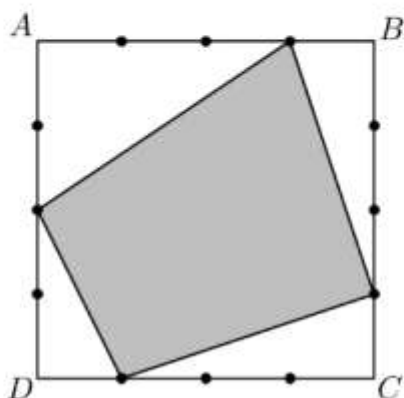


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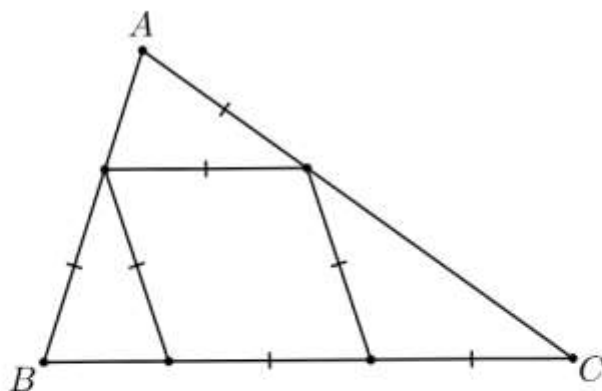
**Problems of 4th Iranian Geometry Olympiad 2017 (Elementary)**

1. Each side of square  $ABCD$  with side length of 4 is divided into equal parts by three points. Choose one of the three points from each side, and connect the points consecutively to obtain a quadrilateral. Which numbers can be the area of this quadrilateral? Just write the numbers without proof.



*Proposed by Hiran Aalipana*

2. Find the angles of triangle  $ABC$ .



*Proposed by Morteza Saghaian*

3. In the regular pentagon  $ABCDE$ , the perpendicular at  $C$  to  $CD$  meets  $AB$  at  $F$ . Prove that  $AE + AF = BE$ .

*Proposed by Alireza Cheraghi*

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4.  $P_1, P_2, \dots, P_{100}$  are 100 points on the plane, no three of them are collinear. For each three points, call their triangle clockwise if the increasing order of them is in clockwise order. Can the number of clockwise triangles be exactly 2017?

*Proposed by Morteza Saghafian*

5. In the isosceles triangle  $ABC$  ( $AB = AC$ ), let  $l$  be a line parallel to  $BC$  through  $A$ . Let  $D$  be an arbitrary point on  $l$ . Let  $E, F$  be the feet of perpendiculars through  $A$  to  $BD, CD$  respectively. Suppose that  $P, Q$  are the images of  $E, F$  on  $l$ . Prove that  $AP + AQ \leq AB$ .

*Proposed by Morteza Saghafian*

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**Problems of 4th Iranian Geometry Olympiad 2017 (Intermediate)**

1. Let  $ABC$  be an acute-angled triangle with  $A = 60^\circ$ . Let  $E, F$  be the feet of altitudes through  $B, C$  respectively. Prove that  $CE - BF = \frac{3}{2}(AC - AB)$ .

*Proposed by Fatemeh Sajadi*

2. Two circles  $\omega_1, \omega_2$  intersect at  $A, B$ . An arbitrary line through  $B$  meets  $\omega_1, \omega_2$  at  $C, D$  respectively. The points  $E, F$  are chosen on  $\omega_1, \omega_2$  respectively so that  $CE = CB, BD = DF$ . Suppose that  $BF$  meets  $\omega_1$  at  $P$ , and  $BE$  meets  $\omega_2$  at  $Q$ . Prove that  $A, P, Q$  are collinear.

*Proposed by Iman Maghsoudi*

3. On the plane,  $n$  points are given ( $n > 2$ ). No three of them are collinear. Through each two of them the line is drawn, and among the other given points, the one nearest to this line is marked (in each case this point occurred to be unique). What is the maximal possible number of marked points for each given  $n$ ?

*Proposed by Boris Frenkin (Russia)*

4. In the isosceles triangle  $ABC$  ( $AB = AC$ ), let  $l$  be a line parallel to  $BC$  through  $A$ . Let  $D$  be an arbitrary point on  $l$ . Let  $E, F$  be the feet of perpendiculars through  $A$  to  $BD, CD$  respectively. Suppose that  $P, Q$  are the images of  $E, F$  on  $l$ . Prove that  $AP + AQ \leq AB$ .

*Proposed by Morteza Saghaian*

5. Let  $X, Y$  be two points on the side  $BC$  of triangle  $ABC$  such that  $2XY = BC$ . ( $X$  is between  $B, Y$ ) Let  $AA'$  be the diameter of the circumcircle of triangle  $AXY$ . Let  $P$  be the point where  $AX$  meets the perpendicular from  $B$  to  $BC$ , and  $Q$  be the point where  $AY$  meets the perpendicular from  $C$  to  $BC$ . Prove that the tangent line from  $A'$  to the circumcircle of  $AXY$  passes through the circumcenter of triangle  $APQ$ .

*Proposed by Iman Maghsoudi*

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### Problems of 4th Iranian Geometry Olympiad 2017 (Advanced)

1. In triangle  $ABC$ , the incircle, with center  $I$ , touches the side  $BC$  at point  $D$ . Line  $DI$  meets  $AC$  at  $X$ . The tangent line from  $X$  to the incircle (different from  $AC$ ) intersects  $AB$  at  $Y$ . If  $YI$  and  $BC$  intersect at point  $Z$ , prove that  $AB = BZ$ .

*Proposed by Hooman Fattahimoghaddam*

2. We have six pairwise non-intersecting circles that the radius of each is at least one. Prove that the radius of any circle intersecting all the six circles, is at least one.

*Proposed by Mohammad Ali Abam – Morteza Saghafian*

3. Let  $O$  be the circumcenter of triangle  $ABC$ . Line  $CO$  intersects the altitude through  $A$  at point  $K$ . Let  $P, M$  be the midpoints of  $AK, AC$  respectively. If  $PO$  intersects  $BC$  at  $Y$ , and the circumcircle of triangle  $BCM$  meets  $AB$  at  $X$ , prove that  $BXOY$  is cyclic.

*Proposed by Ali Daeinabi – Hamid Pardazi*

4. Three circles  $\omega_1, \omega_2, \omega_3$  are tangent to line  $l$  at points  $A, B, C$  ( $B$  lies between  $A, C$ ) and  $\omega_2$  is externally tangent to the other two. Let  $X, Y$  be the intersection points of  $\omega_2$  with the other common external tangent of  $\omega_1, \omega_3$ . The perpendicular line through  $B$  to  $l$  meets  $\omega_2$  again at  $Z$ . Prove that the circle with diameter  $AC$  touches  $ZX, ZY$ .

*Proposed by Iman Maghsoudi – Siamak Ahmadpour*

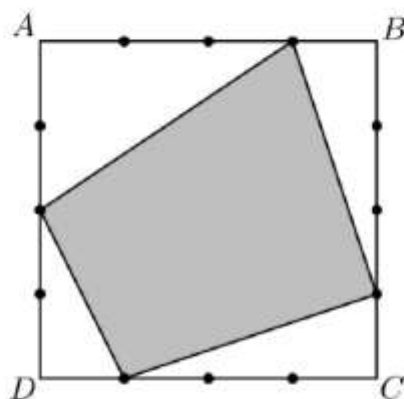
5. Sphere  $S$  touches a plane. Let  $A, B, C, D$  be four points on this plane such that no three of them are collinear. Consider the point  $A'$  such that  $S$  is tangent to the faces of tetrahedron  $A'BCD$ . Points  $B', C', D'$  are defined similarly. Prove that  $A', B', C', D'$  are coplanar and the plane  $A'B'C'D'$  touches  $S$ .

*Proposed by Alexey Zaslavsky (Russia)*

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### Solutions of 4th Iranian Geometry Olympiad 2017 (Elementary)

1. Each side of square  $ABCD$  with side length of 4 is divided into equal parts by three points. Choose one of the three points from each side, and connect the points consecutively to obtain a quadrilateral. Which numbers can be the area of this quadrilateral? Just write the numbers without proof.



*Proposed by Hiran Aalipahan*

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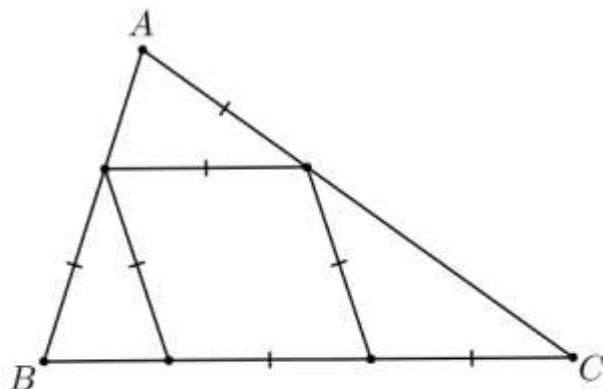
**Solution.**

To find the area of the quadrilaterals, it's enough to find sum of the areas of four right triangles and subtract it from area of the square. Finally, these numbers will be found:

6, 7, 7.5, 8, 8.5, 9, 10

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2. Find the angles of triangle  $ABC$ .

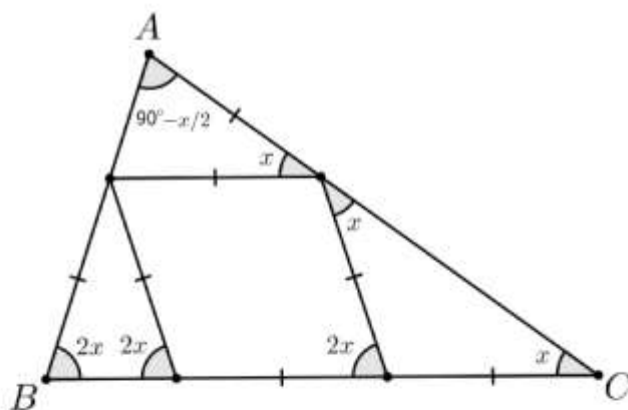


*Proposed by Morteza Saghafian*

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**Solution.**

Let  $\angle ACB = x$ . The quadrilateral with equal sides in triangle  $ABC$  is a rhombus so it has parallel sides.



According to the angles shown in the figure, we can say that sum of the angles of triangle  $ABC$  is equal to  $180^\circ$ .

$$\left(90^\circ - \frac{x}{2}\right) + 2x + x = 180^\circ \Rightarrow x = 36^\circ$$

$$\Rightarrow \angle A = 72^\circ, \angle B = 72^\circ, \angle C = 36^\circ$$

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3. In the regular pentagon  $ABCDE$ , the perpendicular at  $C$  to  $CD$  meets  $AB$  at  $F$ . Prove that  $AE + AF = BE$ .

*Proposed by Alireza Cheraghi*

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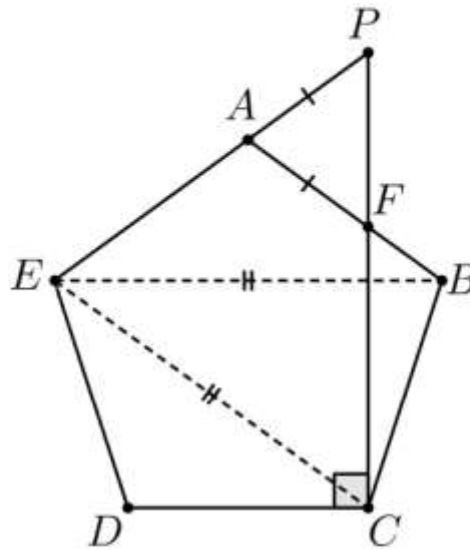
**Solution.**

Suppose that  $P$  is the intersection of  $AE$  and  $FC$ . We know that:

$$\angle ECD = 36^\circ \Rightarrow \angle ECP = 54^\circ, \angle AEC = 72^\circ \Rightarrow \angle EPC = 54^\circ$$

Therefore,  $CE = PE$ . On the other hand, we know that  $\angle ECB = \angle EBC = 72^\circ$ . Therefore,  $BE = CE = PE$ . Also we have  $\angle EAB = 108^\circ$  so  $\angle AFP = \angle APF = 54^\circ$ . It means that  $AF = AP$ .

$$\Rightarrow AE + AF = AE + AP = PE = CE = BE \blacksquare$$



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4.  $P_1, P_2, \dots, P_{100}$  are 100 points on the plane, no three of them are collinear. For each three points, call their triangle clockwise if the increasing order of them is in clockwise order. Can the number of clockwise triangles be exactly 2017?

*Proposed by Morteza Saghafian*

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**Solution.**

First, suppose that  $P_1, P_2, \dots, P_{100}$  are located on a circle in a counter-clockwise order. In this case, the number of clockwise triangles is zero. Now, start to move the points, when a point  $P_i$  passes through a line  $P_jP_k$ , the state of triangle  $P_iP_jP_k$  (clockwise or counter-clockwise) changes and the state of the other triangles does not change (assume that the points move in such a way that no point passes through two lines at the same time). So the number of clockwise triangles changes by one. Suppose that the points  $P_1, P_2, \dots, P_{100}$  move to obtain their clockwise order on a circle. In this case, the number of clockwise triangles is  $\binom{100}{3}$  which is greater than 2017. In this process, the number of clockwise triangles changes one by one. So there is a moment in which the number of clockwise triangles is exactly 2017. ■

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5. In the isosceles triangle  $ABC$  ( $AB = AC$ ), let  $l$  be a line parallel to  $BC$  through  $A$ . Let  $D$  be an arbitrary point on  $l$ . Let  $E, F$  be the feet of perpendiculars through  $A$  to  $BD, CD$  respectively. Suppose that  $P, Q$  are the images of  $E, F$  on  $l$ . Prove that  $AP + AQ \leq AB$ .

*Proposed by Morteza Sagha fian*

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**Solution1.**

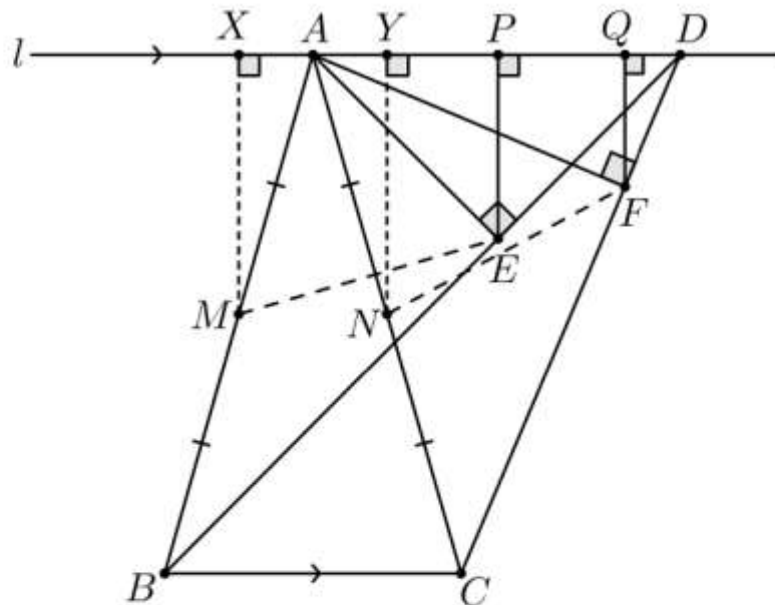
Suppose that  $M, N$  are the midpoints of  $AB, AC$  respectively. We have:

$$\angle AEB = \angle AFC = 90^\circ \Rightarrow ME = NF = \frac{AB}{2} = \frac{AC}{2}$$

Let  $X, Y$  be the feet of perpendiculars through  $M, N$  to line  $l$  respectively. We know that two triangles  $AMA$  and  $ANY$  are equal so  $AX = AY$ .

$$XP = AX + AP \leq ME = \frac{AB}{2} \quad , \quad AQ - AY = YQ \leq NF = \frac{AC}{2}$$

$$\Rightarrow AP + AQ \leq AB \blacksquare$$



Let  $P'$ ,  $E'$ , be the reflection of  $P$ ,  $E$  with respect to the perpendicular bisector of  $BC$ . Therefore, we have  $\angle AE'C = \angle AP'E = 90^\circ$  and  $AP' = AP$ . We have:

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Solutions of 4th Iranian Geometry Olympiad 2017 (Intermediate)

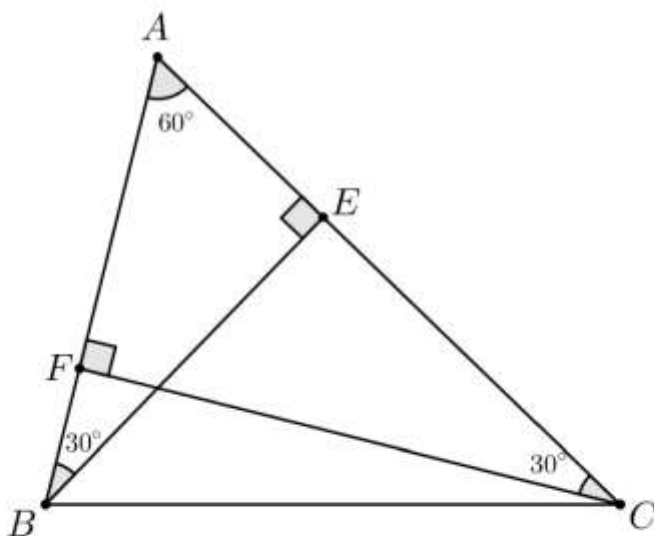
1. Let  $ABC$  be an acute-angled triangle with  $A = 60^\circ$ . Let  $E, F$  be the feet of altitudes through  $B, C$  respectively. Prove that  $CE - BF = \frac{3}{2}(AC - AB)$ .

*Proposed by Fatemeh Sajadi*

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**Solution.**

$$\begin{aligned}\angle A = 60^\circ &\Rightarrow \angle ABF = \angle ACF = 30^\circ \Rightarrow AE = \frac{AB}{2}, \quad AF = \frac{AC}{2} \\ \Rightarrow CE - BF &= (AC - AE) - (AB - AF) \\ &= (AC - AB) + (AF - AE) = \frac{3}{2}(AC - AB) \blacksquare\end{aligned}$$



---

2. Two circles  $\omega_1, \omega_2$  intersect at  $A, B$ . An arbitrary line through  $B$  meets  $\omega_1, \omega_2$  at  $C, D$  respectively. The points  $E, F$  are chosen on  $\omega_1, \omega_2$  respectively so that  $CE = CB, BD = DF$ . Suppose that  $BF$  meets  $\omega_1$  at  $P$ , and  $BE$  meets  $\omega_2$  at  $Q$ . Prove that  $A, P, Q$  are collinear.

*Proposed by Iman Maghsoudi*

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**Solution.**

We know that:

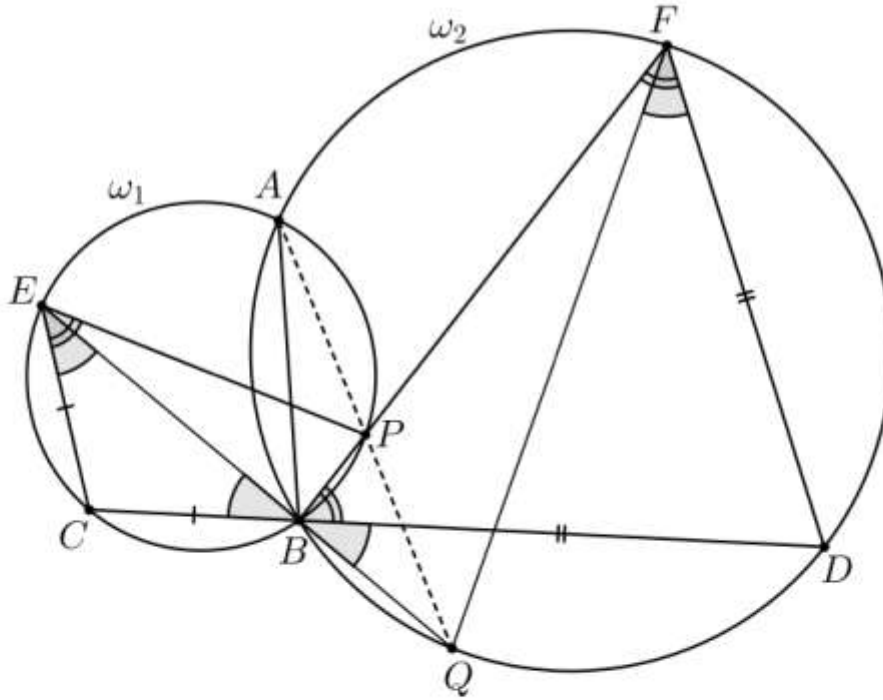
$$\angle BFD = \angle DBF = 180^\circ - \angle CBP = \angle CEP$$

$$\Rightarrow \angle CEB + \angle BEP = \angle BFQ + \angle QFD$$

$$\angle CEB = \angle CBE = \angle QBD = \angle QFD$$

$$\Rightarrow \angle BEP = \angle BFQ \Rightarrow \angle BAP = \angle BEP = \angle BFQ = \angle BAQ$$

So  $A, P, Q$  are collinear. ■



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3. On the plane,  $n$  points are given ( $n > 2$ ). No three of them are collinear. Through each two of them the line is drawn, and among the other given points, the one nearest to this line is marked (in each case this point occurred to be unique). What is the maximal possible number of marked points for each given  $n$ ?

*Proposed by Boris Frenkin (Russia)*

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**Solution.**

The case  $n = 3$  is obvious. For  $n > 4$ , take a regular  $n$ -gon and slightly deform it to make the assumptions of the problem valid. Each vertex is the nearest to the line connecting two adjacent vertices (this is directly proved by computing angles).

If  $n = 4$  then two cases are possible.

1) The given points form a convex quadrilateral (say  $ABCD$ ). Since for each side the nearest vertex is unique, no two sides are parallel. Let lines  $AD$  and  $BC$  meet beyond  $A$  and  $B$ , and lines  $AB$  and  $CD$  meet beyond  $B$  and  $C$  respectively. Then points  $A$ ,  $B$  and  $C$  are marked. Suppose  $D$  is marked as well. Then it is the nearest vertex to  $AC$ , and the area of  $\triangle ACD$  is less than that of  $\triangle ACB$ . On the other hand, draw the line through  $C$  parallel to  $AD$ . Let it meet  $AB$  at point  $E$ . Then  $CE < AD$ , so  $S_{\triangle ACD} > S_{\triangle ACE} > S_{\triangle ACB}$ , a contradiction.

2) One of the given points (say  $O$ ) is inside the triangle  $ABC$  formed by the remaining points. Obviously  $O$  is nearest of the given points to lines  $AB$ ,  $BC$  and  $AC$ . Without loss of generality, the nearest point to line  $AO$  is  $B$ . If the nearest point to line  $BO$  is  $A$  then  $C$  cannot be marked. Suppose the nearest point to line  $BO$  is  $C$ . If the remaining point  $A$  is also marked then it is nearest to line  $CO$ . Let line  $AO$  intersect  $BC$  at point  $A_1$ , and let points  $B_1$ ,  $C_1$  be defined similarly. We have  $S_{\triangle AA_1B} < S_{\triangle AA_1C}$ , hence  $A_1B < A_1C$ . Similarly  $B_1C < B_1A$ ,  $C_1A < C_1B$ . Then  $A_1B \cdot B_1C \cdot C_1A < A_1C \cdot B_1A \cdot C_1B$ , a contradiction with Ceva theorem. ■

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**Comment.**

In fact, it is not necessary to apply Ceva theorem. Let  $M_A$ ,  $M_B$ ,  $M_C$  be the bases of the corresponding medians, and  $M$  be the centroid. Then  $O$  must belong to the interior of each of triangles  $MM_AB$ ,  $MM_BC$ ,  $MM_CA$  which is impossible.

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4. In the isosceles triangle  $ABC$  ( $AB = AC$ ), let  $l$  be a line parallel to  $BC$  through  $A$ . Let  $D$  be an arbitrary point on  $l$ . Let  $E, F$  be the feet of perpendiculars through  $A$  to  $BD, CD$  respectively. Suppose that  $P, Q$  are the images of  $E, F$  on  $l$ . Prove that  $AP + AQ \leq AB$ .

*Proposed by Morteza Saghafian*

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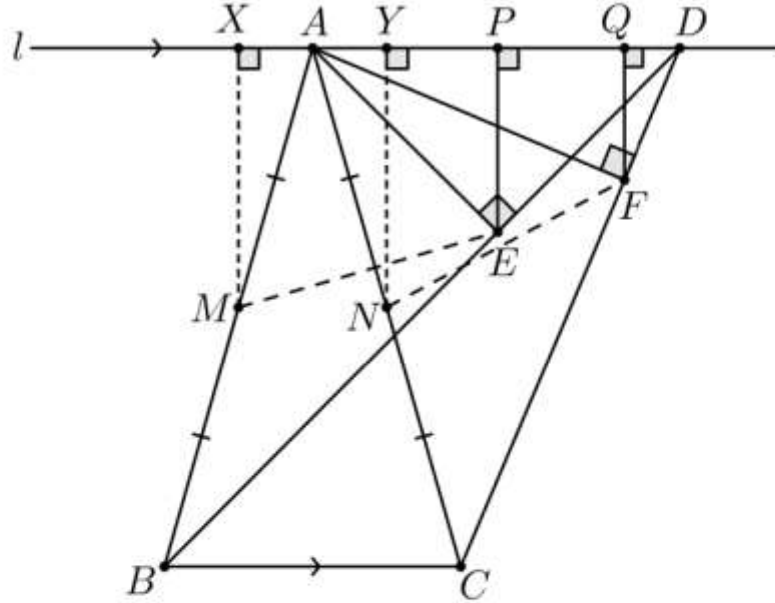
**Solution1.**

Suppose that  $M, N$  are the midpoints of  $AB, AC$  respectively. We have:

$$\angle AEB = \angle AFC = 90^\circ \Rightarrow ME = NF = \frac{AB}{2} = \frac{AC}{2}$$

Let  $X, Y$  be the feet of perpendiculars through  $M, N$  to line  $l$  respectively. We know that two triangles  $AXM$  and  $ANY$  are equal so  $AX = AY$ .

$$\begin{aligned} XP = AX + AP &\leq ME = \frac{AB}{2}, \quad AQ - AY = YQ \leq NF = \frac{AC}{2} \\ &\Rightarrow AP + AQ \leq AB \blacksquare \end{aligned}$$

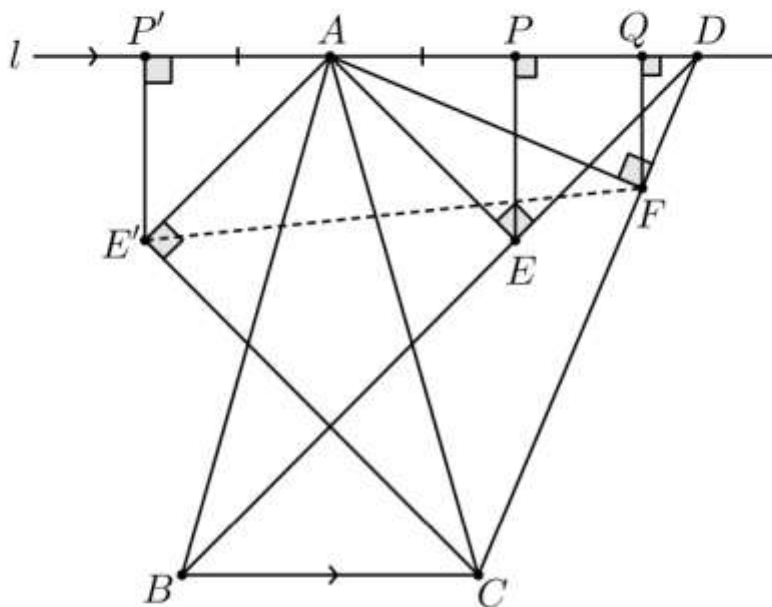


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**Solution2.**

Let  $P'$ ,  $E'$ , be the reflection of  $P$ ,  $E$  respect to the perpendicular bisector of  $BC$ . Therefore, we have  $\angle AE'C = \angle AP'E = 90^\circ$  and  $AP' = AP$ . We have:

$$AP + AQ = AP' + AQ = QP' \leq FE' \leq AC = AB \blacksquare$$

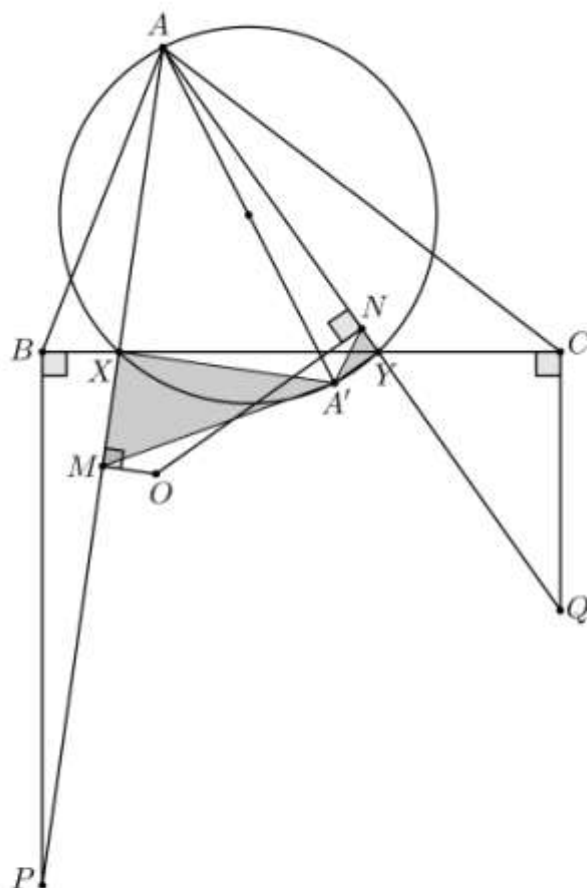


5. Let  $X, Y$  be two points on the side  $BC$  of triangle  $ABC$  such that  $2XY = BC$ . ( $X$  is between  $B, Y$ ) Let  $AA'$  be the diameter of the circumcircle of triangle  $AXY$ . Let  $P$  be the point where  $AX$  meets the perpendicular from  $B$  to  $BC$ , and  $Q$  be the point where  $AY$  meets the perpendicular from  $C$  to  $BC$ . Prove that the tangent line from  $A'$  to the circumcircle of  $AXY$  passes through the circumcenter of triangle  $APQ$ .

*Proposed by Iman Maghsoudi*

**Solution.**

Let  $O, M, N$  be the circumcenter of triangle  $APQ$ , midpoint of  $AP$ , midpoint of  $AQ$  respectively. We know that  $\angle OMA = \angle ONA = 90^\circ$  so the quadrilateral  $AMON$  is cyclic. We have to prove  $\angle OA'A = 90^\circ$ . Therefore, we have to show that  $AMO A'N$  is cyclic. It's enough to show that the quadrilateral  $AMA'N$  is cyclic. We should prove  $\angle A'XM = \angle A'NY$ . We will show that  $\triangle A'NY \sim \triangle A'XM$ .



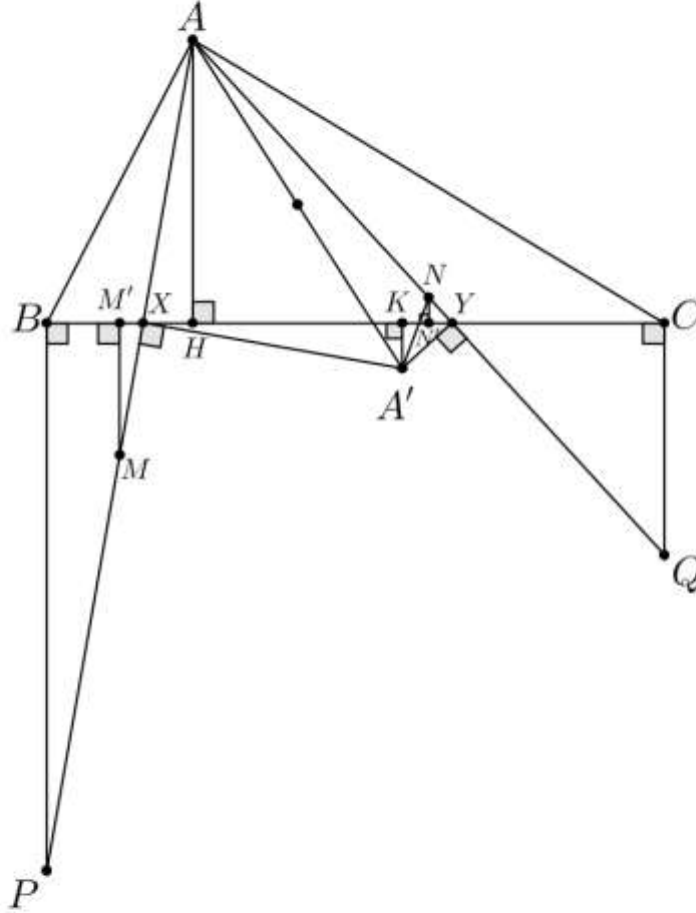
We know that the quadrilateral  $AXA'Y$  is cyclic so  $\angle A'XM = \angle A'YN$ . Therefore, it's enough to show that:

$$\frac{A'X}{MX} = \frac{A'Y}{NY} \quad (1)$$

Let  $H, K, M', N'$  be the feet of perpendiculars through  $A, A', M, N$  to  $BC$  respectively. We know that  $M, N$  are the midpoints of  $AP, AQ$  so we have:

$$BM' = HM' \Rightarrow XM' = HM' - XH = \frac{BH}{2} - XH \quad (2)$$

$$\begin{aligned} CN' = HN' &\Rightarrow YN' = HY - HN' = HY - \frac{CH}{2} = (XY - XH) - \frac{BC - BH}{2} \\ &\Rightarrow YN' = (XY - \frac{BC}{2}) + (\frac{BH}{2} - XH) = \frac{BH}{2} - XH \quad (3) \end{aligned}$$



According to the equations (2), (3) we can say that  $XM' = YN'$ . On the other hand, we have  $AA'$  is the diameter of the circumcircle of triangle  $AXY$ . Therefore:

$$\begin{aligned} \angle A'YA = \angle A'XA = 90^\circ &\Rightarrow \angle KA'Y = \angle NYN' , \quad \angle KA'X = \angle MXM' \\ &\Rightarrow \triangle A'KX \sim \triangle XM'M , \quad \triangle A'KY \sim \triangle YN'N \\ &\Rightarrow \frac{A'X}{MX} = \frac{A'K}{XM'} , \quad \frac{A'K}{NY'} = \frac{A'Y}{NY} \Rightarrow \frac{A'X}{MX} = \frac{A'Y}{NY} \end{aligned}$$

So equation (1) is proved. ■

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### Solutions of 4th Iranian Geometry Olympiad 2017 (Advanced)

1. In triangle  $ABC$ , the incircle, with center  $I$ , touches the side  $BC$  at point  $D$ . Line  $DI$  meets  $AC$  at  $X$ . The tangent line from  $X$  to the incircle (different from  $AC$ ) intersects  $AB$  at  $Y$ . If  $YI$  and  $BC$  intersect at point  $Z$ , prove that  $AB = BZ$ .

*Proposed by Hooman Fattahimoghaddam*

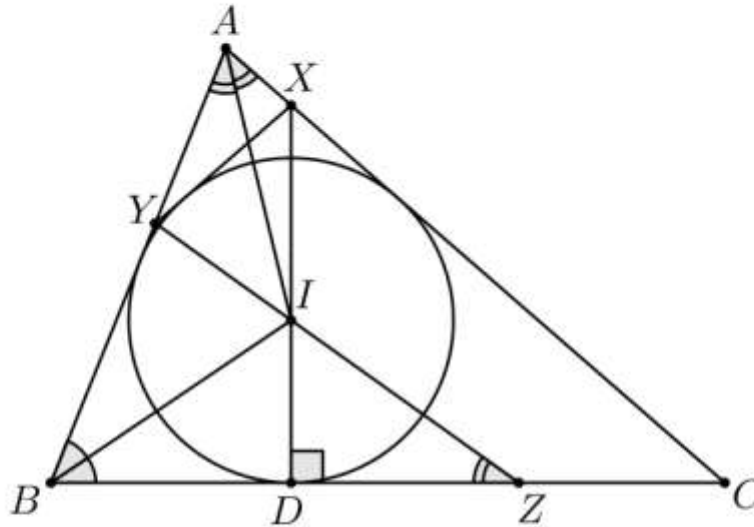
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**Solution.**

In triangle  $AXY$ , the point  $I$  is excenter so  $\angle XIY = 90^\circ - \frac{A}{2}$ .

$$\Rightarrow \angle DIZ = 90^\circ - \frac{A}{2} \Rightarrow \angle IZB = \frac{A}{2} = \angle BAI$$

Also we know that  $\angle IBZ = \angle IBA$  and  $BI = BI$  so two triangles  $ABI$  and  $ZBI$  are equal. Thus,  $AB = BZ$ . ■



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2. We have six pairwise non-intersecting circles that the radius of each is at least one. Prove that the radius of any circle intersecting all the six circles, is at least one.

*Proposed by Mohammad Ali Abam– Morteza Saghafian*

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**Solution.**

Call the centers of these six circles  $O_1, O_2, \dots, O_6$ , and their radii  $R_1, R_2, \dots, R_6$ . Suppose that a circle with center  $O$  and radius  $R$  intersects these six circles. Obviously there exist  $i, j$  such that  $\angle O_i O O_j < 60^\circ$ . The length of  $O_i O_j$  is at least  $R_i + R_j$  and the lengths of  $OO_i, OO_j$  are less than or equal to  $R_i + R, R_j + R$ , respectively. If  $R < 1$  then considering  $R_i, R_j > 1$  we conclude that in triangle  $O_i O O_j$  the longest side is  $O_i O_j$ , so  $\angle O_i O O_j \geq 60^\circ$  which yields a contradiction. So we must have  $R \geq 1$ . ■

---

3. Let  $O$  be the circumcenter of triangle  $ABC$ . Line  $CO$  intersects the altitude through  $A$  at point  $K$ . Let  $P, M$  be the midpoints of  $AK, AC$  respectively. If  $PC$  intersects  $BC$  at  $Y$ , and the circumcircle of triangle  $BCM$  meets  $AB$  at  $X$ , prove that  $BXOY$  is cyclic.

*Proposed by Ali Daeinabi – Hamid Pardazi*

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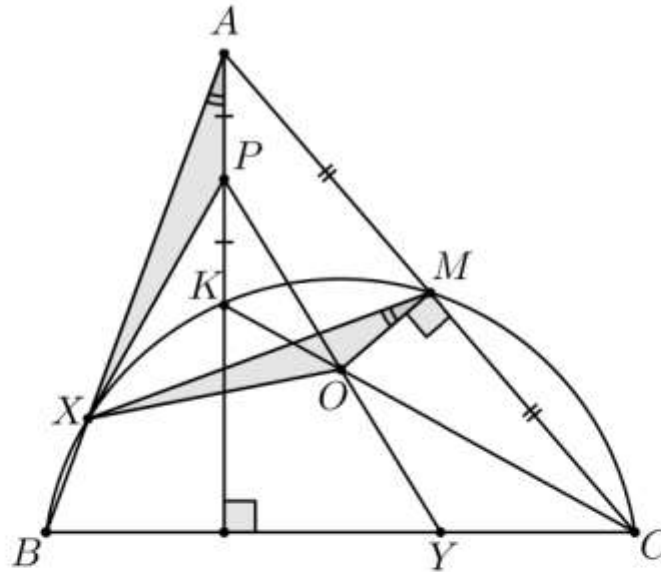
**Solution.**

We want to show that  $\angle XOP = \angle B$ . We know that:

$$\angle B = \angle XMA \Rightarrow \angle XMO = 90^\circ - \angle B = \angle XAK$$

We will prove that two triangles  $XOP$  and  $XMA$  are similar. It's enough to show that two triangles  $XPA$  and  $XOM$  are similar. We have  $\angle XMO = \angle XAK$  so it's enough to prove:

$$\frac{AX}{XM} = \frac{AP}{OM}$$



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The quadrilateral  $BXMC$  is cyclic. Therefore  $\triangle AXM \sim \triangle ACB$ .

$$\Rightarrow \frac{AX}{XM} = \frac{AC}{BC} \quad (1)$$

The point  $O$  is the circumcenter of triangle  $ABC$ . Thus,  $\angle OCA = 90^\circ - \angle B$  and  $\angle AKC = 180^\circ - \angle A$ . According to law of sines in triangles  $AKC$ ,  $OMC$  and  $ABC$  we can say that:

$$\begin{aligned} \frac{AC}{\sin \angle A} &= \frac{AK}{\sin(90^\circ - \angle B)} \quad , \quad OC = \frac{OM}{\sin(90^\circ - \angle B)} \\ \Rightarrow \frac{AK}{OM} &= \frac{AC}{OC \cdot \sin \angle A} \quad \Rightarrow \quad \frac{AP}{OM} = \frac{AC}{2OC \cdot \sin \angle A} = \frac{AC}{BC} \quad (2) \end{aligned}$$

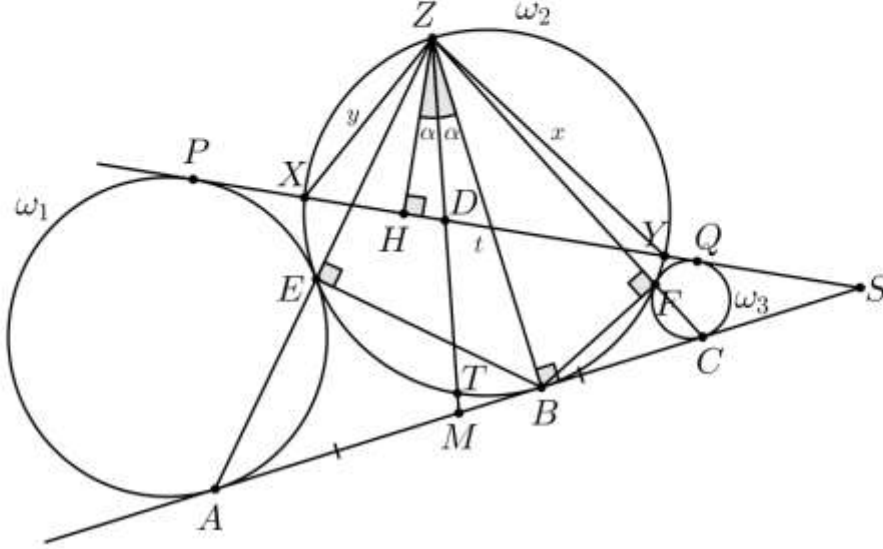
According to equations (1) and (2), we can conclude that two triangles  $XPA$  and  $XOM$  are similar. Therefore two triangles  $XOP$  and  $XMA$  are similar so:

$$\angle XBP = \angle XMA = \angle B \quad \blacksquare$$



According to the equations (1), (2) we can say that  $ZT$  is the radical axis of  $\omega_1, \omega_3$ . Therefore, if  $M$  is the midpoint of  $AC$  then  $Z, T, M$  are collinear because  $M$  lies on radical axis of  $\omega_1, \omega_3$  too. Let  $D$  be the intersection of  $ZM, PQ$  and  $H$  be the feet of perpendiculars through  $Z$  to  $PQ$ . We know that:

$$\angle HZD = \angle MZB = \frac{|\angle ZXY - \angle ZYX|}{2} = \alpha$$



Let  $x = ZY, y = ZX, t = XY$ . We can say that:

$$AM = \frac{ZB}{\cos \alpha} = \frac{ZB}{\frac{ZH}{ZD}} = \frac{ZB}{ZH} \cdot ZD = \frac{ZB}{ZH} \cdot \frac{2xy}{x+y} \cdot \cos \frac{(\angle XZY)}{2}$$

To prove the problem, it's enough to show that the distances of  $M$  from two lines  $ZX, ZY$  are equal to  $\frac{AC}{2}$ . Thus, we have to show that:

$$\begin{aligned} AM \cdot \sin \frac{\angle XZY}{2} &= \frac{AC}{2} \\ \Leftrightarrow 2AM \cdot \sin \frac{(\angle XZY)}{2} &= \frac{ZB}{ZH} \cdot \frac{2xy}{x+y} \cdot \sin(\angle XZY) = AC \end{aligned}$$

On the other hand, we have:

$$S_{\triangle XYZ} = \frac{1}{2} t \cdot ZH = \frac{1}{2} xy \sin(\angle XZY) \Rightarrow t = \frac{xy \sin(\angle XZY)}{ZH}$$

So it's enough to show that:

$$AC = \frac{2t \cdot ZB}{x+y}$$

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We know that:

$$\angle BFZ = \angle BEZ = 90^\circ \Rightarrow ZB^2 = ZE \cdot ZA = ZF \cdot ZC$$

Therefore, lengths of tangents from  $Z$  to  $\omega_1, \omega_3$  are equal to  $ZB$ . Now, According to Casey's theorem (generalization of Ptolemy's theorem) in two cyclic quadrilaterals  $ZY\omega_1X$  and  $ZX\omega_3Y$  we can say that:

$$ZY\omega_1X : x \cdot PX + y \cdot (t + PX) = t \cdot ZB \Rightarrow (x + y)PX + yt = t \cdot ZB$$

$$ZX\omega_3Y : y \cdot YQ + x \cdot (t + YQ) = t \cdot ZB \Rightarrow (x + y)YQ + xt = t \cdot ZB$$

$$\Rightarrow PX + YQ + t = \frac{2t \cdot ZB}{x + y} \Rightarrow AC = \frac{2t \cdot ZB}{x + y} \blacksquare$$

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5. Sphere  $S$  touches a plane. Let  $A, B, C, D$  be four points on this plane such that no three of them are collinear. Consider the point  $A'$  such that  $S$  is tangent to the faces of tetrahedron  $A'BCD$ . Points  $B', C', D'$  are defined similarly. Prove that  $A', B', C', D'$  are coplanar and the plane  $A'B'C'D'$  touches  $S$ .

*Proposed by Alexey Zaslavsky (Russia)*

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**Solution.**

Let  $S$  touch the plane at point  $P$ .  $D$  is the point of concurrency of the planes passing through  $AB, BC, CA$  and touching  $S$  (similarly for  $A', B', C'$ ). Suppose that the plane passing through  $X, Y$  and touching  $S$ , touches it at  $P_{xy}$ . The points  $P, P_{ab}, P_{ac}, P_{ad}$  are on a circle  $W_a$ , because when we connect them to  $P$ , the resulted line touches  $S$ . Similarly we can find other triples which are concyclic with  $P$ , name the circles with  $W_b, W_c, W_d$ . Now use inversion with center  $P$  and arbitrary radius, we get 4 lines with three points on each line. Considering the concurrency of four circles at Michels Point we conclude that the circles  $P_{ab}P_{bc}P_{ca}, P_{ab}P_{bd}P_{da}, P_{ac}P_{cd}P_{da}, P_{bc}P_{cd}P_{db}$  (call them  $W'_d, W'_c, W'_b, W'_a$ ) are concurrent on  $S$  at a point  $P'$ .  $W'_d$  is the locus of feet of tangencies from  $D'$  to  $S$ , so  $D'P'$  is tangent to  $S$  and  $D'$  is on the tangent plane at  $P'$  to  $S$ . Similarly we conclude that  $A', B', C'$  are also on the tangent plane at  $P'$  to  $S$ . So  $A', B', C', D'$  are coplanar and this plane is tangent to  $S$ . ■