SOLUTIONS TO THE PROBLEMS OF THE 1990 ASIAN PACIFIC MATHEMATICS OLYMPIAD

Question 1

In $\triangle ABC$, let D, E, F be the midpoints of BC, AC, AB respectively and let G be the centroid of the triangle.
For each value of $\angle BAC$, how many non-similar triangles are there in which $AEGF$ is a cyclic quadrilateral?

FIRST SOLUTION

Let I be the intersection of AG and EF.
Let $\delta = AI.IG - FI.IE$. Then

1 $$AI = AD/2, \quad IG = AD/6, \quad FI = BC/4 = IE. \qquad (1)$$

Further, applying the cosine rule to triangles ABD, ACD we get

$$\begin{aligned} AB^2 &= BC^2/4 + AD^2 - AD.BC.\cos\angle BDA, \\ AC^2 &= BC^2/4 + AD^2 + AD.BC.\cos\angle BDA, \\ \text{so} \quad AD^2 &= \left(AB^2 + AC^2 - BC^2/2\right)/2. \end{aligned}$$

Hence

$$\begin{aligned} \delta &= \left(AB^2 + AC^2 - 2BC^2\right)/24 \\ &= \left(4AB.AC.\cos\angle BAC - AB^2 - AC^2\right). \end{aligned}$$

3 Now $AEFG$ is a cyclic quadrilateral if and only if $\delta = 0$, i.e. if and only if

$$\begin{aligned} \cos\angle BAC &= \left(AB^2 + AB^2\right)/(4.AB.AC) \\ &= (AB/AC + AC/AB)/4. \end{aligned}$$

5 Now $AB/AC + AC/AB \geq 2$. Hence $\cos\angle BAC \geq 1/2$ and so $\angle BAC \leq 60°$.
For $\angle BAC > 60°$ there is no triangle with the required property.
For $\angle BAC = 60°$ there exists, within similarity, precisely one triangle (which is equilateral) having the required property.
For $\angle BAC < 60°$ there exists, within similarity, again precisely one triangle having the required property (even though for fixed median AE there are two, but one arises from
7 the other by interchanging point B with point C, thus proving them to be similar).

SECOND SOLUTION (Mr Marcus Brazil, La Trobe University, Bundoora, Melbourne, Australia):

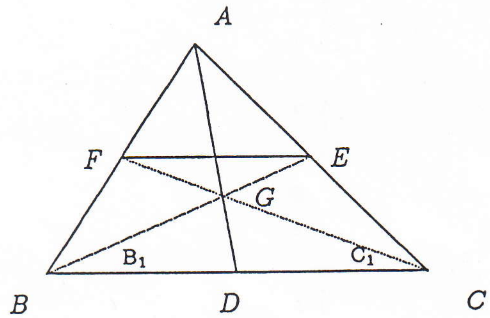
We require, as above,

1 $$AI.IG = EI.IF,$$

4 (which by (1) is equivalent to $AD^2/3 = CD^2$, i.e. $AD = CD.\sqrt{3}$).
Let, without loss of generality, $CD = 1$. Then A lies on the circle of radius $\sqrt{3}$ with centre D. If CD and DA are perpendicular, the angle BAC is $60°$, otherwise it must be
5 less.
7 In this case, for each angle BAC there are two solutions, which are congruent.

THIRD SOLUTION (provided by the Canadian Problems Committee)

In the figure as shown below, we first show that it is necessary that $\angle A$ is less than $90°$ if the quadrilateral $AEGF$ is cyclic.



Now, since $EF \parallel BC$, we get

$$\begin{aligned} \angle EGF &= 180° - (B_1 + C_1) \\ &\geq 180° - (B + C) \\ &= A \,. \end{aligned}$$

1 Thus, if $AEGF$ is cyclic, we would have $\angle EGF + \angle A = 180°$. Therefore it is necessary that $0 < \angle A \leq 90°$.

Continuation "A"

Let O be the circumcentre of ΔAFE. Without loss of generality, let the radius of this circle be 1.

We then let $A = 1, F = z = e^{i\theta}$ and $E = ze^{2i\alpha} = e^{i(\theta+2\alpha)}$.

Then $\angle A = \alpha,\ \ 0 < \alpha \leq 90°$, and $0 < \theta < 360° - 2\alpha$.

Thus,

$$B = 2z - 1$$

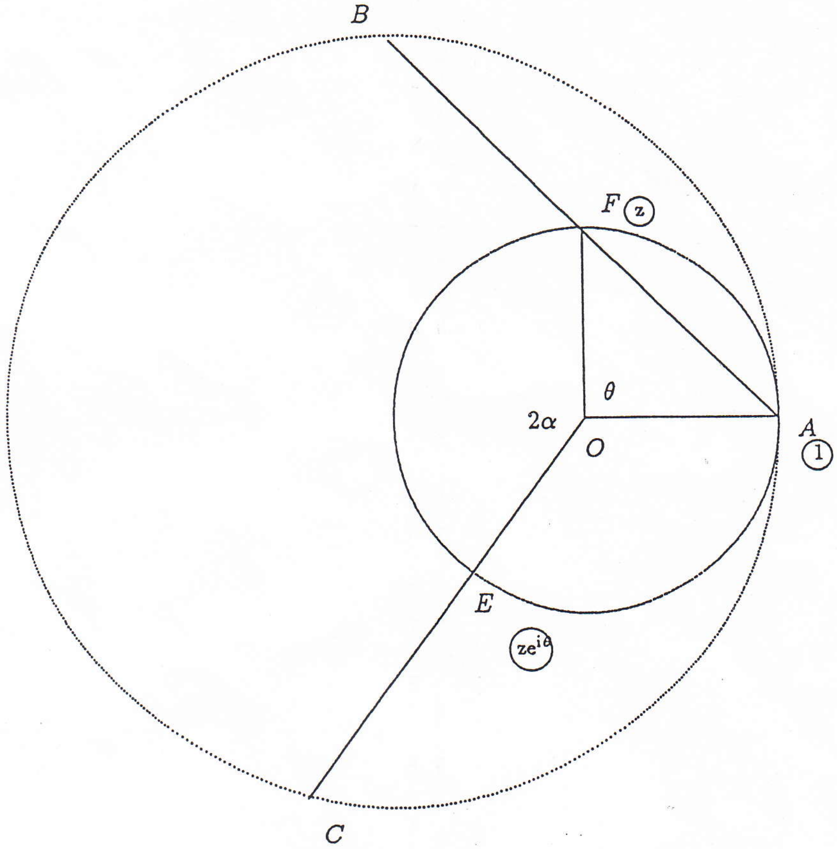
and

$$\begin{aligned} G &= \frac{1}{3}(2z-1) + \frac{2}{3}\left(z\,e^{2i\alpha}\right) \\ &= \frac{1}{3}\left(2e^{i\theta} + 2e^{i(\theta+2\alpha)} - 1\right). \end{aligned}$$

For quadrilateral $AFGE$ to be cyclic, it is now necessary that

3

$$|G| = 1.$$



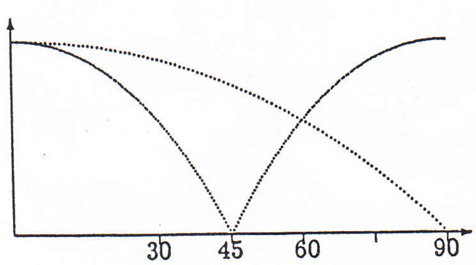
For $|G| = 1$, we must have

$$\begin{aligned}
9 &= (2\cos(\theta) + 2\cos(\theta + 2\alpha) - 1)^2 + (2\sin(\theta) + 2\sin(\theta + 2\alpha))^2 \\
&= 4\left(\cos^2(\theta) + \sin^2(\theta)\right) + 4\left(\cos^2(\theta + 2\alpha) + \sin^2(\theta + 2\alpha)\right) + 1 \\
&\quad +8\left(\cos(\theta)\cos(\theta + 2\alpha) + \sin(\theta)\sin(\theta + 2\alpha)\right) - 4\cos(\theta) - 4\cos(\theta + 2\alpha) \\
&= 9 + 8\cos(2\alpha) - 8\cos(\alpha)\cos(\theta + \alpha)
\end{aligned}$$

so that

$$\cos(\theta + \alpha) = \frac{\cos(2\alpha)}{\cos(\alpha)}.$$

5 Now, $\left|\frac{\cos(2\alpha)}{\cos(\alpha)}\right| \leq 1$ if and only if $\alpha \in (0, 60°]$ in the range of α under consideration, that is $\alpha \in (0, 90°]$. There is equality if and only if $\alpha = 60°$.



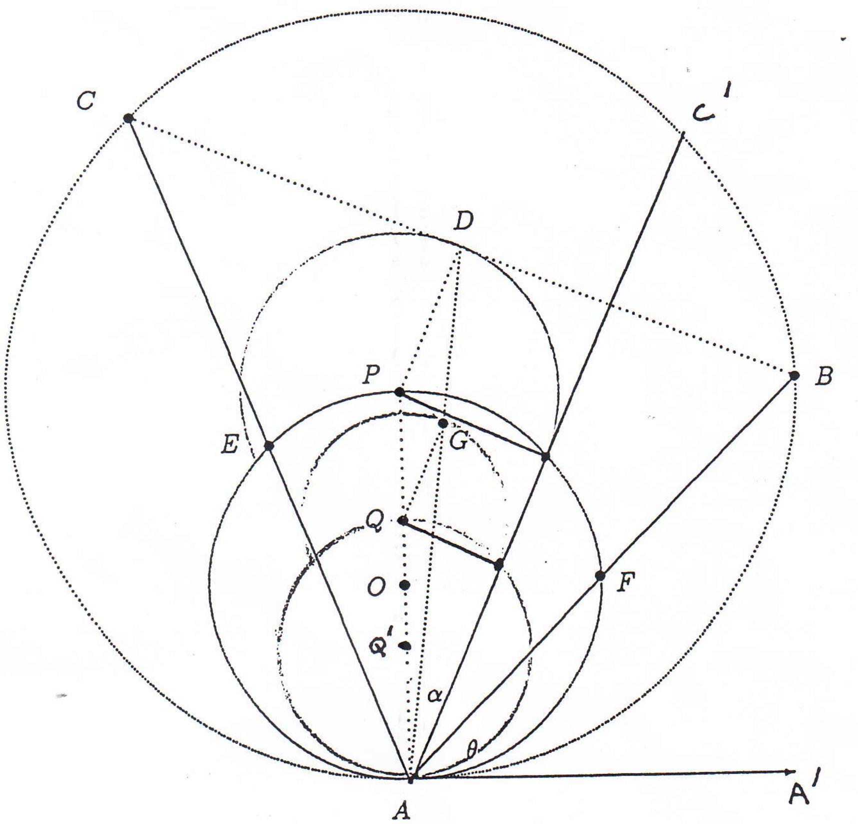
7 Note there is only one solution. The apparent other solution is the mirror image of the first. We are solving for $\alpha + \theta$. The other solution is $360° - \alpha - \theta$.

Continuation "B"

Let O be the circumcentre of triangle AEF. Let AP be a diameter of this circle. Construct the circle with centre P and radius AP. Then B and C lie on this circle.

It is clear that the problem is solved if we allow the angle $\angle BAC = \alpha$ to vary and restrict B and C to the constructed circle.

Let θ be the angle from the drawn axis. Then θ lies in the range $(0, 180° - \alpha)$. We must not forget the necessary restriction of α, that is $\alpha \in (0, 90°]$.



$A'\hat{A}C' = \alpha$
$= \angle C\hat{A}B$

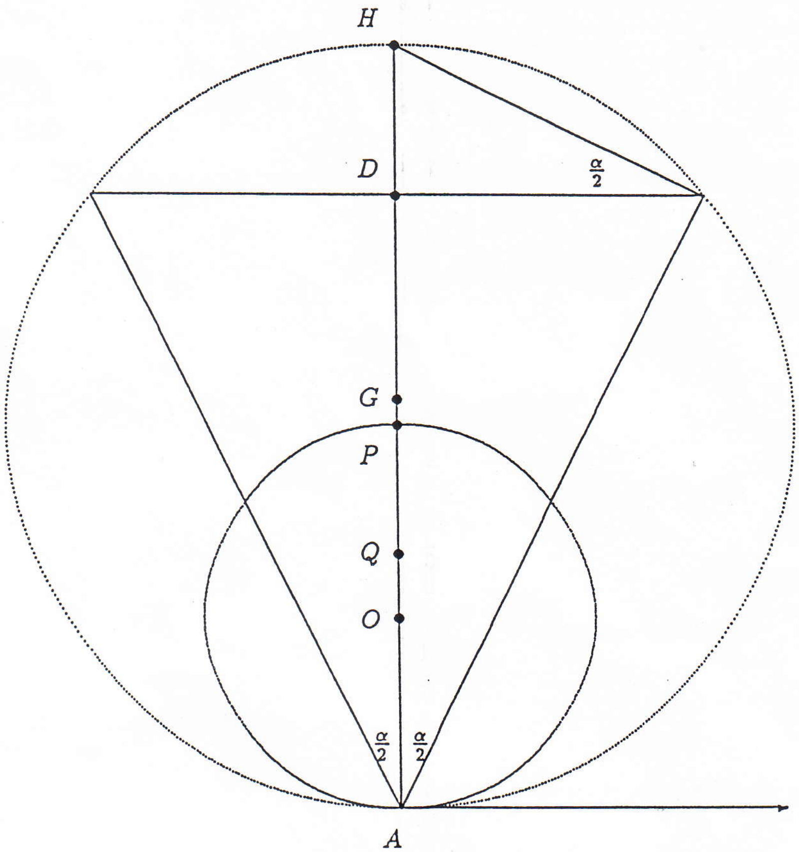
Now, D lies on an arc of a circle, centre P, radius PD exterior to the circle, centre O, radius AO.

By similarity, G, lies on an arc of a circle, centre Q, radius QG where $AQ = \frac{2}{3}AP$ and $QG = \frac{2}{3}PD$.

$\boxed{3}$ For the quadrilateral $AFGE$ to be cyclic, we must have that the radius QG is greater than or equal to QP.

The easiest way to calulate these radii is to consider the case in which the diameter AP bisects the angle $\angle BAC$.

Thus we re-draw the diagram as below. Let AH be a diameter of the larger circle.



Thus we have $AH = 4$ and by similar triangles,

$$\frac{AD}{AB} = \frac{AB}{AH} = \cos\left(\frac{\alpha}{2}\right),$$

so that

$$\begin{aligned} AD &= 4\cos^2\left(\frac{\alpha}{2}\right) \\ &= 2 + 2\cos(\alpha). \end{aligned}$$

Thus $PD = 2\cos(\alpha)$ and $QG = \frac{2}{3}\, 2\cos(\alpha) = \frac{4}{3}\cos(\alpha)$.

The necessary condition for a cyclic quadrilateral is then

$$\frac{4}{3}(1 + \cos(\alpha)) \geq 2,$$

$\boxed{5}$ which reduces to

$$\cos(\alpha) \geq \frac{1}{2}.$$

$\boxed{7}$ Thus it is clear that there is precisely one (up to similarity) solution for $0 < \alpha \leq 60°$ and no solutions otherwise.

Question 2

Let $a_1, a_2, \ldots, a_n$ be positive real numbers, and let S_k be the sum of products of $a_1, a_2, \ldots, a_n$ taken k at a time.
Show that

$$S_k S_{n-k} \geq \binom{n}{k}^2 a_1 a_2 \ldots a_n, \quad \text{for} \quad k = 1, 2, \ldots, n-1$$

.

FIRST SOLUTION

$$\binom{n}{k} a_1 a_2 \ldots a_n$$

2
$$= \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} a_{i_1} a_{i_2} \ldots a_{i_k} . a_1 a_2 \ldots a_n / a_{i_1} a_{i_2} \ldots a_{i_k}$$

(and using the Cauchy-Schwarz inequality)

4
$$\leq \left(\sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} a_{i_1} a_{i_2} \ldots a_{i_k} \right)^{\frac{1}{2}} . \left(\sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} a_1 a_2 \ldots a_n / a_{i_1} a_{i_2} \ldots a_{i_k} \right)^{\frac{1}{2}}$$

5
$$= S_k^{\frac{1}{2}} . S_{n-k}^{\frac{1}{2}} .$$

Therefore

7
$$\binom{n}{k}^2 a_1 a_2 \ldots a_n \leq S_k S_{n-k},$$

q.e.d.

SECOND SOLUTION (provided by the Canadian Problems Committee).

Write S_k as $\sum_{i=1}^{\binom{n}{k}} t_i$. Then

2
$$S_{n-k} = \left(\prod_{m=1}^{n} a_m \right) \left(\sum_{i=1}^{\binom{n}{k}} \frac{1}{t_i} \right),$$

$$\text{so that} \quad S_k S_{n-k} = \left(\prod_{m=1}^{n} a_m \right) \left(\sum_{i=1}^{\binom{n}{k}} t_i \right) \left(\sum_{j=1}^{\binom{n}{k}} \frac{1}{t_j} \right)$$

$$= \left(\prod_{m=1}^{n} a_m \right) \left[\sum_{i=1}^{\binom{n}{k}} 1 + \sum_{i=1}^{\binom{n}{k}} \sum_{j=1_{i \neq j}}^{\binom{n}{k}} \frac{t_i}{t_j} \right]$$

4
$$= \left(\prod_{m=1}^{n} a_m \right) \left[\binom{n}{k} + \sum_{i,j} \left(\frac{t_i}{t_j} + \frac{t_j}{t_i} \right) \right].$$

As there are

$$\frac{\binom{n}{k}^2 - \binom{n}{k}}{2}$$

terms in the sum

5

7

$$
\begin{aligned}
S_k S_{n-k} &\geq \left(\prod_{m=1}^{n} a_m\right)\left[\binom{n}{k} + 2\cdot\frac{\binom{n}{k}^2 - \binom{n}{k}}{2}\right] \\
&= \binom{n}{k}^2\left(\prod_{m=1}^{n} a_m\right)
\end{aligned}
$$

since $\dfrac{t_i}{t_j} + \dfrac{t_j}{t_i} \geq 2$ for $t_i, t_j > 0$.

Question 3

Consider all the triangles ABC which have a fixed base AB and whose altitude from C is a constant h. For which of these triangles is the product of its altitudes a maximum?

SOLUTION:

Let h_a and h_b be the altitudes from A and B, respectively. Then

1

$$\begin{aligned} AB.h.AC.h_b.BC.h_a &= 8.(\text{area of } \triangle ABC)^3 \\ &= (AB.h)^3, \end{aligned}$$

2 which is a constant. So the product $h.h_a.h_b$ attains its maximum when the product $AC.BC$ attains its minimum.

Since

$$\begin{aligned} (\sin C).AC.BC &= BC.h_a \\ &= 2.\text{area of } \triangle ABC, \end{aligned}$$

3 which is a constant, $AC.BC$ attains its minimum when $\sin C$ reaches its maximum.

There are two cases:

(a) $h \leq AB/2$. Then there exists a triangle ABC which has a right angle at C, and for precisely such a triangle $\sin C$ attains its maximum, namely 1.

7 (b) $h > AB/2$. In this case the angle at C is acute and assumes its maximum when the triangle is isosceles.

Note that a solution using calculus obviously exists.

Question 4

A set of 1990 persons is divided into non-intersecting subsets in such a way that

(a) no one in a subset knows all the others in the subset;

(b) among any three persons in a subset, there are always at least two who do not know each other; and

(c) for any two persons in a subset who do not know each other, there is exactly one person in the same subset knowing both of them.

(i) Prove that within each subset, every person has the same number of acquaintances.

(ii) Determine the maximum possible number of subsets.

Note: it is understood that if a person A knows person B, then person B will know person A; an acquaintance is someone who is known. Every person is assumed to know one's self.

SOLUTION:

(i) Let S be a subset of persons satisfying conditions (a), (b) and (c). Let $x \in S$ be one who knows the maximum number of persons in S.

Assume that x knows $x_1, x_2, \ldots, x_n$. By (b), x_i and x_j are strangers if $i \neq j$. For each x_i, let N_i be the set of persons in S who know x_i but not x. Note that, for $i \neq j$, N_i has $\boxed{2}$ no person in common with N_j, otherwise there would be more than one person knowing x_i and x_j, contradicting (c).

By (a) we may assume that N_1 is not empty. Let $y_1 \in N_1$. By (c), for each $k > 1$, there is exactly one person y_k in N_k who knows y_1. This means that y_1 knows n persons, $\boxed{3}$ namely $x_1, y_2, \ldots, y_n$.

Because n is the maximal number of persons in S a person in S can know, y_1 knows exactly n persons in S. By precisely the same reasoning we find that each person in N_i, $\boxed{4}$ $i = 1, 2, \ldots, n$, knows exactly n persons in S.

Letting y_1 take the role of x in our argument, we see that also each x_i knows exactly n persons. Note that, by (c), every person in S other than $x, x_1, \ldots, x_n$, must be in some N_j. Therefore every person in S knows exactly n persons in S and thus has the same $\boxed{5}$ number of acquaintances in S.

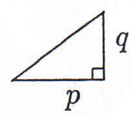
(ii) To maximize the number of subsets, we have to minimize the size of each group. The smallest possible subset is one in which every person knows exactly two persons, and hence there must be exactly five persons in the subset, forming a cycle where two persons stand side by side only if they know each other. Therefore the maximum possible $\boxed{7}$ number of subsets is $1990/5 = 398$.

Question 5

Show that for every integer $n \geq 6$, there exists a convex hexagon which can be dissected into exactly n congruent triangles.

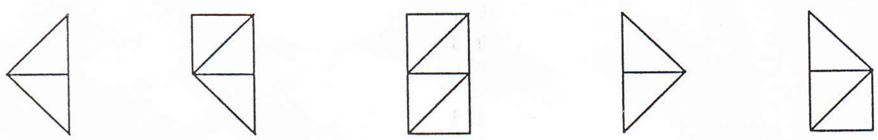
FIRST SOLUTION (provided by the Canadian Problems Committee).

The basic building blocks will be right angled triangles with sides p, q (which are positive integers) adjacent to the right angle.



In the first instance, we take $p = q = 1$ and construct five basic building blocks: L_1, L_2, M, R_1 and R_2.

3

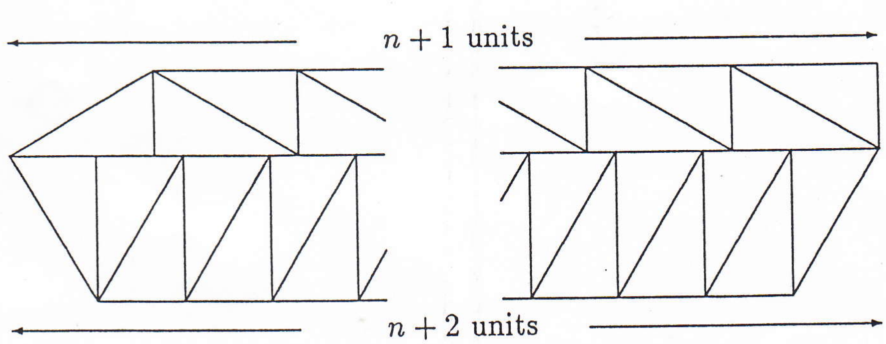


L_1 L_2 M R_1 R_2

We shall now build convex hexagons by taking, on the left, one of the blocks L_i, attaching n copies of the block M, and finally attaching one of the blocks R_j. We must therefore exclude the case when $(i,j) = (2,1)$ for this does not generate a hexagon. Further, for $(i,j) = (1,1)$ or $(i,j) = (1,2)$, we require that $n \geq 1$, whereas for $(i,j) = (2,2)$, we only need require that $n \geq 0$.

Thus, with the obvious interpretation:
$L_1 + nM + R_1$ gives a convex hexagon containing $2 + 4n + 2 = 4n + 4$ $(n \geq 1)$ congruent triangles;
$L_1 + nM + R_2$ gives a convex hexagon containing $2 + 4n + 3 = 4n + 5$ $(n \geq 1)$ congruent triangles; and
$L_2 + nM + R_2$ gives a convex hexagon containing $3 + 4n + 3 = 4n + 6$ $(n \geq 0)$ congruent triangles, or $4n + 2$ $(n \geq 1)$ congruent triangles.

We shall now modify the lengths of the sides of the right triangle to obtain the case of $4n + 3$ $(n \geq 1)$ congruent triangles.



So we have $2n+1$ triangles in the top part and $2n+2$ triangles in the bottom part. In order to match, we need

$$(n+1)p = (n+2)q,$$

so we take

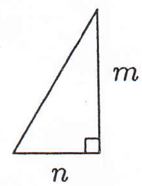
$$q = n+1 \qquad \text{and} \qquad p = n+2.$$

7

This completes the solution.

SECOND SOLUTION (provided by the Canadian Problems Committee):

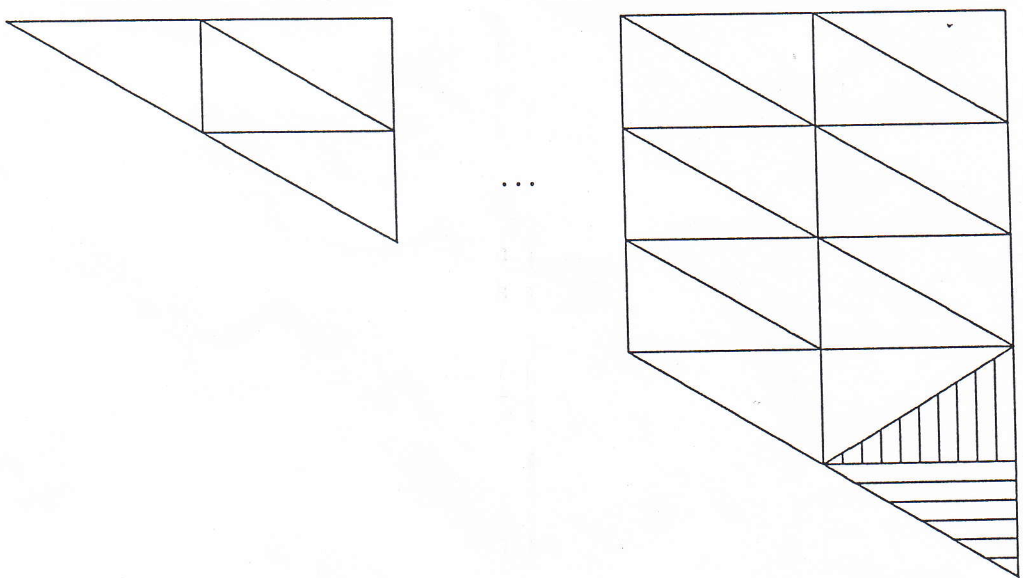
The basic building blocks will be right angled triangles with sides m, n (which are positive integers) adjacent to the right angle.



We construct an "UPPER CONFIGURATION", being a rectangle consisting of m building block units of pairs of triangles with the side of length n as base. This gives a base length of nm across the configuration.

We further construct a "LOWER CONFIGURATION", being a triangle with base up, consisting along the base of n building block units. Again, we have a base length of mn across the configuration.

Two triangles in the upper configuration are shaded horizontally. One triangle in the lower configuration is also shaded horizontally. Another triangle in the lower configuration is shaded vertically.



3

Now consider the figure obtained by joining the two configurations along the base line of common length nm. To create the classes of hexagons defined below, it is necessary that both $n \geq 3$ and $m \geq 3$.

We create a class of convex hexagons (class 1) by omitting the three triangles that are shaded horizontally. The other class of convex hexagons (class 2) is obtained by omitting all shaded triangles.

Now count the total number of triangles in the full configuration.

The upper configuration gives $2m$ triangles. The lower configuration gives

$$\sum_{k=1}^{n}(2k-1) \quad = \quad n^2 \quad \text{triangles.}$$

Thus the total number of triangles in a hexagon in class 1 is

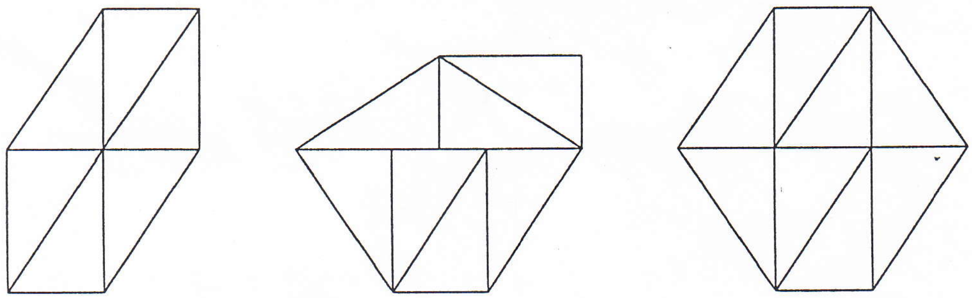
$$2m - 2 + n^2 - 1\,,$$

and the total number of triangle in a hexagon in class 2 is

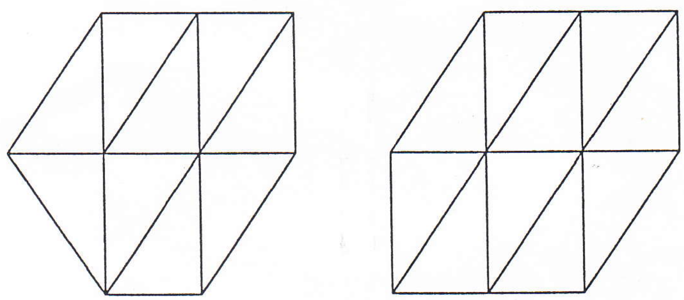
$$2m - 2 + n^2 - 2\,.$$

6 These, together with the restrictions on n and m, generate all positive integers greater than or equal to 11.

For the integers $6, 7, 8, 9$ and 10, we give specific examples:



6 7 8



9 10

7

This completes the solution.

ANSWER KEY

There are $\binom{n}{k}$ products of the a_i taken k at a time. Amongst these products any given a_i will appear $\binom{n-1}{k-1}$ times, since $\binom{n-1}{k-1}$ is the number of ways of choosing the other factors

2 of the product. So the AM/GM inequality gives

4

$$\frac{S_k}{\binom{n}{k}} \geq \left[\prod_{i=1}^{n} a_i^{\binom{n-1}{k-1}}\right]^{\frac{1}{\binom{n}{k}}}.$$

But $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, leading to

6

$$S_k \geq \binom{n}{k}\left(\prod_{i=1}^{n} a_i\right)^{\frac{k}{n}}.$$

Hence

7

$$S_k S_{n-k} \geq \binom{n}{k}\left(\prod_{i=1}^{n} a_i\right)^{\frac{k}{n}}\binom{n}{n-k}\left(\prod_{i=1}^{n} a_i\right)^{\frac{n-k}{n}} = \binom{n}{k}^2\left(\prod_{1}^{n} a_i\right).$$