11TH European Mathematical Cup

10th December 2022 - 18th December 2022

Junior Category

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Problems and Solutions

Problem 1. Find all positive integers n for which there exist three positive divisors a, b, c of n such that $a > b > c$ and

$$a^2 - b^2, \; b^2 - c^2, \; a^2 - c^2$$

are also divisors of n.

(Kims Georgs Pavlovs)

Solution. The answer is all positive integers n which are divisible by 60.

Suppose that n is divisible by 60. Let $a = 4$, $b = 2$, $c = 1$. Then $a^2 - b^2 = 12$, $b^2 - c^2 = 3$, $a^2 - c^2 = 15$. All six of those numbers are divisors of 60, so they're also divisors of n, so any n which is a multiple of 60 is indeed a solution.

2 points.

Now suppose that n is a number for which such a, b, c exist.

Some two numbers among a, b, c have the same parity. Without loss of generality let a and b be such numbers. Then $a - b$ and $a + b$ are both even, so 4 is a divisor of $a^2 - b^2$, so 4 also divides n. We conclude that n is divisible by 4.

2 points.

If any of a, b, c is divisible by 3, then n is also divisible by 3. Otherwise, two numbers among a, b, c give the same remainder upon division by 3. Without loss of generality let a and b be such numbers. Then 3 is a divisor of $a^2 - b^2$, so 3 also divides n. We conclude that in any case, n is divisible by 3.

2 points.

If any of a, b, c is divisible by 5, then n is also divisible by 5. Otherwise, none of a, b, c is divisible by 5. However, a square of a number which isn't divisible by 5 can only give remainder 1 or 4 upon division by 5.

Thus, some two numbers among a^2, b^2, c^2 have the same remainder upon division by 5, so their difference is divisible by 5. We conclude that in any case, n is divisible by 5.

3 points.

Since $3, 4$ and 5 all divide n, then their least common multiple, which is 60, also divides n.

1 point.

Notes on marking:

- If a contestant proves that for some $a > 1$, all numbers of the form $60 \cdot a \cdot k$ for $k \in \mathbb{N}$ are solutions, they should be awarded **1 point** out of possible **2 points** for the first part of the solution.
- If a contestant proves that any n which is a solution is even, they should be awarded **1 point** out of possible **2 points** given for proving divisibility by 4.

Problem 2. Find all pairs of positive real numbers (x, y) such that xy is an integer and

$$x + y = \lfloor x^2 - y^2 \rfloor.$$

(Ivan Novak)

First Solution. Let (x, y) be a pair satisfying the problem's condition.
Note that $x + y$ and xy are both positive integers, so $(x - y)^2 = (x + y)^2 - 4xy$ is also a positive integer.

4 points.

Let $D = (x - y)^2$, and let $a = x + y$. We have

$$a = \lfloor a\sqrt{D} \rfloor,$$

where a and D are both positive integers. Furthermore, note that $D = (x + y)^2 - 4xy$ gives remainder 0 or 1 upon division by 4. If $D > 1$, then $D \geqslant 4$, and $\lfloor a\sqrt{D} \rfloor \geqslant \lfloor 2a \rfloor = 2a > a$, which is a contradiction. Thus, $D = 1$.

3 points.

This means that $x - y = 1$. Since $x + y = a$, we must have

$$x = \frac{a+1}{2},$$
$$y = \frac{a-1}{2}$$

for some positive integer a. Since $y > 0$, we must have $a > 1$. Since xy is an integer, $\frac{a^2-1}{4}$ must be an integer. Thus, a is odd. Let $a = 2n + 1$ for some positive integer n. Then

$$(x, y) = (n + 1, n).$$

It's easy to check that all such pairs satisfy the problem's conditions.

3 points.

Second Solution. Note that $x > y$ since $x^2 - y^2 \geqslant x + y > 0$.
Let $a = x + y$ and $b = xy$. We then have $x^2 - ax + b = 0$ and $y^2 - ay + b = 0$. This means that x and y are the roots of the polynomial $t^2 - at + b$, so, since $x > y$, we have

$$x = \frac{a + \sqrt{D}}{2},$$
$$y = \frac{a - \sqrt{D}}{2},$$

where $D = a^2 - 4b$ is a positive integer.

2 points.

Direct calculation yields $x^2 - y^2 = a\sqrt{D}$.

2 points.

Thus, we again obtain the equality

$$a = \lfloor a\sqrt{D} \rfloor.$$

The rest of the solution is the same as in the First Solution.

6 points.

Third Solution. We let $a = x + y$ (note it must be a positive integer) and then the equality from the statement implies that $a \leqslant x^2 - y^2 < a + 1$ which upon division by a implies

$$1 \leqslant x - y < 1 + \frac{1}{a}.$$

1 point.

Adding $a = x + y$ to both inequalities implies

$$a + 1 \leqslant 2x < a + 1 + \frac{1}{a} \implies \frac{a+1}{2} \leqslant x < \frac{a+1}{2} + \frac{1}{2a}.$$

2 points.

Now, write $x = \frac{a+1}{2} + \varepsilon$ for some ε. We see that $y = \frac{a-1}{2} - \varepsilon$ and also that $0 \leqslant \varepsilon < \frac{1}{2a} \leqslant \frac{1}{2}$.
We can compute

$$xy = \frac{a^2-1}{4} - (\varepsilon^2 + \varepsilon) \in \mathbb{N}$$

and note that $0 \leqslant \varepsilon^2 + \varepsilon < \frac{3}{4}$.

2 points.

If a is even, $a^2 - 1$ gives remainder 3 upon division by 4 so the fractional part of $\frac{a^2-1}{4}$ is $\frac{3}{4}$ and due to $\varepsilon^2 + \varepsilon < \frac{3}{4}$ we have that xy is not an integer.

2 points.

If a is odd, $a^2 - 1$ is divisible by 4 so we must have $\varepsilon^2 + \varepsilon = 0$ and $\varepsilon = 0$ and we obtain the solutions

$$(x, y) = \left(\frac{a+1}{2}, \frac{a-1}{2}\right) = (n+1, n)$$

for any odd integer a, that is, any positive integer n.

3 points.

Fourth Solution. Again, denote $a = x + y$ and $D = (x - y)^2$ (both are integers as in the First Solution or Second Solution). Thus we again have

$$a = \lfloor a\sqrt{D} \rfloor.$$

4 points.

As in the Third Solution, we have that $a = \lfloor a\sqrt{D} \rfloor$ implies that

$$1 \leqslant x - y = \sqrt{D} < 1 + \frac{1}{a}$$

.

1 point.

Applying the inequality of arithmetic and geometric means gives

$$a = x + y \geqslant 2\sqrt{xy} \geqslant 2$$

but as $x \neq y$ equality can not hold. This gives that $a \geqslant 3$ so $\sqrt{D} \leqslant \frac{4}{3}$ and as D is an integer, $D = 1$.

2 points.

The rest of the solution is the same as in the First Solution.

3 points.

Notes on marking:

- In all four solutions, in the final part (last **3 points**), minor flaws should result in a **1 point** deduction.
- If a contestant has not made the steps prior to the final part of the solution, they can score at most **2 points** for the last part.
- Stating that all pairs of the form $(n+1, n)$ are a solution is worth at most **1 point** out of the last **3 points**.
- Points from different solutions are not additive.

Problem 3. Let ABC be an acute-angled triangle with $|BC| < |AC|$. Let I be the incenter and τ the incircle of ABC, which touches BC and AC at points D and E, respectively. The point M is on τ such that BM is parallel to DE and M and B are on the same side of the angle bisector of $\angle BCA$. Let F and H be the intersections of τ with BM and CM different from M, respectively. Let J be a point on the line AC such that JM is parallel to EH. Let K be the intersection of JF and τ different from F.

Prove that the lines ME and KH are parallel.

(Steve Vo Dinh)

First Solution. Let T be the intersection of CF and τ different from F. Consider the homotethy centered at C that sends H to M.
It also sends T to F because of symmetry around the bisector of angle $\angle ACB$.

1 point.

Furthermore, it sends E to J because JM is parallel to EH.

1 point.

Since CE is tangent to τ, CJ is tangent to the circumcircle of $\triangle MFJ$ using the aforementioned homotethy.

4 points.

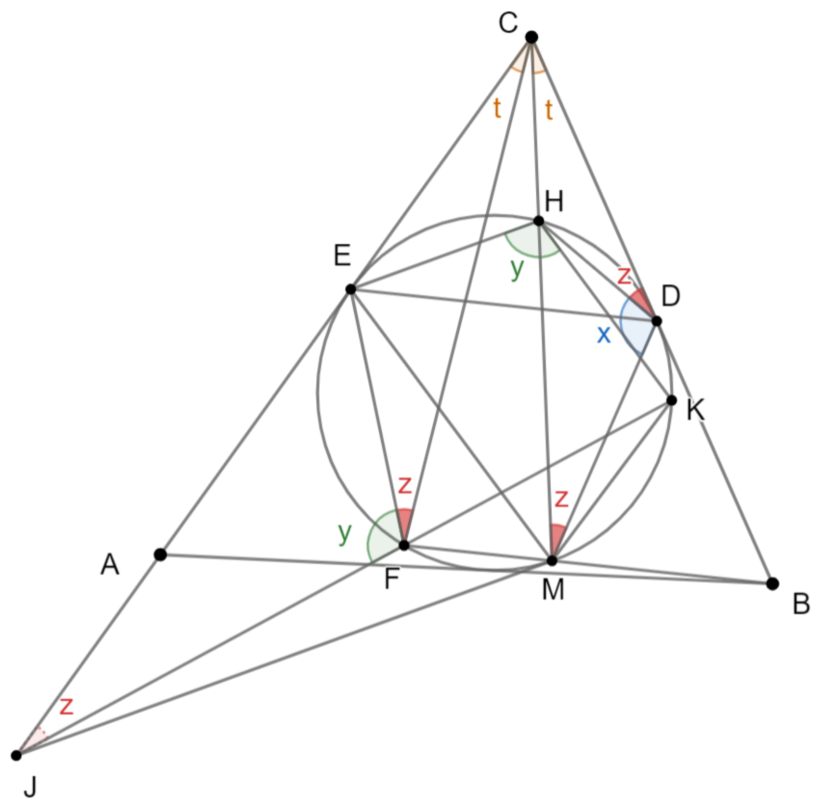
By the tangent chord lemma we have $\angle CJM = \pi - \angle MFJ$.

1 point.

Now notice that $\angle EMH = \angle CEH = \angle CJM = \pi - \angle MFJ = \angle KFM = \angle MHK$.
Thus, ME and KH are parallel, as desired.

3 points.

Second Solution. Let t denote the angle $\angle JCF$, and let z denote the angle $\angle EFC$.



We claim that $\triangle CJF \sim \triangle CDH$.
We first note that $\angle JCF = \angle DCM = t$, $CD = CE$ and $CM = CF$ using the symmetry around the bisector of angle $\angle ACB$. Namely, let ℓ be the mentioned bisector. Then since FM and ED are parallel, we conclude that $EDFM$ is an isosceles trapezoid, so D and E are symmetric around ℓ, and the same holds for F and M.

1 point.

Using the fact that $JM \parallel EH$, the triangles JMC and EHC are similar, and we get $\frac{CJ}{CM} = \frac{CE}{CH}$, or equivalently $\frac{CJ}{CF} = \frac{CD}{CH}$

1 point.

. We are now done by the side-angle-side theorem.

4 points.

Now, notice that from the tangent chord lemma applied to BC and HD, we have $\angle HDC = \angle HMD$. From this, it follows $\angle EFC = \angle CMD = \angle HMD$ because of symmetry around angle $\angle ACB$.

Since we also have $\angle HDC = \angle CJF$ from the aforementioned similarity, we obtain $\angle HDC = \angle CJF = \angle EFC = z$.

1 point.

From triangle JFC, we have $\angle KHE = \angle JFE = \pi - \angle CJF - \angle JCF - \angle EFC = \pi - 2z - t$.

From triangle CDM we have $\angle MDH = \pi - \angle DMH - \angle DCH - \angle HDC = \pi - 2z - t$. Thus, $\angle MDH = \angle EHD$, which implies that $ED = HM$ as the angles across those chords are the same. Thus, $EMKH$ is an isosceles trapezoid, so ME and KH are parallel, as desired.

3 points.

Notes on marking:

- In both solutions, contestants that have done the angle-chase (final part of the solution), but haven't done the previous steps (up to minor flaws), are awarded **1 point** out of **3 points** for the final part, since it can be helpful, but it isn't very useful without the previous steps.
- Points from different solutions are not additive.

Problem 4. Let $X = \{1, 2, 3, \ldots, 300\}$. A collection F of distinct (not necessarily non-empty) subsets of X is *lovely* if for any three (not necessarily distinct) sets A, B, C in F, at most three out of the following eight sets are non-empty:

$$\begin{array}{cccc} A \cap B \cap C, & \overline{A} \cap B \cap C, & A \cap \overline{B} \cap C, & A \cap B \cap \overline{C}, \\ \overline{A} \cap \overline{B} \cap C, & \overline{A} \cap B \cap \overline{C}, & A \cap \overline{B} \cap \overline{C}, & \overline{A} \cap \overline{B} \cap \overline{C}, \end{array}$$

where $\overline{S}$ denotes the set of all elements of X that are not in S.

What is the greatest possible number of sets in a lovely collection?

(Miroslav Marinov)

First Solution. We claim that $|F| \leqslant 8$.

If we apply the condition to the triple (A, B, B) for some distinct sets $A, B \in F$ we obtain that at most three of the sets $A \cap B, B \setminus A, A \setminus B, X \setminus (A \cup B)$ are nonempty. Therefore, for each pair of distinct sets $A, B \in F$ we either have that $A \subset B$ or $B \subset A$ or A, B disjoint or $A \cup B = X$.

1 point.

Now fix an F of maximal size and assume $|F| \geqslant 9$. Consider $G = F \setminus \{\emptyset, X\}$ which has $|G| \geqslant 7$. We wish to show there exist $A, B \in G$ with $A \cup B = X$ and nonempty intersection. Assume the opposite, that all A, B in G are either disjoint or contained in each other.

First, assume a pair of sets with $A \supset B$ exists in G.

If there exists $C \in G$ such that $A \supset B \supset C$ we obtain that

$$\begin{aligned} C &= A \cap B \cap C \\ B \setminus C &= A \cap B \cap \overline{C} \\ A \setminus B &= A \cap \overline{B} \cap \overline{C} \\ X \setminus A &= \overline{A} \cap \overline{B} \cap \overline{C} \end{aligned}$$

are all nonempty, a contradiction. Similarly, A has no superset in G.

1 point.

If there exist distinct $C, D \in G$ such that $A \supset B, C, D$, the previous consideration forces B, C, D to be disjoint and we obtain that

$$\begin{aligned} C &= A \cap \overline{B} \cap C \\ B &= A \cap B \cap \overline{C} \\ X \setminus A &= \overline{A} \cap \overline{B} \cap \overline{C} \\ A \setminus (B \cup C) &= A \cap \overline{B} \cap \overline{C} \end{aligned}$$

and as the first three sets are clearly nonempty, the last one must be empty and we have $B \cup C = A$. However, we now similarly get $D \cup B = D \cup C = A$ and D disjoint from B, C so $B = C$, a contradiction, so A has at most two subsets in G.

1 point.

Now, due to $|G| \geqslant 7$ we can choose distinct $C, D \in G$ with C, D both disjoint from A. We now have the sets

$$\begin{aligned} C &= \overline{A} \cap \overline{B} \cap C \\ B &= A \cap B \cap \overline{C} \\ A \setminus B &= A \cap \overline{B} \cap \overline{C} \\ X \setminus (A \cup C) &= \overline{A} \cap \overline{B} \cap \overline{C} \end{aligned}$$

and as the first three sets are clearly nonempty, the last one must be empty so $C = \overline{A}$. However, now we have a similar consideration for D so $C = D = \overline{A}$, a contradiction. Therefore, any pair of sets $A, B \in F$ are disjoint.

2 points.

Now, take some $A, B, C, D \in F$ and note that

$$A = A \cap \overline{B} \cap \overline{C}$$
$$B = \overline{A} \cap B \cap \overline{C}$$
$$C = \overline{A} \cap \overline{B} \cap C$$
$$X \setminus (A \cup B \cup C) = \overline{A} \cap \overline{B} \cap \overline{C}$$

so as the first three are clearly nonempty, the last one must be empty so we have $A \cup B \cup C = X$ and specially $C = \overline{A \cup B}$. However, the same consideration now applies to D so we obtain $D = C$, a contradiction.

1 point.

Finally, take a pair $A, B \in G$ such that $A \cup B = X$ and $A \cap B \neq \emptyset$. Consider some other $C \in G$. As all three sets $A \cap B$, $A \cap \overline{B}$, $\overline{A} \cap B$ are nonempty, each has to have a nonempty intersection either with C or with $\overline{C}$. As we can have at most three such nonempty intersections, we have that all of them are either a subset of C or a subset of $\overline{C}$. As those three sets partition X, this gives us at most $2^3 = 8$ possible choices for C and as $C \in G = F \setminus \{\emptyset, X\}$ and $C \neq A, C \neq B$, we cannot have that they're all contained in C or in $\overline{C}$ and we cannot have $C = A$ or $C = B$ so there are at most 4 possibilities for C and we obtain $|G| \leqslant 6$, a contradiction.

3 points.

Finally, we have that $|G| \leqslant 6$ so $|F| \leqslant 8$ as desired.

One example of this is F containing $\emptyset, \{1, 2, \ldots, 100\}, \{101, 102, \ldots 200\}, \{201, 300\}$ and their complements. It's easy to check that this is a lovely family.

1 point.

Second Solution. Let F be a lovely family with the maximum possible number of sets. We claim that F contains less than 9 sets. Suppose for the sake of contradiction that it contains more than 8 sets.

Note that the problem's condition is symmetric under taking complements. In other words, the sets A, B, C satisfy the problem's condition if and only if the sets $\overline{A}, B, C$ satisfy it. Because of this, we may assume that for any $A \in F$ we also have $\overline{A} \in F$. Since F consists of least 9 sets, it follows that F contains at least 5 sets which all contain the number 1.

1 point.

We'll prove this is impossible.

We rephrase the problem. Represent each set $A \in F$ with a 300×1 column consisting of zeroes and ones, so that the kth number in the column is 0 if $k \notin A$ and 1 if $k \in A$.

Let A, B, C, D, E be five sets from F which all contain 1. Consider a 300×5 table consisting of 5 columns corresponding to A, B, C, D, E respectively.

Now note that the problem's condition rephrases as follows: for any choice of three columns from the table, the 300×3 table consisting of those three columns contains at most 3 different rows out of possible 8.

1 point.

Lemma. Suppose that there is a row r in the mentioned 300×5 table which doesn't contain all ones and contains at least two ones. Then for any two columns which contain a one in the row r, one of them contains all ones.

Proof. Without loss of generality we may assume that the row r is of the form $110xy$, where $x, y \in \{0, 1\}$. Since the first two columns can't be the same, there must exist a row whose first two entries are distinct. Without loss of generality, this row is of the form $10uvw$ where $u, v, w \in \{0, 1\}$. Now look at the first three columns in the table. They contain rows 111, $10u$, 110. Those are three distinct rows, so the first three entries of any row in the table are 111, $10u$ or 110. This implies that every row must start with a one, which means that the first column contains all ones.

□

3 points.

From this lemma, it immediately follows that there is no row which contains more than 2 and less than 5 ones, and if there is a row which contains exactly 2 ones, then there is a column which contains all ones.

In any case, take a look at the four columns which do not contain all ones. Each of the rows in the corresponding 300×4 table either contains no ones, a single one, or four ones.

Since the columns need to be distinct, there must exist at least three out of four of the following rows:

1000

0100

0010

0001

2 points.

Without loss of generality we can assume that the table contains the first three among them. The table also contains the first row with all ones. However, looking at the first three columns then yields a contradiction, since there exist the following four distinct rows:

111

100

010

001

2 points.

The example which proves that a lovely family can contain 8 sets is the same as in the first solution.

1 point.

Notes on marking:

- In the First Solution, solving the case when there exist A, B whose union is X and intersection is nonempty is worth **3 points**. It is possible to score **1 point** or **2 points** on this part of the problem if a contestant makes partial progress or has a minor flaw in their proof of this case.

11TH EUROPEAN MATHEMATICAL CUP

10th December 2022 - 18th December 2022

Senior Category

Problems and Solutions

Problem 1. Let $n \geqslant 3$ be a positive integer. Alice and Bob are playing a game in which they take turns colouring the vertices of a regular n-gon. Alice plays the first move. Both players start the game with 0 points.

In each turn, player picks a vertex V which hasn't been coloured and colours it. Then, they add k points to their point tally, where k is the number of neighbouring vertices of V which have already been coloured. (Thus, k is either 0, 1 or 2.)

The game ends when all vertices have been coloured, and the player with more points wins. For every $n \geqslant 3$, determine which player has a winning strategy.

(Josip Pupić)

First Solution. We'll prove that Alice wins when n is odd and Bob wins when n is even.

Case 1: n is even.

Bob's strategy is the following. He picks a line ℓ passing through the midpoints of two opposite sides of the n-gon. This splits the n-gon into two halves.

Then, for any move Alice makes, Bob makes a move which mirrors Alice's move with respect to this line.

3 points.

If in some move Alice picks a vertex which isn't an endpoint of a side through which ℓ passes, then Bob wins the same amount of points as her in his next move, because each half of the n-gon has the same vertices coloured before their moves.

If Alice picks a vertex which is an endpoint of a side through which ℓ passes, then Bob wins one point more than Alice. That's because its neighbour on the same side of ℓ is coloured if and only if the mirroring vertex is coloured, so they get the same amount of points for the vertices on the same side of ℓ, but Bob gets one extra point since he picks a vertex adjacent to Alice's.

Since Bob has to pick such a vertex twice, Bob wins by two points difference in the end.

2 points.

Case 2: n is odd.

Alice's strategy is the following. She makes an arbitrary first move, and then considers a line ℓ passing through the vertex she picked and the midpoint of the opposite side of the n-gon. She then uses the same strategy as Bob, mirroring his moves with respect to ℓ.

3 points.

Similarly to the first solution, they get the same amount od points when Bob chooses a vertex which isn't an endpoint of the side through which ℓ passes.

When Bob picks a vertex which is an endpoint of the side through which ℓ passes, Alice wins one more point than Bob for the same reasons as in the first case.

Since Bob has to pick such a vertex, Alice wins by one point difference in the end.

2 points.

Second Solution. We'll prove that Bob wins for even n and Alice wins for odd n.

Case 1: n is even.

Consider vertices labeled from 1 to n in clockwise direction and consider vertices $2a-1$ and $2a$ paired for each $a \in \{1, 2, \ldots, \frac{n}{2}\}$.

Bob's strategy will be to colour a vertex belonging to the same pair as the vertex Alice coloured in her previous turn.

3 points.

After Bob's turn vertices in each pairing will either both be coloured or both uncoloured so Bob will always be able to colour the chosen vertex.

Since Alice always colours a vertex adjacent to an uncolored vertex, she gain's at most 1 point per turn. Similarly Bob always colours a vertex adjacent to a colored vertex so he gains at least 1 point per turn.

They each play for $\frac{n}{2}$ turns. However since Alice plays first, she gains 0 points for her first turn because there are no coloured vertices yet, and Bob plays last so he gains 2 points for his last turn because all vertices are coloured at the end. Bob wins by at least 2 points difference.

2 points.

Case 2: n is odd.

Label the vertices from 0 to $n-1$ in clockwise direction and consider vertices $2a-1$ and $2a$ paired for each $a \in \{1, 2, \ldots, \frac{n-1}{2}\}$.

Alice will firstly colour vertex labeled with 0, and all other turns (similarly to Bob's strategy) colour a vertex belonging to the same pair as the vertex Bob coloured in his previous turn.

3 points.

She will also always be able to colour the chosen vertex.

Alice now gains 0 points for her first turn, 2 points for her last turn and at least 1 point for all other turns, while Bob gains at most 1 point for all turns. Alice scores at least $\frac{n+1}{2}$ points and Bob scores at most $\frac{n-1}{2}$ points so Alice wins by at least 1 point.

2 points.

Third Solution. We'll prove by mathematical induction that Bob wins when n is even and Alice wins when n is odd.

For $n = 3$, Alice wins by 1 point no matter how they play.

For $n = 4$, Bob wins by 2 points by choosing a vertex adjacent to the one Alice chooses in the first move.

Suppose that for all $k < n$, where $n \geqslant 5$, Alice wins by 1 when k is odd and Bob wins by 2 when k is even.

We now prove the same holds for n. We again split into two cases.

Case 1: n is even.

Bob's first move is to pick a vertex adjacent to the one Alice picked. He wins one point by doing this. However, note that if we merge the chosen vertices, we have a same game on an $(n-1)$-gon in which Bob has made the first move and is leading by 1. By inductive hypothesis applied to the $(n-1)$-gon, Bob wins the game on the $(n-1)$-gon by 1 point, so he then wins the game by 2 points.

3 points.

Case 2: n is odd.

Alice's first move is arbitrary.

If Bob plays his first move to a vertex adjacent to the one Alice chose, he scores 1 point, but we can again merge the two vertices together and get a game on an $(n-1)$-gon in which Bob has made the first move, and by inductive hypothesis Alice wins this game on an $(n-1)$-gon by 2 points, so she wins the game on the n-gon by 1 point.

If Bob plays the first move in a vertex such that there is exactly one vertex between the vertices he and Alice chose, then Alice can pick the vertex in between them. She scores 2 points in that move. Merging the three used vertices gives us a game on an $n-2$-gon in which Alice has played the first move, and she wins that game by the inductive hypothesis, so she also wins the game on an n-gon.

1 point.

If Bob plays the first move to a vertex which is at distance at least two from the one Alice chose, he doesn't score any points. If we connect the two chosen vertices, we've split the n-gon into a $(k+1)$-gon and an $(n+1-k)$-gon for some $k \geqslant 3$, $k < n-2$. Merging the two chosen vertices in each of the two polygons gives us a k-gon and an $(n-k)$-gon, where $k \geqslant 3$ and $n-k \geqslant 3$. Note that exactly one among k and $n-k$ is even. Furthermore, also note that the game is now split into two games, a game on a k-gon and a game on an $n-k$-gon, in which the first moves have been played.

2 points.

Note that exactly one among k or $n-k$ is even. Without loss of generality let k be even. Then Alice can make her first move in a k-gon and every other move in the same polygon in which Bob played his move before her. Doing so optimally, using the induction hypotheses for k and $n-k$, Alice wins both the game on the k-gon and the game on the $(n-k)$-gon, since she is the first player on the $(n-k)$-gon and the second player on the k-gon.

4 points.

Notes on marking:

- In the first two solutions, each case depending on the parity of n is worth **5 points** in total. Explaining the strategy is worth **3 points**, while proving that the strategy is winning is worth **2 points**. If a contestant provides a different winning strategy, the same marking scheme applies.
- If a contestant has a minor flaw in their proof that the strategy is winning, they should be deducted **1 point** out of **2 points** for that part of the solution.
- Proving that the total number of points in the game is n is worth **1 point** out of possible **5 points** for the case when n is odd. This is not additive with the rest of the points given for the odd case.
- Providing a non-losing strategy for the case when n is even is worth **1 point**. Proving that the strategy is non-losing should be awarded additional **1 point**. These points are not additive with the rest of the points given for the even case.

Problem 2. We say a positive integer n is *lovely* if there exist a positive integer k and (not necessarily distinct) positive integers $d_1, d_2, \ldots, d_k$ such that $n = d_1 d_2 \ldots d_k$ and

$$d_i^2 \mid n + d_i$$

for all $i \in \{1, \ldots, k\}$.

(a) Are there infinitely many lovely numbers?

(b) Does there exist a lovely number greater than 1 which is a square of an integer?

(Ivan Novak)

Solution. (a) The answer is YES. We prove this using induction. Note that 1 is lovely since $1 = 1^2$ and $1^2 \mid 1 + 1$. Suppose that some number n is lovely and let $d_1 d_2 \ldots d_k$ be the divisors such that $d_i^2 \mid n + d_i$.
We claim that $n(n+1)$ is lovely as well, with the decomposition $n(n+1) = d_1 d_2 \ldots d_k (n+1)$. First note that

$$(n+1)^2 \mid (n(n+1))^2 + n + 1 = (n+1)^2(n^2+1).$$

Furthermore, for any $i \leqslant k$, $d_i^2 \mid n^2$ and $d_i^2 \mid n + d_i$, so also

$$d_i^2 \mid n^2 + n + d_i.$$

Thus, $n(n+1)$ is indeed lovely, and part (a) is solved.

3 points.

(b) The answer is NO. Suppose that some square number $n > 1$ is lovely, with the decomposition $d_1 d_2 \ldots d_k$. Without loss of generality we may assume that all d_i are greater than 1, since adding or removing number 1 from the decomposition doesn't affect the problem's conditions.
Note that the d_i are pairwise coprime, since if some prime number p divides d_i and d_j, then $\nu_p(d_j) < \nu_p(n)$ so $\nu_p(n+d_j) = \nu_p(d_j) < 2\nu_p(d_j)$, which is impossible since $d_j^2 \mid n + d_i$. Consequently, since n is a square, all the d_i are squares.

1 point.

If me multiply the k conditions $d_i^2 \mid n + d_i$ for $i = 1, \ldots, k$, we get

$$d_1^2 d_2^2 \ldots d_k^2 \mid (n + d_1)(n + d_2) \ldots (n + d_k),$$

or equivalently

$$n^2 \mid (n + d_1)(n + d_2) \ldots (n + d_k).$$

1 point.

If we expand the k brackets on the right hand side to get 2^k summands, the only summands which are not divisible by n^2 are $d_1 d_2 \ldots d_k = n$ and those of the form $n \cdot d_1 \cdot \ldots d_{i-1} \cdot d_{i+1} \cdot \ldots \cdot d_k = \frac{n^2}{d_i}$ for some $i \in \{1, \ldots, k\}$. Thus, the condition can be rewritten as

$$n^2 \mid n + \frac{n^2}{d_1} + \ldots + \frac{n^2}{d_k}.$$

2 points.

This means that the number

$$\frac{1}{n} + \frac{1}{d_1} + \ldots + \frac{1}{d_k}$$

is a positive integer. However, note that for every positive integer N we have

$$\begin{aligned} \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{N^2} &< \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{(N-1)N} \\ &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \ldots + \frac{1}{N-1} - \frac{1}{N} \\ &= 1 - \frac{1}{N} < 1. \end{aligned}$$

This means that the sum $\frac{1}{d_1} + \ldots + \frac{1}{d_k} + \frac{1}{n}$ is less than 1 as well since it is a sum of reciprocals of distinct squares greater than 1. This means it cannot be a positive integer and we've reached a contradiction.

3 points.

Notes on marking:

- In part (a), it's possible to score partial points as follows. If a contestant tries constructing a lovely number $d_1 \ldots d_k d_{k+1}$ by starting from a lovely number $d_1 d_2 \ldots d_k$, and writes down that d_{k+1} should be congruent to 1 modulo d_i for $i \leqslant k$, they should receive **1 point** for part (a). If they write down the condition $d_{k+1} \mid d_1 d_2 \ldots d_k + 1$ along with the condition $d_{k+1} \equiv 1 \pmod{d_i}$, they should receive **2 points** for part (a). Only writing down the condition $d_{k+1} \mid d_1 d_2 \ldots d_k + 1$ is worth **0 points** on its own.

- In the induction step in part (a), stating the claim "When n is lovely, then $n(n+1)$ is lovely" without any proof should be awarded **1 point** out of possible **3 points** for part (a). Stating the claim and also providing the decomposition of $n(n+1)$ into d_i, but not checking that they satisfy the conditions should be awarded **2 points** out of possible **3 points**. A check doesn't necessarily need to be explicit, but it has to be written down in some form.
- In the last step of the solution of part (b), stating the inequality $\frac{1}{d_1} + \ldots + \frac{1}{d_k} + \frac{1}{n} < 1$ should be awarded **1 point** out of possible **3 points** for that part. To prove the inequality, one may also (without proof, of course) use the well-known identity $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Problem 3. Let $\mathbb{R}$ denote the set of all real numbers. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x^3)+f(y)^3+f(z)^3=3xyz$$

for all $x,y,z\in\mathbb{R}$ such that $x+y+z=0$.

(Kyprianos-Iason Prodromidis)

First Solution. First of all, for $x=y=z=0$, we get that

$$f(0)+2f(0)^3=0\Leftrightarrow f(0)(2f(0)^2+1)=0\Leftrightarrow f(0)=0.$$

Now, let $x=0$ and $z=-y$. This tells us that

$$f(y)^3+f(-y)^3=0\Leftrightarrow f(-y)=-f(y),\forall y\in\mathbb{R},$$

so f is odd.

1 point.

Moreover, for $z=0$ and $y=-x$, we have

$$f(x^3)+f(-x)^3=0\Leftrightarrow f(x^3)=f(x)^3,\forall x\in\mathbb{R},$$

because f is odd.

1 point.

This means that the original equation becomes

$$f(x)^3+f(y)^3+f(z)^3=3xyz,$$

for all x,y,z such that $x+y+z=0$, and by substituting $z=-x-y$ we can transform this into the equation (using also the fact that f is odd),

$$f(x)^3+f(y)^3+f(-x-y)^3=-3xy(x+y)\Leftrightarrow f(x+y)^3=f(x)^3+f(y)^3+3xy(x+y),\forall x,y\in\mathbb{R}. \tag{1}$$

Lemma: For all $n\in\mathbb{N}$ and $x\in\mathbb{R}$, the following relation holds:

$$f(nx)^3=nf(x)^3+(n^3-n)x^3.$$

Proof. We proceed by induction on n. For $n=1$ the statement is obvious for all x. Next, if this relation holds for some $n\in\mathbb{N}$ and all x, then by putting $y\to nx$ in (1), we conclude that

$$f((n+1)x)^3=f(x)^3+f(nx)^3+3n(n+1)x^3=(n+1)f(x)^3+(n^3-n+3n(n+1))x^3$$
$$(n+1)f(x)^3+((n+1)^3-(n+1))x^3,$$

and the Lemma follows.

□

1 point.

Next, we fix $x\in\mathbb{R}$ and we compute $f(n^3x^3)^3=f(nx)^9$ in two different ways for all $n\in\mathbb{N}$. Of course,

$$f(n^3x^3)^3=n^3f(x^3)^3+(n^9-n^3)x^9=n^9x^9+n^3(f(x)^9-x^9).$$

On the other hand, we have

$$\begin{aligned}f(nx)^9&=(nf(x)^3+(n^3-n)x^3)^3\\&=n^3f(x)^9+3n^2(n^3-n)f(x)^6x^3+3n(n^3-n)^2f(x)^3x^6+(n^3-n)^3x^9\\&=n^9x^9+3n^7(f(x)^3x^6-x^9)+3n^5(f(x)^6x^3-2f(x)^3x^6+x^9)\\&+n^3(f(x)^9-3f(x)^6x^3+3f(x)^3x^6-x^9).\end{aligned}$$

3 points.

This means that the polynomials

$$p_x(y)=x^9y^9+(f(x)^3-x^3)y^3 \quad \text{and}$$
$$q_x(y)=x^9y^9+3x^6(f(x)^3-x^3)y^7+3x^3(f(x)^3-x^3)^2y^5+(f(x)^3-x^3)^3y^3$$

are equal for infinitely many values of y, thus they are equal as polynomials. Comparing the coefficients of y^7, we get that $f(x)^3=x^3\Leftrightarrow f(x)=x,\forall x\in\mathbb{R}$. This function obviously satisfies the condition of the problem.

4 points.

Second Solution. The facts that $f(0) = 0$, f is odd and $f(x^3) = f(x)^3$ are obtained in the same way as in the previous solution.

2 points.

The same way as in the previous solution, we obtain the following equality for all $n \in \mathbb{N}$, $x \in \mathbb{R}$:

$$f(nx)^3 = nf(x)^3 + (n^3 - n)x^3.$$

1 point.

In particular, for $n = 8$ and $x^3 \neq 0$, we have

$$f(8x^3)^3 = 8f(x^3)^3 + (8^3 - 8)x^9.$$

On the other hand, we have

$$f(8x^3)^3 = f(2x)^9 = (2f(x)^3 + (2^3 - 2)x^3)^3.$$

2 points.

Simplifying the equality $(2f(x)^3 + (2^3 - 2)x^3)^3 = 8f(x^3)^3 + (8^3 - 8)x^9$, we obtain:

$$8f(x)^9 + 72f(x)^6x^3 + 216f(x)^3x^6 + 216x^9 = 8f(x)^9 + 504x^9,$$

which, dividing by $72x^3$, can be simplified to

$$f(x)^6 + 3f(x)^3x^3 = 4x^6.$$

This can be factored as

$$(f(x)^3 - x^3)(f(x)^3 + 4x^3) = 0.$$

Thus, for each x, we have either $f(x) = x$ or $f(x) = -\sqrt[3]{4}x$.

2 points.

Suppose that there exists some $y_0 \neq 0$ such that $f(y_0) = -\sqrt[3]{4}y_0$. Then, using the Lemma, we have

$$f(2y_0)^3 = 2f(y_0)^3 + 6y_0^3 = -8y_0^3 + 6y_0^3 = -2y_0^3,$$

which is impossible since $f(2y_0)^3 \in \{2y_0^3, -8y_0^3\}$. Thus, $f(x) = x$ for all $x \in \mathbb{R}$. This function obviously satisfies the problem's condition.

3 points.

Third Solution. The facts that $f(0) = 0$, f is odd and $f(x^3) = f(x)^3$ are obtained in the same way as in the previous solutions.

2 points.

By setting $z = -(x + y)$, the equation can be rephrased as

$$f(x^3) + f(y^3) = f(x + y)^3 - 3xy(x + y).$$

Let $g(x) := f(x)^3 - x^3$. Then

$$g(x + y) = g(x) + g(y).$$

Also, from $f(x^3) = f(x)^3$, we obtain

$$g(x^3) = f(x)^9 - x^9 = g(x)(x^6 + f(x)^6 + x^3f(x)^3),$$

which can be rewritten in terms of g as

$$g(x^3) = g(x)(g(x)^2 + 3x^3g(x) + 3x^6).$$

1 point.

Note that $f(1)^3 = f(1)$, so $f(1) \in \{0, 1, -1\}$, which means $g(1) \in \{0, -1, -2\}$.
We now prove $g(1) = 0$. Note that $g(2) = 2g(1) \in \{0, -2, -4\}$, and

$$4g(2) = g(8) = g(2)(g(2)^2 + 24g(2) + 192).$$

If $g(2) = 0$, then $g(1) = 0$. Otherwise, $g(2)^2 + 24g(2) + 192 = 4$. However, checking for values $g(2) = -2$ and $g(2) = -4$ gives no solution, a contradiction. Thus, $g(2) = 0$ and $g(1) = 0$.
We'll use the following notation. For a function $h : \mathbb{R} \to \mathbb{R}$, $\Delta h(x)$ will denote $h(x + 1) - h(x)$.

Now we express $g((x+1)^3)$ in two different ways. The first way is the following:

$$\begin{aligned} g((x+1)^3) &= g(x+1)(g(x+1)^2 + 3(x+1)^3 g(x+1) + 3(x+1)^2) \\ &= g(x)(g(x)^2 + 3(x+1)^3 g(x) + 3(x+1)^6), \end{aligned}$$

where we used $g(x+1) = g(x) + g(1) = g(x)$.
The second way is the following:

$$\begin{aligned} g((x+1)^3) &= g(x^3) + 3g(x^2) + 3g(x) \\ &= g(x)(g(x)^2 + 3x^3 g(x) + 3x^6) + 3g(x^2) + 3g(x). \end{aligned}$$

Thus, we have the equality

$$g(x)(g(x)^2 + 3(x+1)^3 g(x) + 3(x+1)^2) = g(x)(g(x)^2 + 3x^3 g(x) + 3x^6) + 3g(x^2) + 3g(x).$$

This gives us the following expression for $g(x^2)$ via $g(x)$:

$$g(x^2) = g(x)(g(x)\Delta x^3 + \Delta x^6 - 1).$$

.

2 points.

Now we write $g((x+1)^2)$ in two different ways.
The first way is

$$g((x+1)^2) = g(x)(g(x)\Delta(x+1)^3 + \Delta(x+1)^6 - 1).$$

The second way is

$$\begin{aligned} g((x+1)^2) &= g(x^2) + 2g(x) \\ &= g(x)(g(x)\Delta x^3 + \Delta x^6 + 1). \end{aligned}$$

1 point.

We claim that $g(x) = 0$ for all x. If this isn't the case for some x, then

$$g(x)\Delta x^3 + \Delta x^6 + 1 = g(x)\Delta(x+1)^3 + \Delta(x+1)^6 - 1.$$

However, then the same holds for $x+n$ for every positive integer n since $g(x+n) = g(x) \neq 0$. We conclude that the polynomial

$$p(y) := g(x)\Delta y^3 + \Delta y^6 + 1 - (g(x)\Delta(y+1)^3 + \Delta(y+1)^6 - 1)$$

has infinitely many zeroes. But its degree is 4, which gives us a contradiction. Thus, $g(x) = 0$ for all x, so $f(x)^3 = x^3$, which means $f(x) = x$. It's easy to check that this function satisfies the problem's condition.

4 points.

Notes on marking:

- Points from different solutions are not additive.
- If a contestant doesn't comment that the function $f(x) = x$ is indeed a solution, they can score at most **9 points** on the problem.
- Obtaining $f(1) = 1$ (or equivalently $g(1) = 0$) is worth **1 point**. However, this point is only additive with the first two points, and not additive with the remaining eight points.

Problem 4. Five points A, B, C, D and E lie on a circle τ clockwise in that order such that AB is parallel to CE and $\measuredangle ABC > 90°$. Let k be a circle tangent to AD, CE and τ such that the circles k and τ touch on the arc ED which doesn't contain A, B and C. Let $F \neq A$ be the intersection of τ and the line tangent to k passing through A different from AD.

Prove that there exists a circle tangent to BD, BF, CE and τ.

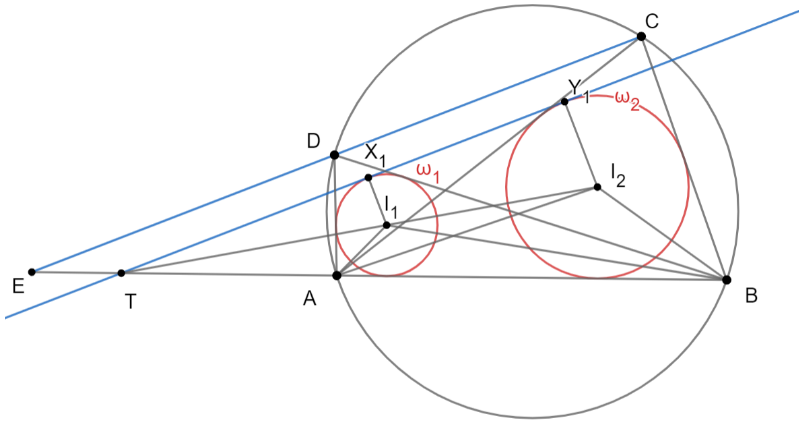
(Steve Vo Dinh)

First Solution. Let us present the main lemma first and we'll see later how the problem statement follows from this lemma.

Lemma Let $ABCD$ be a cyclic quadrilateral and γ_1 and γ_2 be C-mixtilinear and D-mixtilinear incircles of triangles ABD and ABC respectively. The external common tangent t of circles γ_1 and γ_2 which is closer to CD is parallel to CD.

We will divide the proof into a few main claims.

Claim 1 Let $ABCD$ be a cyclic quadrilateral and ω_1 and ω_2 be incircles of triangles ABD and ABC respectively. The external common tangent ℓ of circles ω_1 and ω_2 different from AB is parallel to CD.



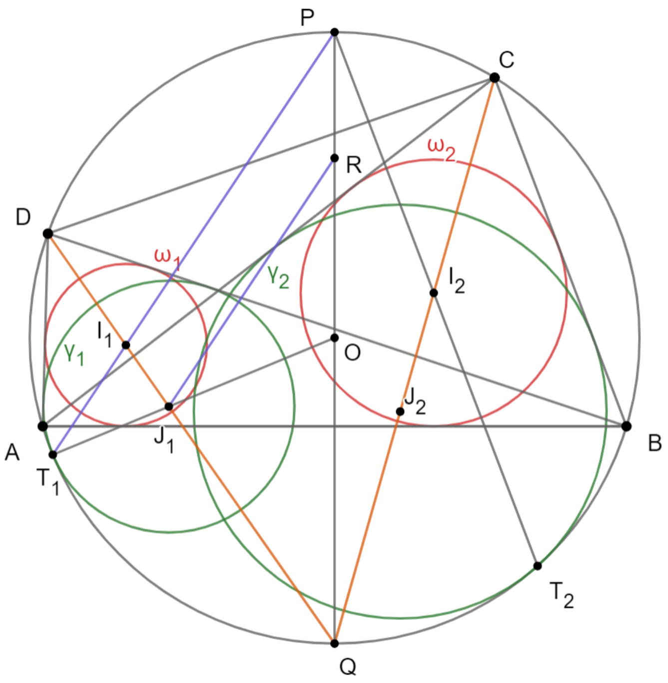
Proof: Let I_1, I_2 be centers of ω_1, ω_2 and let ℓ touch ω_1, ω_2 at X_1, Y_1 respectively. Also let CD intersect AB at E.
We know that $\angle AI_1B = 90° + \frac{\angle ADB}{2} = 90° + \frac{\angle ACB}{2} = \angle AI_2B$ so AI_1I_2B is cyclic quadrilateral. We know that lines AB, X_1Y_1, I_1I_2 concurr at the point T which is the center of homothethy between ω_1 and ω_2. It can be easily seen that line $T - I_1 - I_2$ is the bisector of $\angle X_1TA$ so we have that:

$$\angle X_1TA = 2 \cdot \angle I_1TA = 2 \cdot (\angle I_1AB - \angle TI_1A) = 2 \cdot (\angle I_1AB - \angle ABI_2) = 2 \cdot (\frac{\angle DAB}{2} - \frac{\angle ABC}{2}) = \angle DAB - \angle ABC$$

On the other hand $\angle DAB - \angle ABC = \angle DAB - \angle ADE = \angle AED$. In the end, we have that $\angle X_1TA = \angle AED$ which implies that $\ell \parallel CD$.

2 points.

Claim 2: Consider homotethy $\mathcal{H}_1$ with center at D which sends ω_1 to γ_1 and homotethy $\mathcal{H}_2$ with center at C which sends ω_2 to γ_2. $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same ratio.



Proof: Let J_1, J_2 be centers of γ_1, γ_2 and let γ_1, γ_2 touch $\odot(ABCD)$ at T_1, T_2 respectively. We know that J_1 lies on the bisector of $\angle ADB$ and that J_2 lies on the bisector of $\angle ACB$ so lines $D - I_1 - J_1$ and $C - I_1 - J_1$ intersect at Q, midpoint of an arc AB of $\odot(ABCD)$ which doesn't contain C and D. We know that Q is the center of $\odot(AI_1I_2B)$.
On the other hand it's the well-known property of the mixtilinear incircle that T_1I_1 passes through P, midpoint an arc AB of $\odot(ABCD)$ which contains C and D. Same holds for T_2I_2.
Let O be the center of $\odot(ABCD)$. We know that O, J_1, T_1 are collinear and that O is the midpoint of PQ. Let line passing through J_1 parallel to PT_1 intersect PQ at R. Then T_1J_1RP is an isosceleces trapezoid so $T_1J_1 = PR$. On the other hand from $RJ_1 \parallel PI_1$ we have that $\frac{I_1J_1}{J_1T_1} = \frac{I_1J_1}{PR} = \frac{QI_1}{QP}$.
Analogously we can prove that $\frac{I_2J_2}{J_2T_2} = \frac{QI_2}{QP}$. Because of $QI_1 = QI_2$ we can conclude that $\frac{I_1J_1}{J_1T_1} = \frac{I_2J_2}{J_2T_2} = k$.

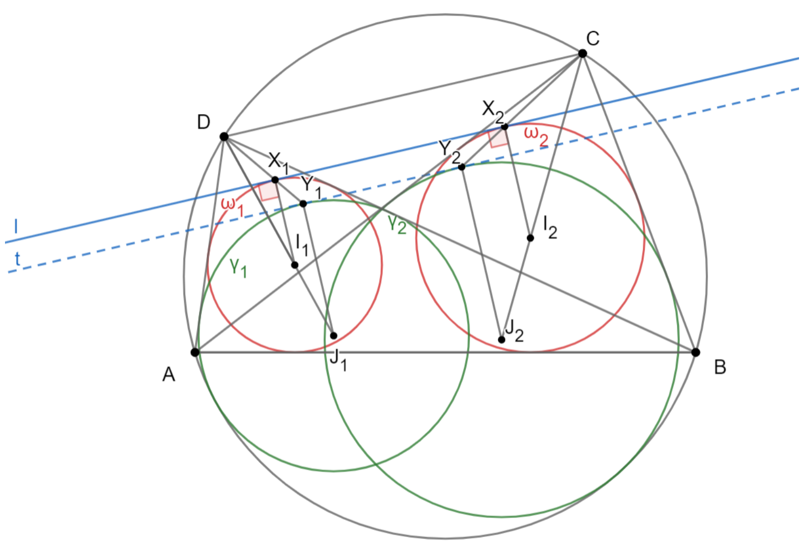
Let r_1, r_2, ρ_1, ρ_2 be radius of circles $\omega_1, \omega_2, \gamma_1, \gamma_2$ respectively. We know that the ratio of $\mathcal{H}_1$ is $\frac{r_1}{\rho_1}$ and the ratio of $\mathcal{H}_2$ is $\frac{r_2}{\rho_2}$. Also we have $DI_1 = \frac{r_1}{\sin\frac{\angle ADB}{2}}, DJ_1 = \frac{\rho_1}{\sin\frac{\angle ADB}{2}}$. We know that:

$$k = \frac{I_1J_1}{J_1T_1} = \frac{DJ_1 - DI_1}{\rho_1} = \frac{\frac{\rho_1 - r_1}{\sin\frac{\angle ADB}{2}}}{\rho_1}$$

which simplifies to $\frac{r_1}{\rho_1} = 1 - k \cdot \sin\frac{\angle ADB}{2}$.
Analogously we can get that $\frac{r_2}{\rho_2} = 1 - k \cdot \sin\frac{\angle ACB}{2}$ so $\frac{r_1}{\rho_1} = \frac{r_2}{\rho_2}$ because of $\angle ADB = \angle ACB$.

3 points.

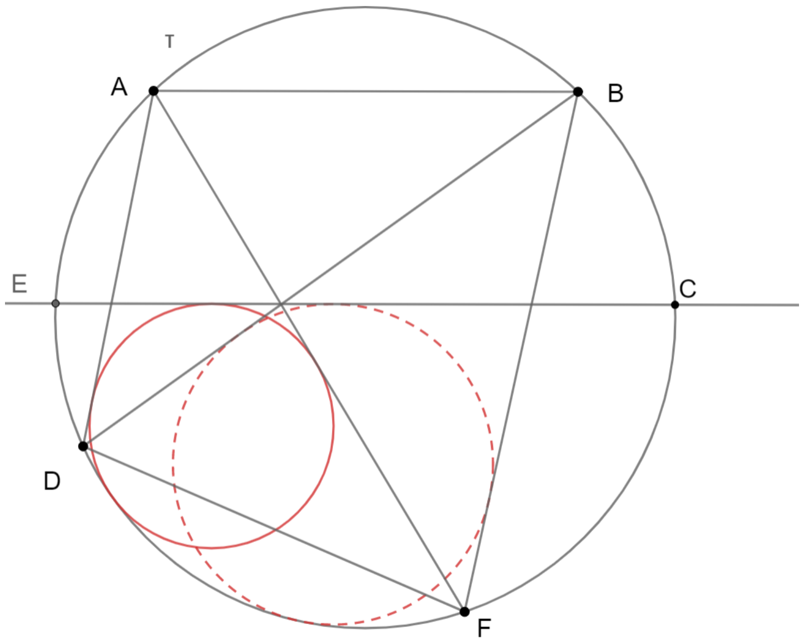
Claim 3: $t \parallel \ell$



Proof: Let ℓ touch ω_1, ω_2 at X_1, X_2 respectively. Let Y_1, Y_2 be points at DX_1, CX_2 respectively such that $I_1X_1 \parallel J_1Y_1$ and $I_2X_2 \parallel J_2Y_2$. From $\frac{DI_1}{DJ_1} = \frac{r_1}{\rho_1}$ and $I_1X_1 \parallel J_1Y_1$ we have that $J_1Y_1 = \rho_1$ so $Y_1 \in \gamma_1$. Analogously $Y_2 \in \gamma_2$. We know that $\frac{DX_1}{DY_1} = \frac{r_1}{\rho_1} = \frac{r_2}{\rho_2} = \frac{DX_2}{DY_2}$. This means that because of $CD \parallel X_1Y_1$ we have that $CD \parallel X_1X_2 \parallel Y_1Y_2$. Now from $I_1X_1 \parallel J_1Y_1$ we have that $I_1X_1 \perp Y_1Y_2$ so Y_1Y_2 is tangent to γ_1. Analogously Y_1Y_2 is tangent to γ_2 so we conclude that line Y_1Y_2 coincides with t.

4 points.

Now let's conclude why the original problem follows from our main lemma.



We know that CE is the tangent to the mixtilinear incircle of the triangle ADF which is parallel to AB. From our main lemma, we can conclude that CE has to be the tangent to the mixtilinear incircle of the triangle BDF as well because there exists a unique tangent line to mixtilinear incircle of the triangle ADF which is parallel to AB and closer to AB than the center of that circle.

1 point.

Second Solution.

Denote by X, Y, Q respectively the points where k touches CE, AF, τ.

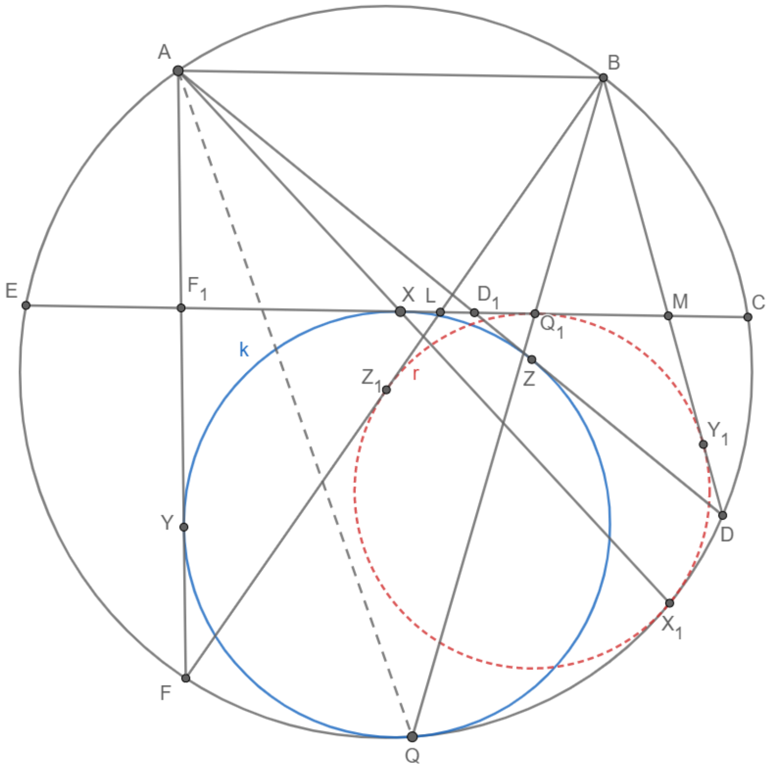
Furthermore, let Z be the point where k touches AD, and X_1 the intersection of τ and AX.

Let r be the circle tangent to BF,BD and CE at points Z_1,Y_1 and Q_1. We claim that X_1 is the touching point of r and τ.

Define:

$L := BF \cap CE$, $M := BD \cap CE$

$F_1 := AF \cap CE$, $D_1 := AD \cap CE$



Claim 1: $\triangle BDF \sim AF_1D_1$ and $\triangle AFD \sim BML$

Proof: We have $\angle D_1AF_1 = \angle FAD = \angle FBD$ and $\angle AD_1F_1 = \angle D_1AB = \angle BAD = \angle BFD$, so the triangles $\triangle BDF$ and $\triangle AF_1D_1$ have the same angles, which proves the first part of the claim.
The second part of the claim follows by the same logic.

1 point.

Claim 2: Q , Q_1 and B are collinear

Proof: Due to the mixtilinear incircle lemma and **Claim 1** applied to $\triangle AFD$ and τ, we know that $\angle LBQ_1 = \angle FAQ$, from which the claim follows.

1 point.

Consider the inversion I with center B and radius $R := \sqrt{BD \cdot BF}$, $I(X) = X^*$.

1 point.

From the properties of inversion we know that $\triangle BFD \sim \triangle BD^*F^*$, so by **Claim 1** we have $AD_1F_1 \sim BD^*F^*$.

Let H be the homothety that sends $\triangle BD^*F^*$ to the $\triangle AD_1F_1$.

Let $f = H \circ I$. This function preserves angles, lines and circles after the inversion.
We can see that $f(B) = A$, $f(D) = D_1$, $f(F) = F_1$.

1 point.

The circle τ goes trough B,F and D, so because of inversion properties and because of the fact that F_1 and D_1 lie on CE, we conclude that $f(\tau) = CE$.

1 point.

$f(L)$ is a point on the line AF, because of the inversion and homothety properties for distance we know that

$$BL \cdot BL^* = BD \cdot BF$$

$$BL^* = \frac{BD \cdot BF}{BL}.$$

We multiply this this by the homothety coefficient $\dfrac{AD_1}{BD^*} = \dfrac{AD_1}{BF}$ and we get

$$BL^* = \frac{BD \cdot BF}{BL} \cdot \frac{AD_1}{BF}$$

$$BL^* = \frac{BD \cdot AD_1}{BL}$$
$$BL^* = \frac{BD \cdot AD_1}{BL}.$$

Beacause of **Claim 1** we have $\frac{BL}{BM} = \frac{AD}{AF}$, so we have

$$BL^* = \frac{BD \cdot AD_1 \cdot AF}{BM \cdot AD}$$

Because $AB \parallel CE \parallel D_1M$, we have $\frac{AD_1}{BM} = \frac{AD}{BD}$

We conclude that $BL^* = AF$. Applying the same logic again, we conclude $BM^* = AD$.

Because of this, we know $f(L) = F$ and $f(M) = D$. This implies that $f(CE) = \tau$.

2 points.

Because of **Claim 2**, we have $\angle MBQ_1 = \angle DBQ = \angle DAQ$, and so by the properties of f we have $f(Q_1) = Q$ because $f(Q_1)$ is on τ and AQ.

1 point.

The function f preserves tangency because H and I also do, so r is sent to a circle tangent to τ at Q, which is also tangent to AF and AD so $f(r) = k$, $f(Z_1) = Z$ and $f(Y_1) = Y$.

The point X_1 is on τ, but we also have $\angle X_1BD = \angle X_1AD$, so we know that $f(X_1)$ is on the lines CE and AX_1, so we conclude $f(X_1) = X$.

Because $f(\tau) = CE$ is tangent to $f(r) = k$ at the point X, the touching point must be preserved, so X_1 is the touching point of r and τ, which proves the claim.

2 points.

Notes on marking:

- Points from different solutions are not additive.