Novi Sad, Serbia & Muntenegru, 28.06.2004.

NUMBER THEORY

NT1. Let a, b, p, q be positive integers such that a and b are relatively prime, ab is even and $p, q \geq 3$. Prove that

$$2a^p b - 2ab^q$$

cannot be a square of an integer number.

Solution. Without loss of generality, assume that a is even and consequently b is odd. Let $a = 2a'$. If

$$2a^p b - 2ab^q = 4a'b(a^{p-1} - b^{q-1})$$

is a square, then a', b and $a^{p-1} - b^{q-1}$ are pairwise coprime.

On the other hand, a^{p-1} is divisible by 4 and b^{q-1} gives the remainder 1 when divided by 4. It follows that $a^{p-1} - b^{q-1}$ has the form $4k+3$, a contradiction.

NT2. Find all four digit numbers A such that

$$\frac{1}{3}A + 2000 = \frac{2}{3}\bar{A},$$

where $\bar{A}$ is the number with the same digits as A, but written in opposite order. (For example, $\overline{1234} = 4321$.)

Solution. Let $A = 1000a + 100b + 10c + d$. Then we obtain the equality

$$\frac{1}{3}(1000a + 100b + 10c + d) + 2000 = \frac{2}{3}(1000d + 100c + 10b + a)$$

or

$$1999d + 190c = 80b + 998a + 6000.$$

It is clear that d is even digit and $d > 2$. So we have to investigate three cases: (i) $d = 4$; (ii) $d = 6$; (iii) $d = 8$.

(i) If $d = 4$. comparing the last digits in the upper equality we see that $a = 2$ or $a = 7$.

If $a = 2$ then $19c = 8b$, which is possible only when $4b = c = 0$. Hence the number $A = 2004$ satisfies the condition.

If $a = 7$ then $19c - 8b = 499$, which is impossible.

(ii) If $d = 6$ then $190c + 5994 = 80b + 998a$. Comparing the last digits we obtain that $a = 3$ or $a = 8$.

If $a = 3$ then $80b + 998a < 80 \cdot 9 + 1000 \cdot 3 < 5994$.

If $a = 8$ then $80b + 998a \geq 998 \cdot 8 = 7984 = 5994 + 1990 > 5994 + 190c$.

(iii) If $d = 8$ then $190c + 9992 = 80b + 998a$. Now $80b + 998a \leq 80 \cdot 9 + 998 \cdot 9 = 9702 < 9992 + 190c$.

Hence we have the only solution $A = 2004$.

NT3. Find all positive integers n, $n \geq 3$, such that $n|(n-2)!$.

Solution. For $n=3$ and $n=4$ we easily check that n does not divide $(n-2)!$.

If n is prime, $n \geq 5$, then $(n-2)! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-2)$ is not divided by n, since n is a prime not included in the set of factors of $(n-2)!$.

If n is composite, $n \geq 6$, then $n = pm$, where p is prime and $m > 1$. Since $p \geq 2$, we have $n = pm \geq 2m$, and consequently $m \leq \frac{n}{2}$. Moreover $m < n-2$, since $n > 4$.

Therefore, if $p \neq m$, both p and m are distinct factors of $(n-2)!$ and so n divides $(n-2)!$.

If $p = m$, then $n = p^2$ and $p > 2$ (since $n > 4$). Hence $2p < p^2 = n$. Moreover $2p < n-1$ (since $n > 4$). Therefore, both p and $2p$ are distinct factors of $(n-2)!$ and so $2p^2|(n-2)!$. Hence $p^2|(n-2)!$, i.e. $n|(n-2)!$. So we conclude that $n|(n-2)!$ if and only if n is composite, $n \geq 6$.

NT4. If the positive integers x and y are such that both $3x+4y$ and $4x+3y$ are perfect squares, prove that both x and y are multiples of 7.

Solution. Let

$$3x + 4y = m^2, \quad 4x + 3y = n^2. \tag{1}$$

Then

$$7(x+y) = m^2 + n^2 \Rightarrow 7|m^2 + n^2. \tag{2}$$

Considering $m = 7k + r$, $r \in \{0,1,2,3,4,5,6\}$ we find that $m^2 \equiv u \pmod 7$, $u \in \{0,1,2,4\}$ and similarly $n^2 \equiv v \pmod 7$, $v \in \{0,1,2,4\}$. Therefore we have either $m^2 + n^2 \equiv 0 \pmod 7$, when $u = v = 0$, or $m^2 + n^2 \equiv w \pmod 7$, $w \in \{1,2,3,4,5,6\}$. However, from (2) we have that $m^2 + n^2 \equiv 0 \pmod 7$ and hence $u = v = 0$ and

$$m^2 + n^2 \equiv 0 \pmod{7^2} \Rightarrow 7(x+y) \equiv 0 \pmod{7^2},$$

and consequently

$$x + y \equiv 0 \pmod 7. \tag{3}$$

Moreover, from (1) we have $x - y = n^2 - m^2$ and $n^2 - m^2 \equiv 0 \pmod{7^2}$ (since $u = v = 0$), so

$$x - y \equiv 0 \pmod 7. \tag{4}$$

From (3) and (4) we have that $x + y = 7k$, $x - y = 7l$, where k and l are positive integers. Hence

$$2x = 7(k+l), \quad 2y = 7(k-l),$$

where $k+l$ and $k-l$ are positive integers. It follows that $7|2x$ and $7|2y$, and finally $7|x$ and $7|y$.

ALGEBRA

A1. Prove that
$$(1+abc)\Big(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\Big) \geq 3+a+b+c,$$
for any real numbers $a, b, c \geq 1$.

Solution. The inequality rewrites as
$$\Big(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\Big)+(bc+ac+ab) \geq 3+a+b+c$$
or
$$\Big(\frac{\frac{1}{a}+\frac{1}{b}}{2}+\frac{\frac{1}{b}+\frac{1}{c}}{2}+\frac{\frac{1}{a}+\frac{1}{c}}{2}\Big)+(bc+ac+ab) \geq 3+a+b+c,$$
which is equivalent to
$$\frac{(2ab-(a+b))(ab-1)}{2ab}+\frac{(2bc-(b+c))(bc-1)}{2bc}+\frac{(2ca-(c+a))(ca-1)}{2ca} \geq 0.$$
The last inequality is true due to the obvious relations
$$2xy-(x+y)=x(y-1)+y(x-1) \geq 0,$$
for any two real numbers $x, y \geq 1$.

A2. Prove that, for all real numbers x, y, z:
$$\frac{x^2-y^2}{2x^2+1}+\frac{y^2-z^2}{2y^2+1}+\frac{z^2-x^2}{2z^2+1} \leq (x+y+z)^2.$$
When the equality holds?

Solution. For $x=y=z=0$ the equality is valid.
Since $(x+y+z)^2 \geq 0$ it is enough to prove that
$$\frac{x^2-y^2}{2x^2+1}+\frac{y^2-z^2}{2y^2+1}+\frac{z^2-x^2}{2z^2+1} \leq 0$$
which is equivalent to the inequality
$$\frac{x^2-y^2}{x^2+\frac{1}{2}}+\frac{y^2-z^2}{y^2+\frac{1}{2}}+\frac{z^2-x^2}{z^2+\frac{1}{2}} \leq 0. \qquad (1)$$
Denote
$$a=x^2+\frac{1}{2},\ \ b=y^2+\frac{1}{2},\ \ c=z^2+\frac{1}{2}.$$

Then (1) is equivalent to

$$\frac{a-b}{a}+\frac{b-c}{b}+\frac{c-a}{c}\le 0. \tag{2}$$

From very well known AG inequality follows that

$$a^2b+b^2c+c^2a\ge 3abc.$$

From the equivalencies

$$a^2b+b^2c+c^2a\ge 3abc \Leftrightarrow \frac{a}{c}+\frac{b}{a}+\frac{c}{b}\ge 3 \Leftrightarrow \frac{a-b}{a}+\frac{b-c}{b}+\frac{c-a}{c}\le 0$$

follows that the inequality (2) is valid, for positive real numbers a, b, c.

A3. Prove that for all real x, y

$$\frac{x+y}{x^2-xy+y^2}\le\frac{2\sqrt{2}}{\sqrt{x^2+y^2}}.$$

Solution. The inequality rewrites as

$$\frac{x+y}{x^2-xy+y^2}\le\frac{\sqrt{2(x^2+y^2)}}{\frac{x^2+y^2}{2}}.$$

Now it is enough to prove the next two simple inequalities:

$$x+y\le\sqrt{2(x^2+y^2)},\quad x^2-xy+y^2\ge\frac{x^2+y^2}{2}.$$

A4. Prove that if $0<\frac{a}{b}<b<2a$ then

$$\frac{2ab-a^2}{7ab-3b^2-2a^2}+\frac{2ab-b^2}{7ab-3a^2-2b^2}\ge 1+\frac{1}{4}\left(\frac{a}{b}-\frac{b}{a}\right)^2.$$

Solution. If we denote

$$u=2-\frac{a}{b},\quad v=2-\frac{b}{a}$$

then the inequality rewrites as

$$\frac{u}{v+uv}+\frac{v}{u+uv}\ge 1+\frac{1}{4}(u-v)^2$$

or

$$\frac{(u-v)^2+uv(1-uv)}{uv(uv+u+v+1)}\ge\frac{(u-v)^2}{4}.$$

Since $u>0$, $v>0$, $u+v\le 2$, $uv\le 1$, $uv(u+v+uv+1)\le 4$, the result is clear.

GEOMETRY

G1. Two circles k_1 and k_2 intersect at points A and B. A circle k_3 centered at A meet k_1 at M and P and k_2 at N and Q, such that N and Q are on different sides of MP and $AB > AM$.

Prove rhat the angles $\angle MBQ$ and $\angle NBP$ are equal.

Solution As $AM = AP$, we have

$$\angle MBA = \frac{1}{2} arcAM = \frac{1}{2} arcAP = \angle ABP,$$

and likewise

$$\angle QBA = \frac{1}{2} arcAQ = \frac{1}{2} arcAN = \angle ABN.$$

Summing these equalities yields $\angle MBQ = \angle NBP$ as needed.

G2. Let E and F be two distinct points inside of a parallelogram $ABCD$. Find the maximum number of triangles with the same area and having the vertices in three of the following five points: A, B, C, D, E, F.

Solution. We shall use the following two well known results:

Lemma 1. Let A, B, C, D be four points lying in the same plane such that the line AB do not intersect the segment CD (in particular $ABCD$ is a convex quadrilateral). If $[ABC] = [ABD]$, then $AB \parallel CD$.

Lemma 2. Let X be a point inside of a parallelogram $ABCD$. Then $[ACX] < [ABC]$ and $[BDX] < [ABD]$. (Here and below the notation $[S]$ stands for the area of the surface of S.)

With the points A, B, C, D, E, F we can form 20 triangles. We will show that at most ten of them can have the same area. In that sense three cases may occur.

Case 1. EF is parallel with one side of the parallelogram $ABCD$.

We can assume that $EF \parallel AD$, E lies inside of the triangle ABF and F lies inside of thr triangle CDE. With the points A, B, C, D, E we can form ten pairs of triangles as follows:

$(\Delta ADE, \Delta AEF)$; $(\Delta ADF, \Delta DEF)$; $(\Delta BCE, \Delta BEF)$; $(\Delta BCF, \Delta CEF)$; $(\Delta CDE. \Delta CDF)$;

$(\Delta ABE, \Delta ABF)$; $(\Delta ADC, \Delta ACE)$; $(\Delta ABC, \Delta ACF)$; $(\Delta ABD, \Delta BDE)$; $(\Delta CBD. \Delta BDF)$.

Using Lemmas 1 – 2 one can easily prove that any two triangles that belong to the same pair have distinct area, so there exists at most ten triangles having the same area.

Case 2. EF is parallel with a diagonal of the paralellogram $ABCD$.

Let us assume $EF \parallel AC$ and that E, F lie inside of the triangle ABC. We consider the following pairs of triangles:

$$(\triangle ABD, \triangle BDE); (\triangle ABC, \triangle BCF); (\triangle ACD, \triangle BCE); (\triangle ABF, \triangle ABE); (\triangle BEF, \triangle DEF);$$

$$(\triangle AEF, \triangle ACE); (\triangle CEF, \triangle ACF); (\triangle ADE, \triangle ADF); (\triangle DCE, \triangle DCF); (\triangle CBD, \triangle BDF).$$

With the same idea as above we deduce that any two triangles that belong to the same pair have distinct area and the conclusion follows.

We also note that if E and F lie on AC then only 16 of 20 triangles are nondegenerate. In this case we consider the following pairs:

$$(\triangle ABE, \triangle ABF); (\triangle ABC, \triangle BCF); (\triangle BCE, \triangle BEF); (\triangle ADE, \triangle ADF);$$

$$(\triangle ACD, \triangle DCF); (\triangle CDE, \triangle EDF); (\triangle BDE, \triangle ABD); (\triangle BDF, \triangle BDC).$$

Case 3. EF is not parallel with any side or diagonal of $ABCD$.

We claim that at most two of the triangles AEF, BEF, CEF, DEF can have the same area. Indeed, supposing the contrary, we may have $[AEF] = [BEF] = [CEF]$. We remark first that A, B, C do not belong to EF (elsewhere, exactly one of the above triangles is degenerate, contradiction!). Hence at least two of the points A, B, C belong to the same side of the line EF. Using now Lemma 1 we get that EF is parallel with AB or BC or AC. This is clearly a contradiction and our claim follows. With the remaining 16 triangles we form 8 pairs as follows:

$$(\triangle ABD, \triangle BDE); (\triangle CDB, \triangle BDF); (\triangle ADC, \triangle ACE); (\triangle ABC, \triangle ACF);$$

$$(\triangle ABE, \triangle ABF); (\triangle BCE, \triangle BCF); (\triangle ADE, \triangle ADF); (\triangle DCE, \triangle DCF).$$

With the same arguments as above, we get at most ten triangles with the same area.

To conclude the proof, it remains only to give an example of points E, F inside of the parallelogram $ABCD$ such that exactly ten of the triangles that can be formed with the vetices A, B, C, D, E, F have the same area.

Denote $AC \cap BD = \{O\}$ and let M, N be the midpoints of AB and CD respectively. Consider E and F the midpoints of MO and NO. Then O, M, N, E, F are collinear and $ME = EO = FO = NF$. Since E and F are the centroids of the triangles ABF and CDE we get $[ABE] = [AEF] = [BEF]$ and $[CEF] = [DEF] = [CDF]$. On the other hand, taking into account that $AECF$ and $BEDF$ are parallelograms we deduce $[AEF] = [CEF] = [ACE] = [ACF]$ and $[BEF] = [DEF] = [BDE] = [BDF]$. From the above equalities we conclude that the triangles

$$\triangle ABE, \triangle CDFE, \triangle ACE, \triangle ACF, \triangle BDE, \triangle BDF, \triangle AEF, \triangle BEF, \triangle CEF, \triangle DEF$$

have the same area. This finishes our proof.

G3. Let ABC be scalene triangle inscribed in the circle k. Circles α, β, γ are internally tangent to k at points A_1, B_1, C_1 respectively, and tangent to the sides BC, CA, AB at points A_2, B_2, C_2 respectively, so that A and A_1 are on opposite sides of BC, B and B_1 are on opposite sides of CA, and C and C_1 are on opposite sides of AB. Lines A_1A_2, B_1B_2 and C_1C_2 meet again the circle k in the points A', B', C' respectively.

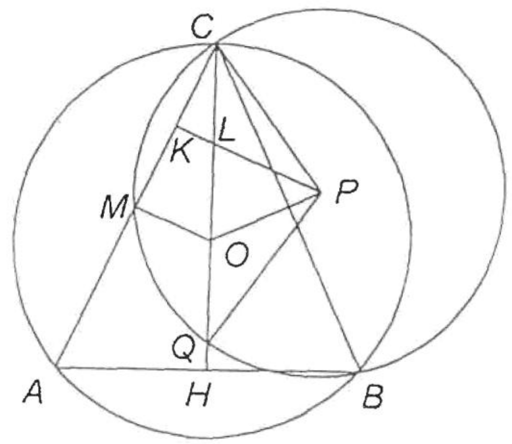
Prove that if M is the intersection point of the lines BB' and CC', the angle $\angle MAA'$ is right.

Solution. The idea is to observe that A', B', C' are the midpoints of the arcs BC, CA and AB of the circle k which do not contain the points A, B, C respectively. To prove this, consider the dilatation with the center A_1 taking α to k. The line BC, which touches α at A_2, is taken to the line t touching k at A'. Since t is parallel to BC, it follows that A' is the midpoint of thr arc BC that do not touch α.

Consequently, lines BB' and CC' are the exterior bisectors of the angles B and C of the scalene triangle ABC and so M is the excenter of ABC. Hence AM and AA' are the interior and exterior bisectors of the angle A, implying $\angle MAA' = 90°$.

G4. Let ABC be isosceles triangle with $AC = BC$, M be the midpoint of AC, BH be the line through C perpendicular to AB. The circle through B, C and M intersects CH in point Q. If $CQ = m$, find the radius of the circumcircle of ABC.

Solution. Let P be the center of circle k_1 through B, C and M; O be the center of the circumcircle of ABC, and KE be the midpoint of MC. Since $AC = BC$, the center O lies on CH. Let KP intersects CH in point L. Since KP and OM are perpendicular to AC, then $KP \parallel OM$. From $MK = KC$ it follows that $OL = CL$. On the other hand OP is perpendicular to BC, hence $\angle LOP = \angle COP = 90° - \angle BCH$. Also we have $\angle OLP = \angle CLK = 90° - \angle ACH$. Since ABC is isosceles and $\angle BCH = \angle ACH$, then $\angle LOP = \angle OLP$ and $LP = OP$. Since $CP = PQ$ we obtain that $\angle CLP = \angle QOP$ and $CL = OQ$. Thus we have $CL = LO = OQ$, so $CO = \frac{2}{3}CQ$. Finally for the radius R of the circumcircle of ABC we obtain $R = \frac{2}{3}m$.



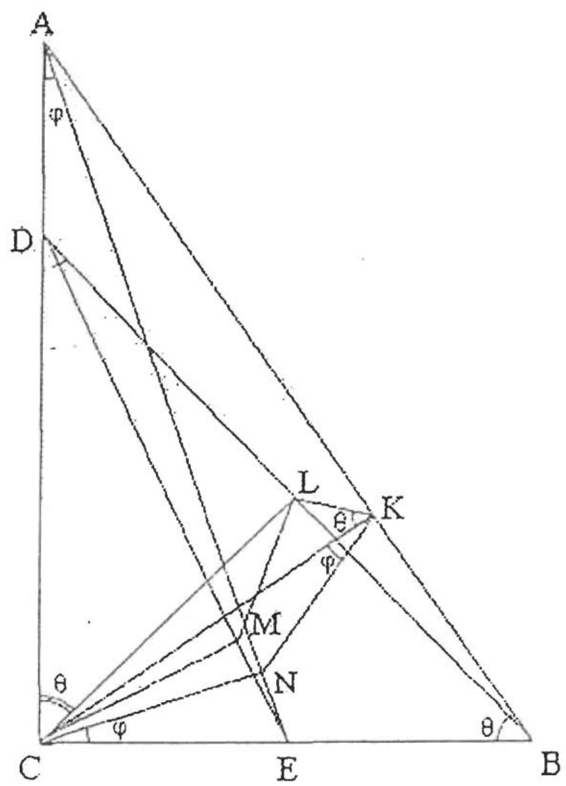
G5. Let ABC be a triangle with $\angle C = 90°$ and $D \in CA$, $E \in CB$, and k_1, k_2, k_3, k_4 semicircles with diameters CA, CB, CD, CE respectively, which have common part with the triangle ABC. Let also,

$$k_1 \cap k_2 = \{C, K\}, k_3 \cap k_4 = \{C, M\}, k_2 \cap k_3 = \{C, L\}, k_1 \cap k_4 = \{C, N\}.$$

Prove that K, L, M and N are cocyclic points.

Solution. The points K, L, M, N belong to the segments AB, BD, DE, EA respectively, where $CK \perp AB$, $CL \perp BD$, $CM \perp DE$, $CN \perp AE$. Then quadrilaterals $CDLM$ and $CENM$ are inscribed. Let $\angle CAE = \varphi$, $\angle DCL = \theta$. Then $\angle EMN = \angle ECN = \varphi$ and $\angle DML = \angle DCL = \theta$. So $\angle DML + \angle EMN = \varphi + \theta$ and therefore $\angle LMN = 180° - \varphi - \theta$. The quadrilaterals $CBKL$ and $CAKN$ are also inscribed and hence $\angle LKC = \angle LBC = \theta$, $\angle CKN = \angle CAN = \varphi$, and so $\angle LKN = \varphi + \theta$, while $\angle LMN = 180° - \varphi - \theta$, which means that $KLMN$ is inscribed.

Note that $KLMN$ is convex because L, N lie in the interior of the convex quadrilateral $ADEB$.



COMBINATORICS

C1. A polygon having n sides is arbitrarily decomposed in triangles having all the vertices among the vertices of the polygon. We paint in black the triangles that have two sides that are also sides of the polygon, in red if only one side of the triangle is side of the polygon and white those triangles that have in common with the poligon only vertices.

Prove that there are 2 more black triangles than white ones.

Solution. Denote by b, r, w the number of black, red white triangles respectively.

It is easy to prove that the polygon is divided into $n-2$ triangles, hence

$$b + r + w = n - 2.$$

Each side of the polygon is a side of exactly one triangle of the decomposition, and thus

$$2b + r = n.$$

Subtracting the two relations yields $w = b - 2$, as needed.

C2. Given $m \times n$ table, each cell signed with "$-$". The following operations are allowed:

(i) to change all the signs in entire row to the opposite, i. e. every "$-$" to "$+$", and every "$+$" to "$-$";

(ii) to change all the signs in entire column to the opposite, i. e. every "$-$" to "$+$", and every "$+$" to "$-$".

(a) Prove that if $m = n = 100$, using the above operations one can not obtain 2004 signs "$+$".

(b) If $m = 1004$, find the least $n > 100$ for which 2004 signs "$+$" can be obtained.

Solution. If we apply (i) to l rows and (ii) to k columns we obtain $(m-k)l + (n-l)k$ signs "$+$".

(a) We have equation $(100-k)l + (100-l)k = 2004$,or $100l + 100k - 2lk = 2004$, i.e.

$$50l + 50k - lk = 1002.$$

Rewrite the last equation as

$$(50 - l)(50 - k) = 2500 - 1002 = 1498.$$

Since $1498 = 2 \cdot 7 \cdot 107$, this equation has no solutions in natural numbers.

(b) Let $n = 101$. Then we have

$$(100 - k)l + (101 - l)k = 2004$$

or

$$100l + 101k - 2lk = 2004,$$

i.e.

$$101k = 2004 - 100l + 2lk = 2(1002 - 50l + lk).$$

Hence $s = 2t$ and we have $101t = 501 - 25l + 2lt$. From here we have

$$t = \frac{501 - 25l}{101 - 2l} = 4 + \frac{97 - 17l}{101 - 2l}.$$

Since t is natural number and $97 - 17l < 101 - 2l$, this is a contradiction, Hence $n \neq 101$.

Let $n = 102$. Then we have

$$(100 - k)l + (102 - l)k = 2004,$$

or

$$100l + 102k - 2lk = 2004,$$

i.e.

$$50l + 51k - lk = 1002.$$

Rewrite the last equation as

$$(51 - l)(50 - k) = 2550 - 1002 = 1548.$$

Since $1458 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 43$ we have $51 - l = 36$ and $50 - k = 43$. From here obtain $l = 15$ and $k = 7$. Indeed,

$$(100 - 7) \cdot 15 + (102 - 15) \cdot 7 = 93 \cdot 15 + 87 \cdot 7 = 1395 + 609 = 2004.$$

Hence, the least n is 102.