Медитеранска математичка олимпијада

12.06.2020 година

Задача 1. Нека a, b, c се позитивни реални броеви такви што $a+b+c=4$. Докажи дека

$$\frac{ab}{\sqrt[4]{3c^2+16}} + \frac{bc}{\sqrt[4]{3a^2+16}} + \frac{ca}{\sqrt[4]{3b^2+16}} \le \frac{4\sqrt[4]{12}}{3}.$$

Задача 2. Нека S е множество од $n \ge 2$ природни броеви. Докажи дека постојат најмалку n^2 природни броеви кои можат да се запишат во облик $x+yz$ каде $x, y, z \in S$.

Задача 3. Определи ги сите природни броеви $m \ge 2$ за кои постои природен број $n \ge 1$ таков што

$$\mathrm{NZD}(m,n)=1 \quad \text{и} \quad \mathrm{NZD}(m,4n+1)=1.$$

Задача 4. Нека P, Q, R се точки од кружницата k_1 такви што $\overline{PQ} = \overline{PR}$ и $\overline{PQ} > \overline{QR}$. Нека k_2 е кружница со центар во P која ги содржи точките Q и R. Нека претпоставиме дека кружницата со центар во Q која минува низ R по вторпат ја сече кружницата k_1 во точката X и по втор пат ја сече k_2 во точката Y.

Точките X и R лежат на различни страни на правата PQ.

Докажи дека точките P, X и Y се колинеарни.

Секоја задача се вреднува по 7 поени.
Време за работа 4:30
Користењето на калкулатор не е дозволено.

XXIII MMC 2020

Problem 1

Let a, b, c be positive real numbers such that $a + b + c = 4$. Prove that

$$\frac{ab}{\sqrt[4]{3c^2 + 16}} + \frac{bc}{\sqrt[4]{3a^2 + 16}} + \frac{ca}{\sqrt[4]{3b^2 + 16}} \le \frac{4}{3}\sqrt[4]{12}.$$

Solution. Squaring both terms, we get

$$\left(\frac{ab}{\sqrt[4]{3c^2 + 1}} + \frac{bc}{\sqrt[4]{3a^2 + 1}} + \frac{ca}{\sqrt[4]{3b^2 + 1}} \right)^2 \le \frac{32}{9}\sqrt{3}.$$

Applying CBS to vectors $\vec{u} = \left(\sqrt{ab}, \sqrt{bc}, \sqrt{ca} \right)$ and

$$\vec{v} = \left(\sqrt{\frac{ab}{\sqrt{3c^2 + 16}}}, \sqrt{\frac{bc}{\sqrt{3a^2 + 16}}}, \sqrt{\frac{ca}{\sqrt{3b^2 + 16}}} \right)$$

we obtain

$$\left(\frac{ab}{\sqrt[4]{3c^2 + 16}} + \frac{bc}{\sqrt[4]{3a^2 + 16}} + \frac{ca}{\sqrt[4]{3b^2 + 16}} \right)^2$$

$$\le (ab + bc + ca) \left(\frac{ab}{\sqrt{3c^2 + 16}} + \frac{bc}{\sqrt{3a^2 + 16}} + \frac{ca}{\sqrt{3b^2 + 16}} \right)$$

From $a + b + c = 4$ and the well-known inequality $(a + b + c)^2 \ge 3(ab + bc + ca)$ immediately follows that $ab + bc + ca \le 16/3$. Equality holds when $a = b = c = 4/3$.

On the other hand,

$$\sqrt{3}\left(\frac{bc}{\sqrt{3a^2 + 16}} \right) = \frac{bc}{\sqrt{a^2 + 16/3}} \le \frac{bc}{\sqrt{a^2 + ab + bc + ca}}$$

$$= \frac{bc}{\sqrt{(a + b)(a + c)}} \le \frac{1}{2}\left(\frac{bc}{a + b} + \frac{bc}{a + c} \right)$$

The last inequality holds on account of AM-GM inequality. Likewise,

$$\sqrt{3}\left(\frac{ca}{\sqrt{3b^2 + 16}} \right) \le \frac{1}{2}\left(\frac{ca}{b + c} + \frac{ca}{b + a} \right)$$

and

$$\sqrt{3}\left(\frac{ab}{\sqrt{3c^2 + 16}} \right) \le \frac{1}{2}\left(\frac{ab}{c + a} + \frac{ab}{c + b} \right)$$

Adding the preceding inequalities, we obtain

$$\sqrt{3}\left(\frac{ab}{\sqrt{3c^2 + 16}} + \frac{bc}{\sqrt{3a^2 + 16}} + \frac{ca}{\sqrt{3b^2 + 16}} \right)$$

$$\leq \frac{1}{2}\left(\frac{ca}{b+c} + \frac{ca}{b+a} + \frac{bc}{a+b} + \frac{bc}{a+c} + \frac{ab}{c+a} + \frac{ab}{c+b}\right) = \frac{1}{2}\left(a+b+c\right) = 2.$$

Therefore,

$$\frac{ab}{\sqrt{3c^2+16}} + \frac{bc}{\sqrt{3a^2+16}} + \frac{ca}{\sqrt{3b^2+16}} \leq \frac{2\sqrt{3}}{3}.$$

Equality holds when $a = b = c = 4/3$. From the preceding, we have

$$\left(\frac{ab}{\sqrt[4]{3c^2+16}} + \frac{bc}{\sqrt[4]{3a^2+16}} + \frac{ca}{\sqrt[4]{3b^2+16}}\right)^2 \leq \frac{16}{3} \cdot \frac{2\sqrt{3}}{3} = \frac{32}{9}\sqrt{3}.$$

from which

$$\frac{bc}{\sqrt[4]{3a^2+16}} + \frac{ca}{\sqrt[4]{3b^2+16}} + \frac{ab}{\sqrt[4]{3c^2+16}} \leq \frac{4}{3}\sqrt[4]{12}$$

follows. Equality holds when $a = b = c = 4/3$, and we are done.

Problem 2

Let S be a set of $n \geq 2$ positive integers. Prove that there exist at least n^2 integers that can be written in the form $x + yz$ with $x, y, z \in S$.

Solution

Let m denote the largest number in S. We claim that for all pairs $(x_1, y_1), (x_2, y_2) \in S \times S$ with $(x_1, y_1) \neq (x_2, y_2)$ we have $x_1 + y_1 m \neq x_2 + y_2 m$. Indeed, suppose otherwise and rewrite the equation $x_1 + y_1 m = x_2 + y_2 m$ into

$$(y_1 - y_2)m = x_2 - x_1.$$

If $y_1 = y_2$, we get $x_2 - x_1 = 0$ and the contradiction $(x_1, y_1) = (x_2, y_2)$. If $y_1 \neq y_2$, we get $m = (x_2 - x_1)/(y_1 - y_2)$. Since $|y_1 - y_2| \geq 1$ and $|x_2 - x_1| \leq |m - 1|$, this inequality implies $m \leq m - 1$ which is ridiculous.

Consequently, the n^2 integers of the form $x + ym$ with $(x, y) \in S \times S$ are pairwise distinct.

Problem 3

Determine all integers $m \geq 2$ for which there exists an integer $n \geq 1$ with

$$\gcd(m, n) = 1 \quad \text{and} \quad \gcd(m, 4n + 1) = 1.$$

Solution

The answer is all integers $m \geq 2$. We first discuss the cases where m is a prime number.

- If m is a prime with $m \neq 3$, then any number n with $n \equiv -1 \bmod m$ works: Indeed, let $n = km - 1$. Then $\gcd(m, n) = \gcd(m, km - 1) = 1$ holds and furthermore

$$\gcd(m, 4n + 1) = \gcd(m, 4km - 3) = \gcd(m, 3) = 1.$$

- If $m = 3$, then any number n with $n \equiv 1 \bmod m$ works: Let $n = km + 1$. Then $\gcd(m, n) = \gcd(m, km + 1) = 1$ and $\gcd(m, 4n + 1) = \gcd(m, 4km + 5) = \gcd(m, 5) = 1$.

Finally, let $m \geq 2$ be an arbitrary integer, and let $p_1 < p_2 < \cdots < p_s$ be an enumeration of the prime factors of m. We construct a system of congruences for n with $1 \leq i \leq s$: If $p_i = 3$, then we impose $n \equiv 1 \bmod p_i$. If $p_i \neq 3$, then we impose $n \equiv -1 \bmod p_i$. By the Chinese Remainder Theorem, there exists an integer n that simultaneously satisfies all these congruences and that by the above discussion satisfies $\gcd(m, n) = 1$ and $\gcd(m, 4n + 1) = 1$ as desired.

Problem 4

Let P, Q and R three points on a circle k_1, such that PQ = PR and PQ>QR. Let k_2 be the circle with centre P that passes through Q and R. Suppose that the circle with centre Q and passing through R intersect k_1 again at X and k_2 again at Y.

The points X and R lie on different sides of the line PQ.

Prove that P, X and Y lie on a line.

Solution 1

All the angles to be directed. The result follows immediately from

$$\measuredangle YPQ = \measuredangle QPR = \measuredangle XPQ \ .$$

The first equality is due to QR = QY, and so these equal chords subtend equal angles in k_1. The second is due to QR = QX, and so these equal chords subtend equal angles at the centre of k_2 .∎

Solution 2

We will need to prove that $\angle QPX = \angle QPY$.

  a) Since Q, Y, P, R are concyclic, $\angle QPY = \angle QRY$ and $\angle QPR = \angle QYR$.
  b) Since the triangle QRY is isosceles, $\angle QRY = \angle QYR$.
  c) Since the triangles QPR and QPX are congruent (3 equal sides), $\angle QPX = \angle QPR$.

From the 3 previous items, we obtain

$$\angle QPY = \angle QRY = QYR = \angle QPR = \angle QPX ,$$

and we are done.∎