathbb{N}$ and observe that $p!=kmp$. Consider $a\in\mathbb{N}$ so that $p^a\|2^k-1$ and observe that it suffices to prove $p^{a+1}|2^{p!}-1$. The case $a=0$ is solved as the case $k=p$. If else, write $2^k=1+p^a\cdot l, l\in\mathbb{N}$ and rising at the power mp gives

$$2^{p!}=(1+p^a\cdot l)^{mp}=1+mp\cdot p^a\cdot l+Mp^{2a},$$

where Mn stands for a multiply of n. Now it is clear that $p^{a+1}|2^{p!}-1$, as claimed.

Comment .The case k=p can be included in the case $a=0$.

NT4. Find all the three digit numbers $\overline{abc}$ such that

$$\overline{abc} = abc(a+b+c)$$

Solution

We will show that the only solutions are 135 and 144.

We have $a > 0, b > 0, c > 0$ and

$$9(11a + b) = (a+b+c)(abc-1).$$

- If $a+b+c \equiv 0 \pmod 3$ and $abc - 1 \equiv 0 \pmod 3$,

then $a \equiv b \equiv c \equiv 1 \pmod 3$ and $11a + b \equiv 0 \pmod 3$.

It follows now that

$a+b+c \equiv 0 \pmod 9$ or $abc - 1 \equiv 0 \pmod 9$

- If $abc - 1 \equiv 0 \pmod 9$

we have $11a + b = (a+b+c)k$, where k is an integer and is easy to see that we must have $1 < k \le 10$.

So we must deal only with the cases $k = 2,3,4,5,6,7,8,9,10$.

If $k = 2.4,5,6,8,9,10$ then $9k+1$ has prime divisors greater than 9 ,so we must see only the cases $k = 3,7$.

- If $k = 3$ we have

$$8a = 2b + 3c \quad \text{and} \quad abc = 28$$

It is clear that c is even, $c = 2c_1$ and that both b and c_1 are odd.

From $abc_1 = 14$ it follows that $a=2, b=7, c_1=1$ or $a=2, b=1, c_1=7$ and it is clear that there exists no solution in this case.

- If $k=7$ we have c-even, $c = 2c_1$, $2a = 3b + 7c_1$, $abc_1 = 32$, then both b and c_1 are even.

But this is impossible, because they imply that $a>9$.

Now we will deal with the case when $a+b+c \equiv 0 \pmod 9$ or $a+b+c = 9l$, where l is an integer.

- If $l \ge 2$ we have $a+b+c \ge 18$, $\max\{a,b,c\} \ge 6$ and it is easy to see that $abc \ge 72$ and $abc(a+b+c) > 1000$,so the case $l \ge 2$ is impossible.
- If $l = 1$ we have

$$11a + b = abc - 1 \text{ or } 11a + b + 1 = abc \le \left(\frac{a+b+c}{3}\right)^3 = 27.$$

So we have only two cases: $a = 1$ or $a = 2$.

- If $a = 1$, we have $b + c = 8$ and $11 + b = bc - 1$ or $b + (c-1) = 7$ and $b(c-1) = 12$ and the solutions are $(a,b,c) = (1,3,5)$ and $(a,b,c) = (1,4,4)$, and the answer is 135 and 144.
- If $a=2$ we have $b(2c-1) = 23$ and there is no solution for the problem.

NT5. Let p be a prime number and let a be an integer. Show that if n^2-5 is not divisible by p for any integer n, there exist infinitely many integers m so that p divides m^5+a.

Solution

We start with a simple fact:

Lemma: If b is an integer not divisible by p then there is an integer s so that sb has the remainder *1* when divided by p.

For a proof, just note that numbers $b, 2b, \ldots, (p-1)b$ have distinct non-zero remainders when divided by p, and hence one of them is equal to 1.

We prove that if $x, y=0, 1, 2, \ldots, p-1$ and p divides x^5-y^5, then $x=y$.

Indeed, assume that $x \neq y$. If $x=0$, then $p|y^5$ and so $y=0$, a contradiction.

To this point we have $x, y \neq 0$. Since

$$p|(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4) \text{ and } p\nmid(x-y),$$

we have

$$p|\left(x^2+y^2\right)^2+xy\left(x^2+y^2\right)-x^2y^2, \text{and so}$$

$$p|\left(2(x^2+y^2)+xy\right)^2-5x^2y^2$$

As $p\nmid xy$, from the lemma we find an integer s so that $sxy=kp+1, k\in\mathbb{N}$. Then

$$p|\left[s(2x^2+2y^2+xy)\right]^2-5\left(k^2p^2+2kp+1\right),$$

and so $p|z^2-5$, where $z=s\left(2x^2+2y^2+xy\right)$, a contradiction.

Consequently $x=y$.

Since we have proved that numbers $0^5, 1^5, \ldots, (p-1)^5$ have distinct remainders when divided by p, the same goes for the numbers $0^5+a, 1^5+a, \ldots, (p-1)^5+a$ and the conclusion can be reached easily.

Comments

1. For beauty we may choose $a=-2$ or any other value.
2. Moreover, we may ask only for one value of m, instead of "infinitely many".
3. A simple version will be to ask for a proof that the numbers $0^5, 1^5, \ldots, (p-1)^5$ have distinct remainders when divided by p.

Combinatorics

C1. A triangle with area 2003 is divided into non-overlapping small triangles. The number of all the vertices of all those triangles is 2005. Show that at ~~most~~ least one of the smaller triangles has area less or equal to 1.

Solution

Since all the vertices are 2005, and the vertices of the big triangle are among them, it follows that the number of the small triangles is at least 2003. So, it follows that at least one of the small triangles has area at most 1

C2. Government of Greece wants to change the building. So they will move in building with square shape 2005×2005. Only demand is that every room has exactly 2 doors and doors cannot lead outside the building. Is this possible? What about a building 2004×2005?

Solution

The key idea is to color the table 2005×2005 in two colors, like chessboard. One door connects two adjacent squares, so they have different colors. We can count the number of doors in two ways. The number of doors is twice the number of white squares and it is also twice the number of black squares. This is, of course, impossible, because we have one more black square on the table. So, if n is odd, we cannot achieve that every room has two doors.

For table 2004×2005 this is possible, and one of many solutions is presented on the picture below.

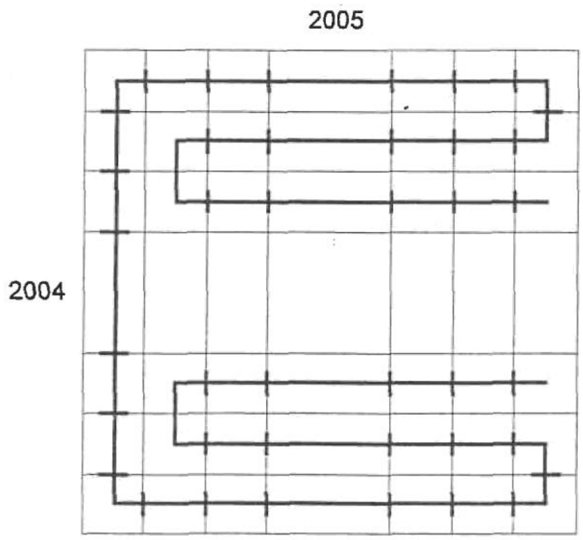

Figure 14

Alternative statement of the problem

Is it possible to open holes on two distinct sides of each cell of a 2005×2005 square grid so that no hole is on the perimeter of the grid? What about a 2004×2005 grid?

C3. Denote with $R(n)$ the number of right angle triangles with vertices – n points in the plane. Prove that:

a) There exist 5 points, such that $R(5) = 8$.
b) There exist 65 points, such that $R(65) > 2005$.

Solution

a) The 5 points are the vertices of a square and its center.
b) Let k be a circle with center O and 1, 2, 3,..., n are points on k such that 12, 34, 56, ... are diameters of k and the diameter (12) is perpendicular to diameter (34), ..., the diameter $(j, j+1)$ is perpendicular to diameter $(j+2, j+3)$.
Let $K(m)$ be the number of right angle triangles with vertices – the point O and $4m$ points on k. We have:

$$K(1) = 8.$$

Since the points 5, 6, 7, 8 together with O form 8 triangles, each of the points 5, 6, 7, 8 form 2 triangles with the diameters (12) and (34), and each of the points 1, 2, 3, 4 form 2 triangles with the diameters (56) and (78) we obtain

$$K(2) = K(1) + 8 + 4\cdot 2 + 4\cdot 2 = 4\cdot 8 = 8\cdot 2^2.$$

In the same way we have

$$K(3) = K(2) + 8 + 4\cdot 2\cdot 2 + 8\cdot 2 = 9.8 = 8\cdot 3^2.$$

and generally:

$$K(m) = K(m-1) + 8 + 4\cdot 2(m-1) + 4(m-1)\cdot 2 = 8(m-1)^2 + 8(2m-1) = 8m^2.$$

From $8m^2 > 2005$ we obtain $m = 16$. Hence for $n = 4\cdot 16 + 1 = 65$ we have

$$R(65) = K(16) = 2048 > 2005.$$

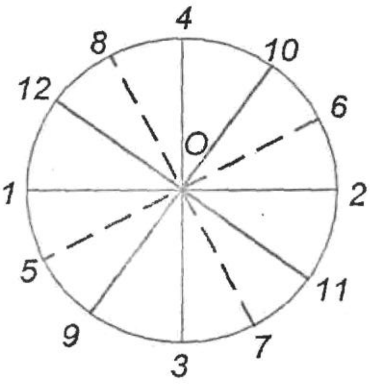

Figure 15

C4. Let $p_1, p_2, ..., p_{2005}$ be different prime numbers. Let S be a set of natural numbers which elements have the property that their simple divisors are some of the numbers $p_1, p_2, ..., p_{2005}$ and product of any two elements from S is not perfect square.
What is the maximum number of elements in S?

Solution

Let a, b be two arbitrary numbers from S. They can be written as

$$a = p_1^{a_1} p_2^{a_2} \cdots p_{2005}^{a_{2005}} \text{ and } b = p_1^{\beta_1} p_2^{\beta_2} \cdots p_{2005}^{\beta_{2005}}.$$

In order for the product of the elements a and b to be a square all the sums of the corresponding exponents need to be even from where we can conclude that for every i, a_i and β_i have the same parity. If we replace all exponents of a and b by their remainders modulo 2, then we get two numbers a', b' whose product is a perfect square if and only if ab is a perfect square.
In order for the product $a'b'$ not to be a perfect square, at least one pair of the corresponding exponents modulo 2, need to be of opposite parity.
Since we form 2005 such pairs modulo 2, and each number in these pairs is 1 or 2, we conclude that we can obtain 2^{2005} distinct products none of which is a perfect square.
Now if we are given $2^{2005}+1$ numbers, thanks to Dirichlet's principle, there are at least two with the same sequence of modulo 2 exponents, thus giving a product equal to a square.
So, the maximal number of the elements of S is 2^{2005}.