19th JUNIOR BALKAN MATHEMATICAL OLYMPIAD

SUGGESTED PROBLEMS

Short List

Belgrade, Serbia
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Problem Selecting Committee

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ALGEBRA

A1 MLD

Let x, y, z be real numbers, satisfying the relations

$$\begin{cases} x \geq 20, \\ y \geq 40, \\ z \geq 1675, \\ x+y+z=2015. \end{cases}$$

Find the greatest value of the product $P = x \cdot y \cdot z$.

Solution 1:
By virtue of $z \geq 1675$ we have

$$y+z<2015 \iff y < 2015-z \leq 2015-1675 < 1675.$$

It follows that $(1675-y)\cdot(1675-z) \leq 0 \iff y \cdot z \leq 1675 \cdot (y+z-1675)$.

By using the inequality $u \cdot v \leq \left(\frac{u+v}{2}\right)^2$ for all real numbers u, v we obtain

$$P = x \cdot y \cdot z \leq 1675 \cdot x \cdot (y+z-1675) \leq 1675 \cdot \left(\frac{x+y+z-1675}{2}\right)^2 =$$

$$1675 \cdot \left(\frac{2015-1675}{2}\right)^2 = 1675 \cdot 170^2 = 48407500.$$

We have $P = x \cdot y \cdot z = 48407500 \iff \begin{cases} x+y+z=2015, \\ z=1675, \\ x=y+z-1675. \end{cases} \iff \begin{cases} x=170, \\ y=170 \\ z=1675. \end{cases}$

So, the greatest value of the product is $P = x \cdot y \cdot z = 48407500$.

Solution 2:
Let $S = \{(x,y,z) \mid x \geq 20, y \geq 40, z \geq 1675, x+y+z=2015\}$ and $\Pi = \{x \cdot y \cdot z \mid (x,y,z) \in S\}$
We have to find the biggest element of Π. By using the given inequalities we obtain:

$$\begin{cases} 20 \leq x \leq 300, \\ 40 \leq y \leq 320, \\ 1675 \leq z \leq 1955, \\ y < 1000 < z \end{cases}$$

Let $z = 1675+d$. Since $x \leq 300$ so $(1675+d)\cdot x = 1675x + dx \leq 1675x + 1675d = 1675 \cdot (x+d)$
That means that if $(x, y, 1675+d) \in S$ then $(x+d, y, 1675) \in S$, and
$x \cdot y \cdot (1675+d) \leq (x+d) \cdot y \cdot 1675$. Therefore $z = 1675$ must be for the greatest product.

Furthermore, $x \cdot y \leq \left(\frac{x+y}{2}\right)^2 = \left(\frac{2015-1675}{2}\right)^2 = \left(\frac{340}{2}\right)^2 = 170^2$. Since $(170,170,1675) \in S$ that means that the biggest element of Π is $170 \cdot 170 \cdot 1675 = 48407500$

A2 ALB

3) If $x^3-3\sqrt{3}x^2+9x-3\sqrt{3}-64=0$, find the value of $x^6-8x^5+13x^4-5x^3+49x^2-137x+2015$.

Solution

$x^3-3\sqrt{3}x^2+9x-3\sqrt{3}-64=0 \Leftrightarrow (x-\sqrt{3})^3=64 \Leftrightarrow (x-\sqrt{3})=4 \Leftrightarrow x-4=\sqrt{3} \Leftrightarrow x^2-8x+16=3 \Leftrightarrow$
$x^2-8x+13=0$

$x^6-8x^5+13x^4-5x^3+49x^2-137x+2015=(x^2-8x+13)(x^4-5x+9)+1898=0+1898=1898$

A3 MNE

Let a,b,c be positive real numbers. Prove that

$$\frac{a}{b}+\sqrt{\frac{b}{c}}+\sqrt[3]{\frac{c}{a}}>2.$$

Solution:

Starting from the double expression on the left-hand side of given inequality, and applying twice the Arithmetic-Geometric mean inequality, we find that

$$\begin{aligned}
2\frac{a}{b}+2\sqrt{\frac{b}{c}}+2\sqrt[3]{\frac{c}{a}} &= \frac{a}{b}+\left(\frac{a}{b}+\sqrt{\frac{b}{c}}+\sqrt{\frac{b}{c}}\right)+2\sqrt[3]{\frac{c}{a}} \\
&\geq \frac{a}{b}+3\sqrt[3]{\frac{a}{b}\sqrt{\frac{b}{c}}\sqrt{\frac{b}{c}}}+2\sqrt[3]{\frac{c}{a}} \\
&= \frac{a}{b}+3\sqrt[3]{\frac{a}{c}}+2\sqrt[3]{\frac{c}{a}} \\
&= \frac{a}{b}+\sqrt[3]{\frac{a}{c}}+2\left(\sqrt[3]{\frac{a}{c}}+\sqrt[3]{\frac{c}{a}}\right) \\
&\geq \frac{a}{b}+\sqrt[3]{\frac{a}{c}}+2\cdot 2\sqrt{\sqrt[3]{\frac{a}{c}}\sqrt[3]{\frac{c}{a}}} \\
&= \frac{a}{b}+\sqrt[3]{\frac{a}{c}}+4 \\
&> 4,
\end{aligned}$$

which yields the given inequality.

A4 GRE

Let a,b,c be positive real numbers such that $a+b+c=3$. Find the minimum value of

$$A=\frac{2-a^3}{a}+\frac{2-b^3}{b}+\frac{2-c^3}{c}.$$

Solution:

We rewrite A as follows:

$$A=\frac{2-a^3}{a}+\frac{2-b^3}{b}+\frac{2-c^3}{c}=2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-a^2-b^2-c^2=$$

$$2\left(\frac{ab+bc+ca}{abc}\right)-(a^2+b^2+c^2)=2\left(\frac{ab+bc+ca}{abc}\right)-((a+b+c)^2-2(ab+bc+ca))=$$

$$2\left(\frac{ab+bc+ca}{abc}\right)-(9-2(ab+bc+ca))=2\left(\frac{ab+bc+ca}{abc}\right)+2(ab+bc+ca)-9=$$

$$2(ab+bc+ca)\left(\frac{1}{abc}+1\right)-9$$

Recall now the well-known inequality $(x+y+z)^2\geq 3(xy+yz+zx)$ and set $x=ab, y=bc, z=ca$, to obtain $(ab+bc+ca)^2\geq 3abc(a+b+c)=9abc$ where we have used $a+b+c=3$. By taking the square roots on both sides of the last one we obtain:

$ab+bc+ca\geq 3\sqrt{abc}$. (1)

Also by using AM-GM inequality we get that

$\frac{1}{abc}+1\geq 2\sqrt{\frac{1}{abc}}$. (2)

Multiplication of (1) and (2) gives:

$$(ab+bc+ca)\left(\frac{1}{abc}+1\right)\geq 3\sqrt{abc}\cdot 2\sqrt{\frac{1}{abc}}=6.$$

So $A\geq 2\cdot 6-9=3$ and the equality holds if and only if $a=b=c=1$, so the minimum value is 3.

Remark: Note that if $f(x)=\frac{2-x^3}{x}, x\in(0,3)$ then $f''(x)=\frac{4}{x^3}-2$, so the function is convex on $x\in(0,\sqrt[3]{2})$ and concave on $x\in(\sqrt[3]{2},3)$. This means that we cannot apply Jensen's inequality.

A5 MKD

Let x, y, z be positive real numbers that satisfy the equality $x^2+y^2+z^2=3$. Prove that

$$\frac{x^2+yz}{x^2+yz+1}+\frac{y^2+zx}{y^2+zx+1}+\frac{z^2+xy}{z^2+xy+1}\leq 2.$$

Solution:

We have

$$\frac{x^2+yz}{x^2+yz+1}+\frac{y^2+zx}{y^2+zx+1}+\frac{z^2+xy}{z^2+xy+1}\le 2 \Leftrightarrow$$

$$\frac{x^2+yz+1}{x^2+yz+1}+\frac{y^2+zx+1}{y^2+zx+1}+\frac{z^2+xy+1}{z^2+xy+1}\le 2+\frac{1}{x^2+yz+1}+\frac{1}{y^2+zx+1}+\frac{1}{z^2+xy+1}\Leftrightarrow$$

$$3\le 2+\frac{1}{x^2+yz+1}+\frac{1}{y^2+zx+1}+\frac{1}{z^2+xy+1}\Leftrightarrow$$

$$1\le \frac{1}{x^2+yz+1}+\frac{1}{y^2+zx+1}+\frac{1}{z^2+xy+1}$$

$$\frac{1}{x^2+yz+1}+\frac{1}{y^2+zx+1}+\frac{1}{z^2+xy+1}\ge \frac{9}{x^2+yz+1+y^2+zx+1+z^2+xy+1}=$$

$$\frac{9}{x^2+y^2+z^2+xy+yz+zx+3}\ge \frac{9}{2\ x^2+y^2+z^2\ +3}=1$$

The first inequality: AM-GM inequality (also can be achieved with Cauchy-Bunjakowski-Schwarz inequality). The second inequality: $xy+yz+zx$ £ $x^2+y^2+z^2$ (there are more ways to prove it, AM-GM, full squares etc.)

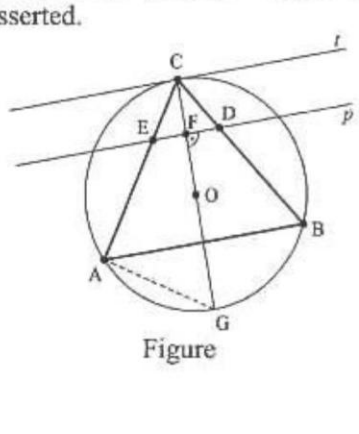
GEOMETRY

G1 MNE

Around the triangle ABC the circle is circumscribed, and at the vertex C tangent t to this circle is drawn. The line p which is parallel to this tangent intersects the lines BC and AC at the points D and E, respectively. Prove that the points A, B, D, E belong to the same circle.

Solution:

Let O be the center of a circumscribed circle k of the triangle ABC, and let F and G be the points of intersection of the line CO with the line p and the circle k, respectively (see Figure). From $p\,||\,t$ it follows that $p\perp CO$. Furthermore, $\angle ABC=\angle AGC$, because these angles are peripheral over the same chord. The quadrilateral $AGFE$ has two right angles at the vertices A and F, and hence, $\angle AED+\angle ABD=\angle AEF+\angle AGF=180^\circ$. Hence, the quadrilateral $ABDE$ is cyclic, as asserted.



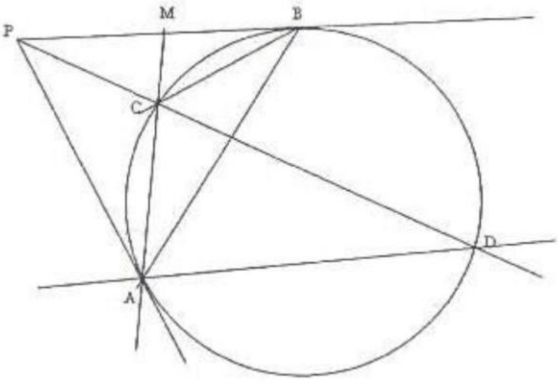
Figure

G2 MLD

The point P is outside of the circle Ω. Two tangent lines, passing from the point P, touch the circle Ω at the points A and B. The median AM, $M \in (BP)$, intersects the circle Ω at the point C and the line PC intersects again the circle Ω at the point D. Prove that the lines AD and BP are parallel.

Solution:

Since $\angle BAC = \angle BAM = \angle MBC$, we have $\Delta MAB \cong \Delta MBC$.



We obtain $\frac{MA}{MB} = \frac{MB}{MC} = \frac{AB}{BC}$. The equality $MB = MP$ implies $\frac{MA}{MP} = \frac{MP}{MC}$ and $\angle PMC \equiv \angle PMA$ gives the relation $\Delta\ PMA \cong \Delta\ CMP$. It follows that $\angle BPD \equiv \angle MPC \equiv \angle MAP \equiv \angle CAP \equiv \angle CDA \equiv \angle PDA$. So, the lines AD and BP are parallel.

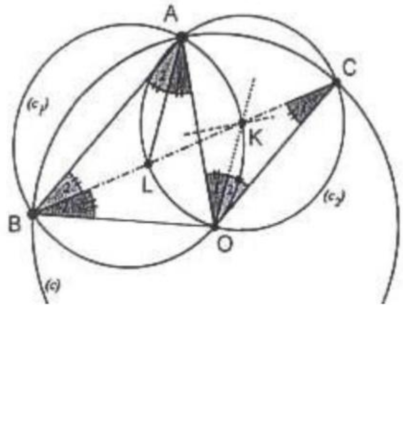
G3 GRE

Let $c \equiv c(O,R)$ be a circle with center O and radius R and A,B be two points on it, not belonging to the same diameter. The bisector of the angle $A\hat{B}O$ intersects the circle c at point C, the circumcircle of the triangle AOB, say c_1 at point K and the circumcircle of the triangle AOC, say c_2, at point L. Prove that the point K is the circumcenter of the triangle AOC and the point L is the incenter of the triangle AOB.

Solution:

The segments OB,OC are equal, as radii of the circle c. Hence OBC is an isosceles triangle and

$$\hat{B}_1 = \hat{C}_1 = \hat{x}. \qquad (1)$$



The chord BC is the bisector of the angle $O\hat{B}A$, and hence

$$\hat{B}_1 = \hat{B}_2 = \hat{x} \qquad (2)$$

The angles $\hat{B}_2$ and $\hat{O}_1$ are inscribed to the same arc OK of the circle c_1 and hence

$$\hat{B}_2 = \hat{O}_1 = \hat{x} \qquad (3)$$

The segments KO, KC are equal, as radii of the circle c_2. Hence the triangle KOC is isosceles and so

$$\hat{O}_2 = \hat{C}_1 = \hat{x} \qquad (4)$$

From equalities (1), (2), (3) we conclude that

$$\hat{O}_1 = \hat{O}_2 = \hat{x},$$

and so OK is the bisector, and hence perpendicular bisector of the isosceles triangle OAC. The point K is the middle of the arc OK (since BK bisects the angle $O\hat{B}A$). Hence the perpendicular bisector of the chord AO of the circle c_1 is passing through point K. It means that K is the circumcenter of the triangle OAC.

From equalities (1), (2), (3) we conclude that $\hat{B}_2 = \hat{C}_1 = \hat{x}$ and so $AB // OC \Rightarrow O\hat{A}B = A\hat{O}C$, that is $\hat{A}_1 + \hat{A}_2 = \hat{O}_1 + \hat{O}_2$ and since $\hat{O}_1 = \hat{O}_2 = \hat{x}$, we conclude that

$$\hat{A}_1 + \hat{A}_2 = 2\hat{O}_1 = 2\hat{x}. \qquad (5)$$

The angles $\hat{A}_1$ and $\hat{C}_1$ are inscribed into the circle c_2 and correspond to the same arc OL. Hence

$$\hat{A}_1 = \hat{C}_1 = \hat{x}. \qquad (6)$$

From (5) and (6) we have $\hat{A}_1 = \hat{A}_2$, i.e. AL is the bisector of the angle $B\hat{A}O$.

G4 CYP

Let ΔABC be an acute triangle. The lines (ε_1), (ε_2) are perpendicular to AB at the points A, B, respectively. The perpendicular lines from the midpoint M of AB to the sides of the triangle AC, BC intersect the lines (ε_1), (ε_2) at the points E, F, respectively. If I is the intersection point of EF, MC, prove that

$$\angle AIB = \angle EMF = \angle CAB + \angle CBA$$

Solution:

Let H, G be the points of intersection of ME, MF, with AC, BC respectively. From the similarity of triangles ΔMHA and ΔMAE we get

$$\frac{MH}{MA} = \frac{MA}{ME}$$

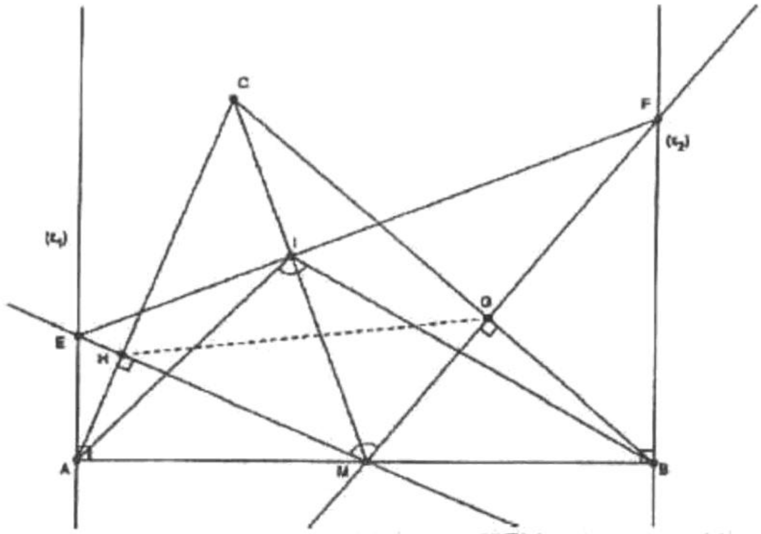
thus, $MA^2 = MH \cdot ME$ (1).

Similarly, from the similarity of triangles ΔMBG and ΔMFB we get

$$\frac{MB}{MF} = \frac{MG}{MB}$$

thus, $MB^2 = MF \cdot MG$ (2).

Since $MA = MB$, from (1), (2), we have that the points E, H, G, F are concyclic.



Therefore, we get that $\angle FEH = \angle FEM = \angle HGM$. Also, the quadrilateral $CHMG$ is cyclic, so $\angle CMH = \angle HGC$. We have

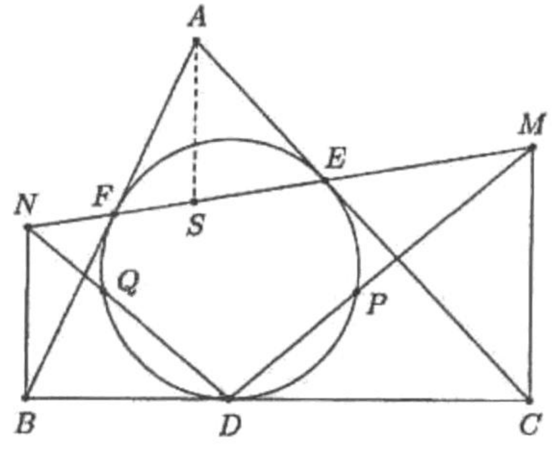
$$\angle FEH + \angle CMH = \angle HGM + \angle HGC = 90°$$

Thus $CM \perp EF$. Now, from the cyclic quadrilaterals $FIMB$ and $EIMA$, we get that $\angle IFM = \angle IBM$ and $\angle IEM = \angle IAM$. Therefore, the triangles ΔEMF and ΔAIB are similar, so $\angle AIB = \angle EMF$. Finally,

$$\angle AIB = \angle AIM + \angle MIB = \angle AEM + \angle MFB = \angle CAB + \angle CBA.$$

G5 ROU

Let ABC be an acute triangle with $AB \neq AC$. The incircle ω of the triangle touches the sides BC, CA and AB at D, E and F, respectively. The perpendicular line erected at C onto BC meets EF at M, and similarly, the perpendicular line erected at B onto BC meets EF at N. The line DM meets ω again in P, and the line DN meets ω again at Q. Prove that $DP = DQ$.



Solution:

Proof 1.1.

Let $\{T\} = EF \cap BC$. Applying Menelaus' theorem to the triangle ABC and the transversal line $E - F - T$ we obtain $\frac{TB}{TC} \cdot \frac{EC}{EA} \cdot \frac{FA}{FB} = 1$, i.e. $\frac{TB}{TC} \cdot \frac{s-c}{s-a} \cdot \frac{s-a}{s-b} = 1$, or $\frac{TB}{TC} = \frac{s-b}{s-c}$, where the notations are the usual ones.

This means that triangles TBN and TCM are similar, therefore $\frac{TB}{TC} = \frac{BN}{CM}$. From the above it follows $\frac{BN}{CM} = \frac{s-b}{s-c}$, $\frac{BD}{CD} = \frac{s-b}{s-c}$, and $\angle DBN = \angle DCM = 90^\circ$, which means that triangles BDN and CDM are similar, hence angles BDN and CDM are equal. This leads to the arcs DQ and DP being equal, and finally to $DP = DQ$.

Proof 1.2.

Let S be the meeting point of the altitude from A with the line EF. Lines BN, AS, CM are parallel, therefore triangles BNF and ASF are similar, as are triangles ASE and CME. We obtain $\frac{BN}{AS} = \frac{BF}{FA}$ and $\frac{AS}{CM} = \frac{AE}{EC}$.

Multiplying the two relations, we obtain $\frac{BN}{CM} = \frac{BF}{FA} \cdot \frac{AE}{EC} = \frac{BF}{EC} = \frac{BD}{DC}$ (we have used that $AE = AF$, $BF = BD$ and $CE = CD$).

It follows that the right triangles BDN and CDM are similar (SAS), which leads to the same ending as in the first proof.

NUMBER THEORY

NT1 SAU

What is the greatest number of integers that can be selected from a set of 2015 consecutive numbers so that no sum of any two selected numbers is divisible by their difference?

Solution:

We take any two chosen numbers. If their difference is 1, it is clear that their sum is divisible by their difference. If their difference is 2, they will be of the same parity, and their sum is divisible by their difference. Therefore, the difference between any chosen numbers will be at least 3. In other words, we can choose at most one number of any three consecutive numbers. This implies that we can choose at most 672 numbers.

Now, we will show that we can choose 672 such numbers from any 2015 consecutive numbers. Suppose that these numbers are a, $a + 1$, ..., $a + 2014$. If a is divisible by 3, we can choose $a + 1$, $a + 4$, ..., $a + 2014$. If a is not divisible by 3, we can choose a, $a + 3$,..., $a + 2013$.

NT2 BUL

A positive integer is called *a repunit*, if it is written only by ones. The repunit with n digits will be denoted by $\underbrace{11...1}_{n}$. Prove that:

a) the repunit $\underbrace{11...1}_{n}$ is divisible by 37 if and only if n is divisible by 3;

b) there exists a positive integer k such that the repunit $\underbrace{11...1}_{n}$ is divisible by 41 if and only if n is divisible by k.

Solution:

a) Let $n = 3m + r$, where m and r are non-negative integers and $r < 3$.

Denote by $\underbrace{00...0}_{p}$ a recording with p zeroes and $\underbrace{abcabc...abc}_{p\times abc}$ recording with p times abc. We have:

$$\underbrace{11...1}_{n}=\underbrace{11...1}_{3m+r}=\underbrace{11...1}_{3m}\cdot\underbrace{00...0}_{r}+\underbrace{11...1}_{r}=111\cdot\underbrace{100100...100}_{(m-1)\times 100}\underbrace{100...0}_{r}+\underbrace{11...1}_{r}.$$

Since $111=37\cdot 3$, the numbers $\underbrace{11...1}_{n}$ and $\underbrace{11...1}_{r}$ are equal modulo 37. On the other hand the numbers 1 and 11 are not divisible by 37. We conclude that $\underbrace{11...1}_{n}$ is divisible by 37 if only if $r = 0$, i.e. if and only if n is divisible by 3.

b) Using the idea from a), we look for a repunit, which is divisible by 41. Obviously, 1 and 11 are not divisible by 41, while the residues of 111 and 1111 are 29 and 4, respectively. We have $11111=41\cdot 271$. Since 11111 is a repunit with 5 digits, it follows in the same way as in a) that $\underbrace{11...1}_{n}$ is divisible by 41 if and only if n is divisible by 5.

NT3 ALB

a) Show that the product of all differences of possible couples of six given positive integers is divisible by 960 (original from Albania).

b) Show that the product of all differences of possible couples of six given positive integers. is divisible by 34560 (modified by problem selecting committee).

Solution:

a) Since we have six numbers then at least two of them have a same residue when divided by 3, so at least one of the differences in our product is divisible by 3.

Since we have six numbers then at least two of them have a same residue when divided by 5, so at least one of the differences in our product is divisible by 5.

We may have:

a) six numbers with the same parity

b) five numbers with the same parity

c) four numbers with the same parity

d) three numbers with the same parity

There are $C_6^2 = 15$ different pairs, so there are 15 different differences in this product.

a) The six numbers have the same parity; then each difference is divisible by 2, therefore our product is divisible by 2^{15}.

b) If we have five numbers with the same parity, then the couples that have their difference odd are formed by taking one number from these five numbers, and the second will be the sixth one. Then $C_5^1 = 15 = 5$ differences are odd. So $15 - 5 = 10$ differences are even, so product is divisible by 2^{10}.
c) If we have four numbers with the same parity, then the couples that have their difference odd are formed by taking one number from these four numbers, and the second will be from the two others numbers. Then $2 \cdot C_4^1 = 8$ differences are odd. So $15 - 8 = 7$ differences are even, so our product is divisible by 2^7.
d) If we have three numbers with the same parity, then the couples that have their difference odd are formed by taking one from each triple. Then $C_3^1 \cdot C_3^1 = 9$ differences are odd, therefore $15 - 9 = 6$ differences are even, so our product is divisible by 2^6.
Thus, our production is divisible by $2^6 \cdot 3 \cdot 5 = 960$.

b) Let a_1, a_2, a_3, a_4, a_5, a_6 be these numbers. Since we have six numbers then at least two of them when divided by 5 have the same residue, so at least one of these differences in our product is divisible by 5.
Since we have six numbers, and we have three possible residues at the division by 3, then at least three of them replies the residue of previous numbers, so at least three of these differences in our product are divisible by 3.
Since we have six numbers, and we have two possible residues at the division by 2, then at least four of them replies the residue of previous numbers, and two of them replies replied residues, so at least six of these differences in our product are divisible by 2.
Since we have six numbers, and we have four possible residues at the division by 4, then at least two of them replies the residue of previous numbers, so at least two of these differences in our product are divisible by 4. That means that two of these differences are divisible by 4 and moreover four of them are divisible by 2.
Thus, our production is divisible by $2^4 \cdot 4^2 \cdot 3^3 \cdot 5 = 34560$.

NT4

Find all prime numbers a, b, c and integers k which satisfy the equation $a^2 + b^2 + 16 \cdot c^2 = 9 \cdot k^2 + 1$.

Solution:
The relation $9 \cdot k^2 + 1 \equiv 1 \pmod 3$ implies

$$a^2 + b^2 + 16 \cdot c^2 \equiv 1 \pmod 3 \Leftrightarrow a^2 + b^2 + c^2 \equiv 1 \pmod 3.$$

Since $a^2 \equiv 0, 1 \pmod 3$, $b^2 \equiv 0, 1 \pmod 3$, $c^2 \equiv 0, 1 \pmod 3$, we have:

a^2	0	0	0	0	1	1	1	1
b^2	0	0	1	1	0	0	1	1
c^2	0	1	0	1	0	1	0	1
$a^2 + b^2 + c^2$	0	1	1	2	1	2	2	0

From the previous table it follows that two of three prime numbers a, b, c are equal to 3.
Case 1. $a = b = 3$. We have

$$a^2 + b^2 + 16 \cdot c^2 = 9 \cdot k^2 + 1 \Leftrightarrow 9 \cdot k^2 - 16 \cdot c^2 = 17 \Leftrightarrow (3k - 4c) \cdot (3k + 4c) = 17.$$

If $\begin{cases} 3k-4c=1, \\ 3k+4c=17, \end{cases}$ then $\begin{cases} c=2, \\ k=3, \end{cases}$ and $(a,b,c,k)=(3,3,2,3)$.

If $\begin{cases} 3k-4c=-17, \\ 3k+4c=-1, \end{cases}$ then $\begin{cases} c=2, \\ k=-3, \end{cases}$ and $(a,b,c,k)=(3,3,2,-3)$.

Case 2. $c=3$. If $(3,b_0,c,k)$ is a solution of the given equation, then $(b_0,3,c,k)$ is a solution, too.

Let $a=3$. We have

$$a^2+b^2+16\cdot c^2=9\cdot k^2+1 \Leftrightarrow 9\cdot k^2-b^2=152 \Leftrightarrow (3k-b)\cdot(3k+b)=152.$$

Both factors shall have the same parity and we obtain only 4 cases:

If $\begin{cases} 3k-b=2, \\ 3k+b=76, \end{cases}$ then $\begin{cases} b=37, \\ k=13, \end{cases}$ and $(a,b,c,k)=(3,37,3,13)$.

If $\begin{cases} 3k-b=4, \\ 3k+b=38, \end{cases}$ then $\begin{cases} b=17, \\ k=7, \end{cases}$ and $(a,b,c,k)=(3,17,3,7)$.

If $\begin{cases} 3k-b=-76, \\ 3k+b=-2, \end{cases}$ then $\begin{cases} b=37, \\ k=-13, \end{cases}$ and $(a,b,c,k)=(3,37,3,-13)$.

If $\begin{cases} 3k-b=-38, \\ 3k+b=-4, \end{cases}$ then $\begin{cases} b=17, \\ k=-7, \end{cases}$ and $(a,b,c,k)=(3,17,3,-7)$.

In addition, $(a,b,c,k)\in\{(37,3,3,13),\ (17,3,3,7),\ (37,3,3,-13),\ (17,3,3,-7)\}$.

So, the given equation has 10 solutions:

$$S=\begin{cases} (37,3,3,13),\ (17,3,3,7),\ (37,3,3,-13),\ (17,3,3,-7),\ (3,37,3,13),\ (3,17,3,7),\ (3,37,3,-13), \\ (3,17,3,-7),\ (3,3,2,3),\ (3,3,2,-3). \end{cases}$$

NT5 MNE

Does there exist positive integers a, b and a prime p such that

$$a^3-b^3=4p^2?$$

Solution:

The given equality may be written as

(1) $$(a-b)(a^2+ab+b^2)=4p^2.$$

Since $a-b<a^2+ab+b^2$, it follows from (1) that

(2) $$a-b<2p.$$

Now consider two cases: 1. $p=2$, and 2. p is an odd prime.

***Case* 1**: $p=2$. Then (1) becomes

(3) $$(a-b)(a^2+ab+b^2)=16:$$

In view of (2) and (3), it must be $a-b=1$ or $a-b=2$. If $a-b=1$, then substituting $a=b+1$ in (3) we obtain

$$b(b+1)=5,$$

which is impossible since $b(b+1)$ is an even integer.

If $a-b=2$, then substituting $a=b+2$ in (3) we get

$$3b(b+2)=4,$$

which is obviously impossible.

***Case* 2**: p is an odd prime. Then (1) yields $a-b\mid 4p^2$. This together with the facts that p is a prime and that by (2) $a-b<2p$, yields $a-b\in\{1,2,4,p\}$.

If $a-b=1$, then substituting $a=b+1$ in (1) we obtain

$$3b(b+1)+1=4p^2,$$

which is impossible since $3b(b+1)+1$ is an odd integer.
If $a-b=2$, then substituting $a=b+2$ in (1) we obtain

$$3b^2+6b+4=2p^2,$$

whence it follows that

$$2(p^2-2)=3(b^2+2b)\equiv 0 \pmod 3,$$

and hence

(5) $$p^2\equiv 2 \pmod 3.$$

Since $p^2\equiv 1 \pmod 3$ for each odd prime $p>3$ and $3^2\equiv 0 \pmod 3$, it follows that the congruence (5) is not satisfied for any odd prime p.
If $a-b=4$, then substituting $a=b+4$ in (1) we obtain

$$3b^2+12b+16=p^2,$$

whence it follows that b is an odd integer such that

$$3b^2\equiv p^2 \pmod 4,$$

whence since $p^2\equiv 1 \pmod 4$ for each odd prime p, we have

(6) $$3b^2\equiv 1 \pmod 4.$$

However, since $b^2\equiv 1 \pmod 4$ for each odd integer b, it follows that the congruence (6) is not satisfied for any odd integer b.
If $a-b=p$, then substituting $a=b+p$ in (1) we obtain

$$p(3b^2+3bp+p^2-4p)=0,$$

i.e.,

(7) $$3b^2+3bp+p^2-4p=0.$$

If $p\geq 5$, then $p^2-4p>0$, and thus (7) cannot be satisfied for any positive integer b. If $p=3$, then (7) becomes

$$3(b^2+3b-1)=0,$$

which is obviously not satisfied for any positive integer b.
Hence, there does not exist positive integers a, b and a prime p such that $a^3-b^3=4p^2$.

COMBINATORICS

C1 BUL

A board $n\times n$ $(n\geq 3)$ is divided into n^2 unit squares. Integers from 0 to n included are written down: one integer in each unit square, in such a way that the sums of integers in each 2×2 square of the board are different. Find all n for which such boards exist.

Solution:
The number of the 2×2 squares in a board $n\times n$ is equal to $(n-1)^2$. All possible sums of the numbers in such squares are $0, 1, \ldots, 4n$. A necessary condition for the existence of a board with the required property is $4n+1\geq (n-1)^2$ and consequently $n(n-6)\leq 0$. Thus $n\leq 6$. The examples show the existence of boards $n\times n$ for all $3\leq n\leq 6$.

1	1	1
1	0	0
0	0	0

0	0	2	0
0	0	0	2
0	1	2	2
2	2	2	2

0	0	2	0	4
0	0	0	2	1
0	1	2	2	4
2	2	3	3	4
4	3	4	4	4

6	6	6	6	5	5
6	6	5	5	5	5
1	2	3	4	4	5
3	5	0	5	0	5
1	0	2	1	0	0
1	0	1	0	0	0

C2 SAU

2015 points are given in a plane such that from any five points we can choose two points with distance less than 1 unit. Prove that 504 of the given points lie on a unit disc.

Solution:

Start from an arbitrary point A and draw a unit disc with center A. If all other points belong to this disc then we are done. Otherwise, take any point B outside of the disc. Draw a unit disc with center B. If two drawn discs cover all 2015 points, by PHP, one of the discs contains at least 1008 points.

Suppose that there is a point C outside of the two drawn discs. Draw a unit disc with center C. If three drawn discs cover all 2015 points, by PHP, one of the discs contains at least 672 points.

Finally, if there is a point D outside of the three drawn discs, draw a unit disc with center D. By the given condition, any other point belongs to one of the four drawn discs. By PHP, one of the discs contains at least 504 points, concluding the solution.

C3 ALB

Positive integers are put into the following table

1	3	6	10	15	21	28	36		
2	5	9	14	20	27	35	44		
4	8	13	19	26	34	43	53		
7	12	18	25	33	42				
11	17	24	32	41					
16	23								
...									
...									

Find the number of the line and column where the number 2015 stays.

Solution 1:

We shall observe straights lines as on the next picture. We can call these lines diagonals.

1	3	6	10	15	21	28	36	
2	5	9	14	20	27	35	44	
4	8	13	19	26	34	43	53	
7	12	18	25	33	42			
11	17	24	32	41				

16	23							
...								
...								

On the first diagonal is number 1.
On the second diagonal are two numbers: 2 and 3.
On the 3rd diagonal are three numbers: 4, 5 and 6
...

On the n-th diagonal are n numbers. These numbers are greater then $\frac{(n-1)n}{2}$ and not greater than $\frac{n(n+1)}{2}$ (see the next sentence!).

On the first n diagonals are $1+2+3+\ldots+n=\frac{n(n+1)}{2}$ numbers.

If m is in the k-th row l-th column and on the n-th diagonal, then it is $m=\frac{(n-1)n}{2}+l$ and $n+1=k+l$. So, $m=\frac{(k+l-2)(k+l-1)}{2}+l$.

We have to find such numbers n, k and l for which:

$$\frac{(n-1)n}{2}<2015\le\frac{n(n+1)}{2} \qquad (1)$$

$$n+1=k+l \qquad (2)$$

$$2015=\frac{(k+l-2)(k+l-1)}{2}+l \qquad (3)$$

(1), (2), (3) $\Rightarrow n^2-n<4030\le n^2+n \Rightarrow n=63,\ k+l=64, 2015=\frac{(64-2)(64-1)}{2}+l\Rightarrow$

$l=2015-31\cdot 63=62$, $k=64-62=2$

Therefore 2015 is located in the second row and 62-th column.

Solution 2:

Firstly, we can see that the first elements of the columns are triangular numbers. If a_i is the first element of the line i, we have $a_i=\frac{i(i-1)}{2}$.

The second elements of the first row obtained by adding the first element 2.
The second elements of the second row is obtained by adding the first element 3.
And so on, then the second element on the n-th row is obtained by adding the first element $n+1$.
Then the third element of the n-th row is obtained by adding n+2, and the k-th element of it is obtained by adding k.

Since the first element of the n-th row is $\frac{(n-1)n}{2}+1$, the second one is

$\frac{(n-1)n}{2}+1+(n+1)=\frac{n(n+1)}{2}+2$.

The third one $\frac{n(n+1)}{2}+1+(n+2)=\frac{(n+1)(n+2)}{2}+3$ so the k-th one should be

$\frac{(n+k-2)(n+k-1)}{2}+k$.

$\frac{(n+k-2)(n+k-1)}{2}+k=2015 \Leftrightarrow n^2+n(2k-3)+k^2-k-4028=0$

To have a positive integer solution $(2k-3)^2-4(k^2-k-4028)=16121-8k$ must be a perfect square.

From $16121-8k=x^2$, it is noticed that the maximum of x is 126 (since k> 0).

Simultaneously can be seen that x is odd, so $x \le 125$.

$16121-8k=x^2 \Leftrightarrow 125^2+496-8k=x^2$

So $496-8k=0$, form that $k=62$.

From that we can find $n=2$.

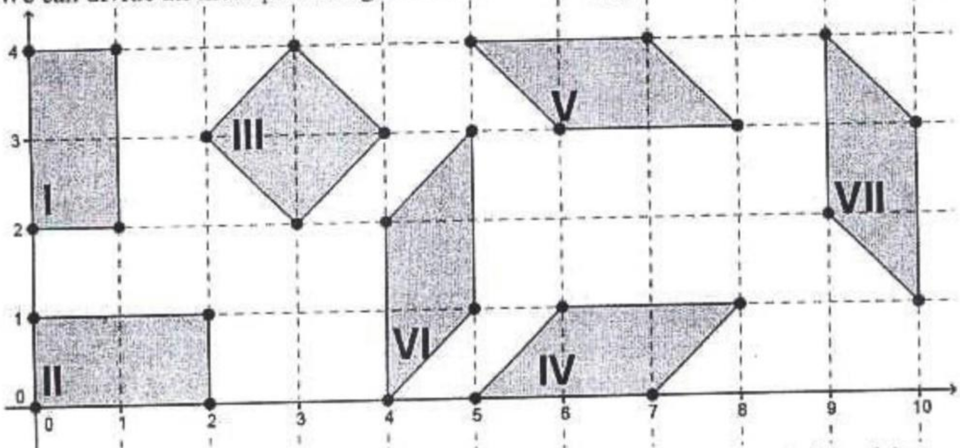
So 2015 is located on the second row and 62-th column.

C4 GRE

Let $n \ge 1$ be a positive integer. A square of side length n is divided by lines parallel to each side into n^2 squares of side length1. Find the number of parallelograms which have vertices among the vertices of the n^2 squares of side length 1, with both sides smaller or equal to 2, and which have the area equal to 2.

Solution:

We can divide all these parallelograms into 7 classes (types I – VII), according to Figure.



Type I: There are n ways to choose the strip for the horizontal (shorter) side of the parallelogram, and $(n-1)$ ways to choose the strip (of the width 2) for the vertical (longer) side. So there are $n(n-1)$ parallelograms of the type I.

Type II: There are $(n-1)$ ways to choose the strip (of the width 2) for the horizontal (longer) side, and n ways to choose the strip for the vertical (shorter) side. So the number of the parallelogram of this type is also $n(n-1)$.

Type III: Each parallelogram of this type is a square inscribed in a unique square 2×2 of our grid. The number of such squares is $(n-1)^2$. So there are $(n-1)^2$ parallelograms of type III.

For each of the types IV, V, VI, VII, the strip of the width 1 in which the parallelogram is located can be chosen in n ways and for each such choice there are $n-2$ parallelograms located in the chosen strip.

Summing we obtain that the total number of parallelograms is:

$$2n(n-1)+(n-1)^2+4n(n-2)=7n^2-12n+1$$

C5 C4P

We have a 5×5 chessboard and a supply of L-shaped triominoes, i.e. 2×2 squares with one corner missing. Two players ***A*** and ***B*** play the following game: A positive integer $k \le 25$ is chosen. Starting with ***A***, the players take alternating turns marking squares of the chessboard until they mark a total of k squares. (In each turn a player has to mark exactly one new square.)

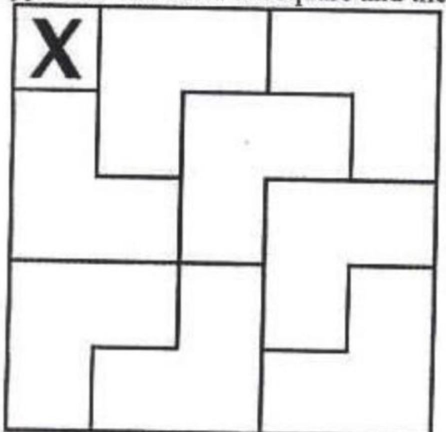
At the end of the process, player ***A*** wins if he can cover without overlapping all but at most 2 unmarked squares with L-shaped triominoes, otherwise player ***B*** wins. It is not permitted any marked squares to be covered.

Find the smallest k, if it exists, such that player ***B*** has a winning strategy.

Solution:

We will show that player ***A*** wins if $k=1$, 2 or 3, but player ***B*** wins if $k=4$. Thus the smallest k for which ***B*** has a winning strategy exists and is equal to 4.

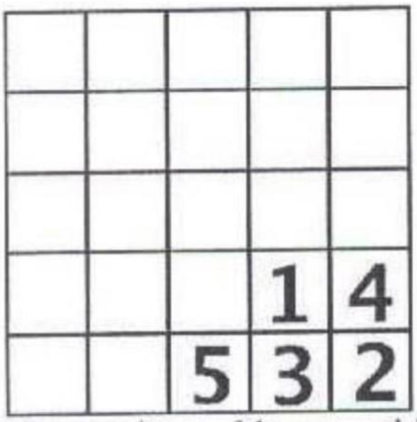
If $k=1$, player ***A*** marks the upper left corner of the square and then fills it as follows.



If $k=2$, player ***A*** marks the upper left corner of the square. Whatever square player ***B*** marks, then player ***A*** can fill in the square in exactly the same pattern as above except that he doesn't put the triomino which covers the marked square of ***B***. Player ***A*** wins because he has left only two unmarked squares uncovered.

For $k=3$, player ***A*** wins by following the same strategy. When he has to mark a square for the second time, he marks any yet unmarked square of the triomino that covers the marked square of ***B***.

Let us now show that for $k=4$ player ***B*** has a winning strategy. Since there will be 21 unmarked squares, player ***A*** will need to cover all of them with seven L-shaped triominoes. We can assume that in his first move, player ***A*** does not mark any square in the bottom two rows of the chessboard (otherwise just rotate the chessboard). In his first move player ***B*** marks the square labeled 1 in the following figure.



If player A in his next move does not mark any of the squares labeled 2, 3 and 4 then player B marks the square labeled 3. Player B wins as the square labeled 2 is left unmarked but cannot be covered with an L-shaped triomino.
If player A in his next move marks the square labeled 2, then player B marks the square labeled 5. Player B wins as the square labeled 3 is left unmarked but cannot be covered with an L-shaped triomino.
Finally, if player A in his next move marks one of the squares labeled 3 or 4, player B marks the other of these two squares. Player B wins as the square labeled 2 is left unmarked but cannot be covered with an L-shaped triomino.
Since we have covered all possible cases, player B wins when $k = 4$.