The 25th JUNIOR BALKAN MATHEMATICAL OLYMPIAD

Chișinău, Republic of Moldova

June 29 – July 5, 2021

SHORTLIST

OF SELECTED PROBLEMS

AND SOLUTIONS

The 25th Junior Balkan Mathematical Olympiad (JBMO 2021)

NOTE OF CONFIDENTIALITY

The shortlisted problems should be kept strictly confidential until JBMO 2022.

CONTRIBUTING COUNTRIES

The Organizing Committee and the Problem Selection Committee of the JBMO 2021 thank the following 7 countries for contributing 22 problem proposals:

Azerbaijan

Bosnia and Herzegovina

Bulgaria

Cyprus

France

North Macedonia

Turkey

PROBLEM SELECTION COMMITTEE

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Members:	***Costaș Ana***
	Croitoru Dorian
	Cuznețov Eugeniu
	Guțu Valeriu
	Lopotenco Alexandru
	Rusu Galina

ALGEBRA

A1. Consider the equation

$$2\left\lfloor \frac{1}{2x} \right\rfloor - n + 1 = (n+1)(1-nx),$$

where n is a positive integer and x is the unknown nonzero real variable.

a) Solve the equation when: i) $n = 8$; ii) $n = 51$.

b) Prove that for some integer N and all integers $n \geq N$ the equation has at least 2021 solutions.

(*For a real number y with $\lfloor y \rfloor$ we denote the largest integer K such that $K \leq y$.*)

(Proposed by Bulgaria)

A1B[1]**.** Let n ($n \geq 1$) be an integer. Consider the equation

$$2 \cdot \left\lfloor \frac{1}{2x} \right\rfloor - n + 1 = (n+1)(1-nx),$$

where x is the unknown real variable.

a) Solve the equation for $n = 8$.

b) Prove that there exists an integer k ($k \geq 1$) such that for all integers $n \geq k$ the equation has at least 2021 solutions.

(For any real number y by $\lfloor y \rfloor$ we denote the largest integer m such that $m \leq y$.)

A2. Let $n \geqslant 3$ be a positive integer. Find all integers k such that $1 \leqslant k \leqslant n$ and for which the following property holds:

> If $x_1, \ldots, x_n$ are n real numbers such that $x_i + x_{i+1} + \ldots + x_{i+k-1} = 0$ for all integers $i \geqslant 1$ (indexes are taken modulo n), then $x_1 = \ldots = x_n = 0$.

(Proposed by Vincent Jugé and Théo Lenoir, France)

A2B[2]**.** Let $n \geqslant 3$ be an integer. Find all integers k ($1 \leqslant k \leqslant n$) for which the following property holds:

> If n real numbers are placed around the circle such that the sum of any k consecutively placed of these numbers equals 0, then all these numbers are necessarily 0.

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[1]Proposed by PSC (Problem Selection Committee).

[2]Proposed by PSC.

A3. Let n be a positive integer. A finite set of integers is called n-*divided* if there is exactly n ways to partition this set into two subsets with equal sums.

For example, the set $\{1, 3, 4, 5, 6, 7\}$ is 2-divided because the only ways to partition it into two subsets with equal sums is by dividing it into $\{1, 3, 4, 5\}$ and $\{6, 7\}$, or $\{1, 5, 7\}$ and $\{3, 4, 6\}$.

Find all the integers $n \geqslant 0$ for which there exists a n-divided set.

(Proposed by Martin Rakovsky, France)

A3B[3]. Let n ($n \geq 1$) be an integer. A finite set of integers is called n-*divided* if there are exactly n different partitions of this set into two subsets with equal sums of their elements.

For example, the set $\{1, 3, 4, 5, 6, 7\}$ is 2-divided because the are only two its partitions into two subsets with equal sums, namely the first is: $\{1, 3, 4, 5\}$ and $\{6, 7\}$, the second is: $\{1, 5, 7\}$ and $\{3, 4, 6\}$.

Find all the integers $n \geq 1$ for which there exists a n-divided set.

(Two partitions of the set are considered to be different if at least one subset of one partition does not coincide with any subset of another partition.)

[3] Proposed by PSC.

GEOMETRY

G1. Let ABC be an acute scalene triangle with circumcenter O. Let D be foot of an A-altitude in triangle ABC and let E denote intersection of lines BC and AO. Let l be a line through E perpendicular to AO. Let l intersect AB and AC at K, L respectively. Denote by ω circumcircle of triangle AKL. Line AD intersects ω again at X. Prove that circumcles of triangles ABC, DEX and ω have a common point.

(Proposed by Boris Stanković, Bosnia and Herzegovina)

G1B.[4] Let ABC be an acute scalene triangle with circumcenter O. Let D be the foot of the altitude from A to the side BC. The lines BC and AO intersect at E. Let l be the line through E perpendicular to AO. The line l intersects AB and AC at K and L, respectively. Denote by ω the circumcircle of triangle AKL. Line AD intersects ω again at X. Prove that ω and the circumcircles of triangles ABC and DEX have a common point.

G2. Let P be an interior point of the isosceles triangle ABC with $\widehat{A} = 90^\circ$. If

$$\widehat{PAB} + \widehat{PBC} + \widehat{PCA} = 90^\circ,$$

prove that $AP \perp BC$.

(Proposed by Mehmet Akif Yıldız, Turkey)

G2B[5]**.** Let P be a point inside the isosceles triangle ABC with $\angle BAC = 90^\circ$. Prove that the lines AP and BC are perpendicular if and only if

$$\angle PAB + \angle PBC + \angle PCA = 90^\circ.$$

G3. Let ABC be an acute triangle with circumcircle ω and circumcenter O. The perpendicular from A to BC intersects BC and ω at D and E, respectively. Let F be a point on the segment AE, such that $2 \cdot \overline{FD} = \overline{AE}$. Let l be the perpendicular to OF through F. Prove that l, the tangent to ω at E and the line BC are concurrent.

(Proposed by Stefan Lozanovski, North Macedonia)

G3B[6]**.** Let ABC be an acute triangle with circumcircle ω and circumcenter O. The perpendicular from A to BC intersects BC at D, and it intersects ω again at E. Let F be a point on the segment AE, such that $AE = 2 \cdot FD$. Let l be the perpendicular to OF through F. Prove that l, the line BC and the tangent to ω at E have a common point.

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[4]Proposed by PSC.
[5]Proposed by PSC.
[6]Proposed by PSC.

G4. Let $ABCD$ be a convex quadrilateral with $\angle B = \angle D = 90^\circ$. Let E be the point of intersection of BC with AD and let M be the midppoint of AE. On the extension of CD, beyond the point D, we pick a pint Z such that $MZ = \frac{AE}{2}$. Let U and V be the projections of A and E respectively on BZ. The circumcircle of the triangle DUV meets again AE at the point L. If I is the point of intersection of BZ with AE, prove that the lines BL and CI intersect on the line AZ.

(Proposed by Cyprus)

G4B[7]**.** Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC = 90^\circ$. The perpendiculars from A and C to BD intersect BD at U and V, respectively. The perpendicular from D to AC intersects AC and BC at T and X, respectively. The circumcircle of triangle TUV intersects AC again at R. The lines AC and BD intersect at F. Prove that the lines XF, BR and AD have a common point.

G5. Let ABC be an acute scalene triangle with circumcircle ω. A line parallel to BC cuts AB and AC in P and Q respectively. Let L be a point on ω such that $AL \parallel BC$. Denote by S the intersection of segments BQ and CP. If K is the intersection of LS with ω, prove that $\angle BKP = \angle CKQ$.

(Proposed by Ervin Macić, Bosnia and Herzegovina)

G5B[8]**.** Let ABC be an acute scalene triangle with circumcircle ω. Let P and Q be interior points of the sides AB and AC, respectively, such that PQ is parallel to BC. Let L be a point on ω such that AL is parallel to BC. The segments BQ and CP intersect at S. The line LS intersects ω at K. Prove that $\angle BKP = \angle CKQ$.

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[7] Proposed by PSC.
[8] Proposed by PSC.

NUMBER THEORY

NT1. Find all positive integers a, b, c such that $ab+1$, $bc+1$ and $ca+1$ are all equal to factorials of some positive integers.

(Proposed by Nikola Velov, North Macedonia)

NT2. The real numbers x, y and z are such that $x^2+y^2+z^2=1$.

a) Determine the smallest and the largest possible values of $xy+yz-xz$.

b) Prove that there does not exist a triple (x, y, z) of rational numbers, which attains any of the two values in a).

(Proposed by Bulgaria)

NT2B[9]**.** The real numbers x, y and z are such that $x^2+y^2+z^2=1$. Let s and L be the smallest and the largest possible values of $xy+yz-xz$.

a) Determine s and L.

b) Prove that there does not exist a triple (x, y, z) of rational numbers, which attains any of the two values s and L.

NT3. For given set $A=\{x_1, x_2, x_3, x_4, x_5\}$ of five distinct positive integers denote sum of its elements with S_A. Let T_A denote number of triplets (i, j, k) with $1 \leqslant i<j<k \leqslant 5$ for which $x_i+x_j+x_k$ divides S_A. Among all sets of five distinct positive integers find, with proof, maximum value that T_A can attain.

(Proposed by Boris Stanković, Bosnia and Herzegovina)

NT3B[10]**.** For any set $A=\{x_1, x_2, x_3, x_4, x_5\}$ of five distinct positive integers denote by S_A the sum of its elements, and denote by T_A the number of triplets (i, j, k) with $1 \leqslant i<j<k \leqslant 5$ for which $x_i+x_j+x_k$ divides S_A. Find the largest possible value of T_A.

[9] Proposed by PSC.

[10] Proposed by PSC.

NT4. Dragoş, the early ruler of Moldavia, and Maria the Oracle play the following game. Firstly, Maria chooses a set S of prime numbers. Then Dragoş gives an infinite sequence $x_1, x_2, \ldots$ of distinct positive integers. Then Maria picks a positive integer M and a prime number p from her set S. Finally, Dragoş picks a positive integer N and the game ends. Dragoş wins if and only if for all integers $n \geq N$ the number x_n is divisible by p^M; otherwise, Maria wins.

Who has a winning strategy if the set S must be: a) finite; b) infinite?

(Proposed by Bulgaria)

NT4B[11]**.** Dragoş and Maria play the following game. Firstly, Maria chooses a nonempty set S of prime numbers. Dragoş, knowing the chosen set S, gives an infinite sequence $x_1, x_2, \ldots$ of distinct positive integers.

Dragoş wins if and only if for any positive integer m and any prime number $p \in S$, there exists a positive integer k such that for any integer $n \geq k$ the number x_n is divisible by p^m; otherwise, Maria wins.

Who has a winning strategy if:

a) the set S is finite; b) the set S is infinite?

NT5. Find all pairs of integers (x, y) such that $x^2 + 5y^2 = 2021y$.

(Proposed by Bulgaria)

NT6. Given a positive integer $n \geqslant 2$, we define $f(n)$ to be the sum of all remainders obtained by dividing n by all positive integers less than n. For example dividing 5 with $1, 2, 3$ and 4 we have remainders equal to $0, 1, 2$ and 1 respectively. Therefore $f(5) = 0 + 1 + 2 + 1 = 4$.

Find all positive integers $n \geqslant 3$ such that $f(n) = f(n-1) + (n-2)$.

(Proposed by Cyprus)

NT6B[12]**.** Given a positive integer $n \geqslant 2$, we denote by f_n the sum of all remainders obtained by dividing n by all positive integers less than n. For example, dividing 5 with 1, 2, 3 and 4, we have remainders equal to 0, 1, 2 and 1 respectively. Therefore $f_5 = 0 + 1 + 2 + 1 = 4$.

Find all positive integers $n \geqslant 2$ such that $f_{n+1} - f_n = n - 1$.

[11] Proposed by PSC.

[12] Proposed by PSC.

NT7. Alice chooses a prime number $p > 2$ and then Bob chooses a positive integer n_0. Alice, in the first move, chooses an integer $n_1 > n_0$ and calculates the expression $s_1 = n_0^{n_1} + n_1^{n_0}$; then Bob, in the second move, chooses an integer $n_2 > n_1$ and calculates the expression $s_2 = n_1^{n_2} + n_2^{n_1}$; etc. one by one. (Each player knows the numbers chosen by the other in the previous moves.) The winner is the one who first chooses the number n_k such that p divides $s_k(s_1 + 2s_2 + \cdots + ks_k)$. Who has a winning strategy?

(Proposed by Borche Joshevski, North Macedonia)

NT7B[13]. Alice and Bob play a game. At her initial move, Alice chooses a prime number $p > 2$ and then Bob chooses a positive integer n_0. After that, Alice, in her standard move, chooses an integer $n_1 > n_0$ and calculates the expression $s_1 = n_0^{n_1} + n_1^{n_0}$. Then Bob, in his standard move, chooses an integer $n_2 > n_1$ and calculates the expression $s_2 = n_1^{n_2} + n_2^{n_1}$. They repeat one by one their standard moves. Each player knows the numbers chosen by the other in the previous moves. The winner is the player who first chooses the number n_k such that p divides $s_k(s_1 + 2s_2 + \cdots + ks_k)$. Who has a winning strategy?

[13]Proposed by PSC.

COMBINATORICS

C1+NT. In Mathcity, there are infinitely many buses and infinitely many stations. The stations are indexed by the powers of 2: $1, 2, 4, 8, 16, \ldots$ Each bus goes by finitely many stations, and the bus number is the sum of all the stations it goes by. For simplifications, the mayor of Mathcity wishes that the bus numbers form an arithmetic progression with common difference r and whose first term is the favourite number of the mayor.

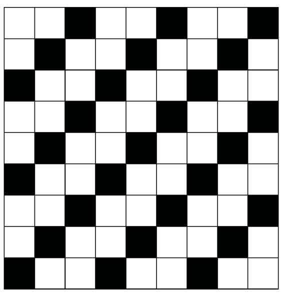
For which positive integers r is it always possible that, no matter the favourite number of the mayor, given any m stations, there is a bus going by all of them?

(Proposed by Savinien Kreczman and Martin Rakovsky, France)

C1B+NT[14]**.** In Mathcity, there are infinitely many buses and infinitely many stations. All stations are indexed by the different powers of 2: $S = \{1, 2, 4, 8, 16, \ldots\}$, such that each element of S is the index for some station. Each bus stops by finitely many distinct stations, and the bus number is the sum of indices of all the stations it stops by. The bus numbers form an arithmetic progression with common difference r and whose first term is the favorite number of the mayor.

Find all positive integers r such that for any finite set of stations, no matter the favorite number of the mayor, there is always a bus stopping by these stations (and, possibly, other stations)?

C2. Let n be a positive integer. We are given a $3n \times 3n$ board whose unit squares are colored in black and white in such way that starting with the top left square, every third diagonal is colored in black and the rest of the board is in white. In one move, one can take a 2×2 square and change the color of all its squares in such way that white squares become orange, orange ones become black and black ones become white. Find all n for which, using a finite number of moves, we can make all the squares which were initially black white, and all squares which were initially white black.



(Proposed by Boris Stanković and Marko Dimitrić, Bosnia and Herzegovina)

[14] Proposed by PSC.

C3. We have a set of 343 closed jars, each containing blue, yellow and red marbles with the number of marbles from each color being at least 1 and at most 7. No two jars have exactly the same contents. Initially all jars are with the caps up. To flip a jar will mean to change its position from cap-up to cap-down or vice versa. It is allowed to choose a triple of positive integers $(b; y; r) \in \{1; 2; ...; 7\}^3$ and flip all the jars whose number of blue, yellow and red marbles differ by not more than 1 from b, y, r, respectively. After n moves all the jars turned out to be with the caps down. Find the number of all possible values of n, if $n \leq 2021$.

(Proposed by Bulgaria)

C3B[15]**.** There are 343 closed jars, each containing blue, yellow and red marbles with the number of marbles from each color being at least 1 and at most 7. No two jars have exactly the same contents. Initially all jars are with the caps up. To flip a jar will mean to change its position from cap-up to cap-down or vice versa. It is allowed to choose a triple of positive integers (b, y, r) with $b, y, r \in \{1, 2, \ldots, 7\}$ and flip all jars whose numbers of blue, yellow and red marbles differ by not more than 1 from b, y and r, respectively. After n moves all the jars turned out to be with the caps down.

a) Find the smallest possible value of n.

b) Is it possible that $n = 2021$?

C4. Alice and Bob play a game together as a team on a 100×100 board with all unit squares initially white. Alice sets up the game by coloring exactly k of the unit squares red at the beginning. After that, a legal move for Bob is to choose a row or column with at least 10 red squares and color all of the remaining squares in it red. What is the smallest k such that Alice can set up a game in such a way that Bob can color the entire board red after finitely many moves.

(Proposed by Nikola Velov, North Macedonia)

C4B[16]**.** Alice plays a game on a 100×100 board with all unit squares 1×1 initially white. First, Alice chooses k unit squares and colors them in red. After, at each her move, Alice chooses a row or a column with at least 10 red squares (if there is) and colors in red all of the remaining white squares (if there are) in this row or column. It is known that after finitely many moves Alice succeeded to color in red all the unit squares of the table. Find the smallest possible value of k.

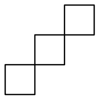
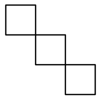
C5. Let A be a subset of the set of 2021 integers $\{1, 2, 3, \ldots, 2021\}$ such that whenever a, b, c are three not necessarily distinct elements of A, then $|a + b - c| > 10$.

What is the largest possible number of elements of A?

(Proposed by Cyprus)

[15] Proposed by PSC.

[16] Proposed by PSC.

C6. Given an $m \times n$ table consisting of mn unit cells. Alice and Bob play the following game: Alice goes first and the one who makes move colors one of the empty cells with one of the given three colors. Alice wins if there is a  or  figure having three different colors. Otherwise Bob is the winner. Determine the winner for all cases of m and n where $m, n \geq 3$.

(Proposed by Toghrul Abbasov, Azerbaijan)

SOLUTIONS

ALGEBRA. SOLUTIONS

A1. Let n be a positive integer. Consider the equation

$$2\left\lfloor\frac{1}{2x}\right\rfloor - n + 1 = (n+1)(1-nx),$$

where x is the nonzero real variable.

a) Solve the equation when: i) $n = 8$; ii) $n = 51$.

b) Prove that for some integer N and all integers $n \geq N$ the equation has at least 2021 solutions.

(*For a real number y, $\lfloor y\rfloor$ denotes the largest integer K such that $K \leq y$.*)

(Proposed by Bulgaria)

Remark.[17] Feel free to remove/add suitable parts – e.g. if in your opinion b) becomes too obvious after having solved a) ii), you could remove a) ii).

Solution. Let $A = \lfloor\frac{1}{2x}\rfloor$ – then the equation gives $x = \frac{2(n-A)}{n(n+1)}$ and now substituting in the definition of A yields

$$A = \left\lfloor\frac{n(n+1)}{4(n-A)}\right\rfloor.$$

The latter equality is a necessary and sufficient condition for the corresponding x to be a solution to the equation. Let us also observe that A is an integer and that $1 \leq A \leq n-1$ for $n \geq 3$ – indeed, if $A = 0$, then $0 = \lfloor\frac{n+1}{4}\rfloor \geq 1$; if $A = n$, the right-hand side is undefined; and if $A < 0$ or $A > n$, then the sides have different signs.

a) i) For $n = 8$ we want $A = \lfloor\frac{18}{8-A}\rfloor$. By the above, A is an integer between 1 and 7 inclusive. A direct verification shows that only $A = 3$ and $A = 4$ are solutions, with the corresponding x being $x = \frac{5}{36}$ and $x = \frac{1}{9}$.

ii) For $n = 51$ we want $A = \lfloor\frac{663}{51-A}\rfloor$ for integers $1 \leq A \leq 50$. This holds if and only if $A \leq \frac{663}{51-A} < A+1$. The left inequality is equivalent to $(2A-51)^2 + 51 \geq 0$ and holds for all A. The right one is equivalent to $(A-25)^2 < 13$ and hence has only $22 \leq A \leq 28$ as solutions. Hence all solutions are $x = \frac{51-A}{26\cdot 51}$ for $22 \leq A \leq 28$.

b) It suffices to have at least 2021 integer solutions $1 \leq A \leq n-1$ to $A \leq \frac{n(n+1)}{4(n-A)} < A+1$ whenever $n \geq N$ for some suitable N. The left inequality is equivalent to $(2A-n)^2+n \geq 0$ and holds for all A. The right inequality is equivalent to $(2A-n+1)^2 < n+1$ and hence holds precisely for $\frac{n-1-\sqrt{n+1}}{2} < A < \frac{n-1+\sqrt{n+1}}{2}$. Observe that this range for A is tighter than $1 \leq A \leq n-1$ for $n \geq 6$, as $(n-3)^2 > n+1$ and $(n-1)^2 > (n+1)$ for these n. Finally, the difference between the endpoints of the interval $(\frac{n-1-\sqrt{n+1}}{2}, \frac{n-1+\sqrt{n+1}}{2})$ is $\sqrt{n+1}$ and hence for sufficiently large n this interval must contain at least 2021 integers. This completes the proof. □

[17] All remarks are made by the problem authors.

A2. Let $n \geqslant 3$ be a positive integer. Find all integers k such that $1 \leqslant k \leqslant n$ and for which the following property holds:

> If $x_1, \ldots, x_n$ are n real numbers such that $x_i + x_{i+1} + \ldots + x_{i+k-1} = 0$ for all integers $i \geqslant 1$ (indexes are taken modulo n), then $x_1 = \ldots = x_n = 0$.

(Proposed by Martin Rakovsky, France)

Answer: All integers k such that $1 \leqslant k \leqslant n$ and k is coprime with n.

Solution. First, if some integer $d \geqslant 2$ divides both k and n, the sequence

$$x_1, x_2, \ldots, x_n = \underbrace{1, 0, \ldots, 0, -1}_{d \text{ numbers}}, \underbrace{1, 0, \ldots, 0, -1}_{d \text{ numbers}}, \ldots, \underbrace{1, 0, \ldots, 0, -1}_{d \text{ numbers}}$$

is such that $x_i + x_{i+1} + \ldots + x_{i+k-1} = 0$ for all integers $i \geqslant 1$, but it contains non-zero terms. Thus, if k is not coprime to n, it cannot be a solution of the problem.

Now, consider some integer $k \in \{1, 2, \ldots, n\}$ that is coprime with n, and let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_i + x_{i+1} + \ldots + x_{i+k-1} = 0$ for all integers $i \geqslant 1$. Given any integer i, we have

$$x_i = -x_{i+1} - \ldots - x_{i+k-1} = x_{i+k}.$$

Thus, the sequence $(x_m)_{m \geqslant 1}$ is periodic, with period k. Since k is coprime with n, there exists an integer ℓ such that $k\ell \equiv 1 \pmod{n}$. It follows that $x_i = x_{i+k\ell} = x_{i+1}$ for all $i \geqslant 1$, i.e., that the real numbers x_i are all equal. Hence, $0 = x_1 + x_2 + \ldots + x_k = kx_1$ and $0 = x_1 = x_2 = \ldots = x_n$.

In conclusion, the solutions are the integers k such that $1 \leqslant k \leqslant n$ and k is coprime with n. $\square$

A3. Let n be a nonnegative integer. A finite set of integers is called n-*divided* if there are exactly n ways to partition this set into two subsets with equal sums.

For example, the set $\{1, 3, 4, 5, 6, 7\}$ is 2-divided because the only ways to partition it into two subsets with equal sums is by dividing it into $\{1, 3, 4, 5\}$ and $\{6, 7\}$, or $\{1, 5, 7\}$ and $\{3, 4, 6\}$.

Find all the integers $n \geqslant 0$ for which there exists a n-divided set.

(Proposed by Vincent Jugé and Théo Lenoir, France)

Answer: All integers $n \geqslant 0$.

Solution. First, note that the set $\{1\}$ is 0-divided and the set $\{1, 2, 3\}$ is 1-divided.

Now consider an integer $n \geqslant 2$ and let us show that the set

$$E = \{k\colon 3n \leqslant k \leqslant 4n-1\} \cup \{k\colon 4n+1 \leqslant k \leqslant 5n\} \cup \{8n(n-2)\}$$

is n-divided. Indeed, if Σ_X denotes the sum of the elements of a set X, choosing a way to divide the set E into two subsest with equal sums corresponds to selecting a subset X of E containing the number $8n(n-2)$, and for which

$$\Sigma_X = \frac{\Sigma_E}{2} = \frac{1}{2}\left(8n(n-2) + \sum_{\ell=1}^{n}((4n-\ell)+(4n+\ell))\right) = \frac{1}{2}\left(8n(n-2)+8n^2\right) = 8n(n-1).$$

In other words, it corresponds to selecting a subset Y of the set $E' = E \setminus \{8n(n-2)\}$ for which $\Sigma_Y = 8n$.

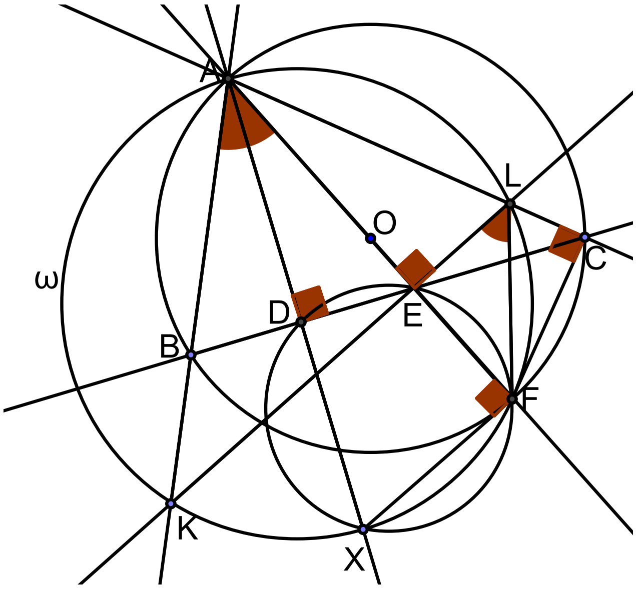
Since $\max E' < 8n < 3\min E'$, such a set Y contains exactly 2 elements. So the corresponding sets Y are the sets of the form $\{4n-\ell, 4n+\ell\}$ with $1 \leqslant \ell \leqslant n$. There exist n such sets, so E is n-divided as desired. $\square$

GEOMETRY. SOLUTIONS

G1. Let ABC be an acute scalene triangle with circumcenter O. Let D be foot of an A-altitude in triangle ABC and let E denote intersection of lines BC and AO. Let l be a line through E perpendicular to AO. Let l intersect AB and AC at K, L respectively. Denote by ω circumcircle of triangle AKL. Line AD intersects ω again at X. Prove that the circumcircles of triangles ABC, DEX and ω have a common point.

(Proposed by Boris Stanković, Bosnia and Herzegovina)

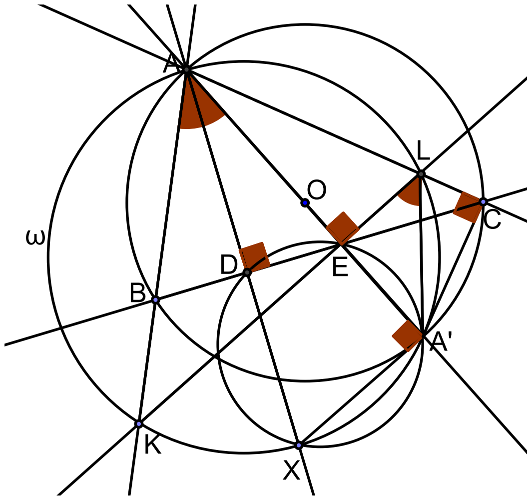
Solution 1.



Let us denote angles of triangle ABC with α, β, γ in a standard way. We easily get that $\angle BAD = 90^\circ - \beta = \angle OAC$ and $\angle CAD = \angle BAO = 90^\circ - \gamma$.

Using the fact that lines AE and AX are isogonal with respect to $\angle KAL$ we can conclude that X is an A-antipode on ω. (This fact can be purely angle-chased, for example we have $\angle KAX + \angle AXK = \angle KAX + \angle ALK = 90^\circ - \beta + \beta = 90^\circ$ which implies $\angle AKX = 90^\circ$) Now let us denote F point on line AE such that $XF \perp AE$. Using that AX is a diameter of ω and $\angle EDX = 90^\circ$ it's clear that F is the intersection point of ω and the circumcircle of DEX. Now it suffices to show that $ABFC$ is cyclic. Now we have that $\angle KLF = \angle KAF = 90^\circ - \gamma$ and from $\angle FEL = 90^\circ$ we have that $\angle EFL = \gamma = \angle ECL$ so quadrilateral $EFCL$ is cyclic. Now we have that $\angle AFC = \angle EFC = 180^\circ - \angle ELC = \angle ELA = \beta$ (where last equality holds because of $\angle AEL = 90^\circ$ and $\angle EAL = 90^\circ - \beta$).$\square$

Solution 2.



As in the first solution we have that $\angle BAD = 90^\circ - \beta = \angle OAC$ and that AX is the diameter of ω. Also we note that $\angle ALK = \beta$, $\angle KLC = 180^\circ - \beta = \angle KBC$ so $BKCL$ is cyclic. Let AO intersect circumcircle of ABC again at A'. We will show that A' is the desired concurrence point. Obviously AA' is the diameter of circumcircle of triangle ABC so $\angle A'CA = 90^\circ$ which implies that $A'CLE$ is cyclic. From power of point E we have that $EK \cdot EL = EB \cdot EC = EA \cdot EA'$ so we can conclude that $A' \in \omega$. Now using the fact that AX being the diameter of ω implies $\angle AXA' = 90^\circ$ we have that $DXA'E$ is cyclic because of $\angle EDX = 90^\circ$ which finishes the proof. $\square$

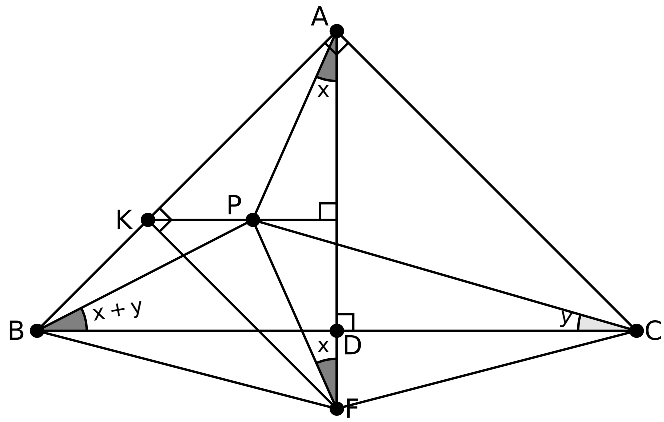
G2. Let P be an interior point of the isosceles triangle ABC with $\widehat{A} = 90^\circ$. If

$$\angle PAB + \angle PBC + \angle PCA = 90^\circ,$$

prove that $AP \perp BC$.

(Proposed by Mehmet Akif Yıldız, Turkey)

Solution.



Let D be the point on BC with $AD \perp BC$. If AP is not perpendicular to BC, without loss of generality, assume P is inside the triangle ABD. Write $\angle PBC = x + y$ and $\angle PCB = y$. From the given angle equality, it is easy to see that $\angle PAD = x$. Let F be the point on AD with $|PA| = |PF|$. Firstly, we have $\angle PFD = x$. Then, when we consider the triangle PBC, we have

$$|FB| = |FC| \text{ and } \angle PFC - \angle PFB = 2x = 2 \cdot (\angle PBC - \angle PCB)$$

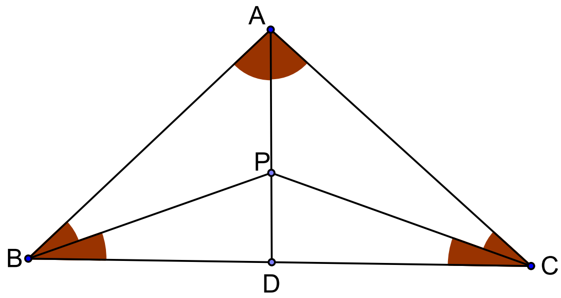
This implies F is the circumcenter of the triangle PBC, and hence we get $|FB| = |FP|$. On the other hand, let K be the point on AB such that $PK \perp AD$. Since $\angle AKF = 90^\circ$ and $KP \perp AF$, we get $|FB| > |FK| > |FP|$, which leads to a contradiction. As a result, we conclude that $AP \perp BC$. □

G2B[18]**.** Let P be a point inside the isosceles triangle ABC with $\angle BAC = 90^\circ$. Prove that the straight lines AP and BC are perpendicular if and only if

$$\angle PAB + \angle PBC + \angle PCA = 90^\circ.$$

Solution 1. $'' \Rightarrow ''$ Let D be the midpoint of the side BC. Then the straight lines AD and BC are perpendicular. Suppose that the straight lines AP and BC are perpendicular. It follows that the point P is a point of the segment AD (see figure).

[18]Proposed by PSC.



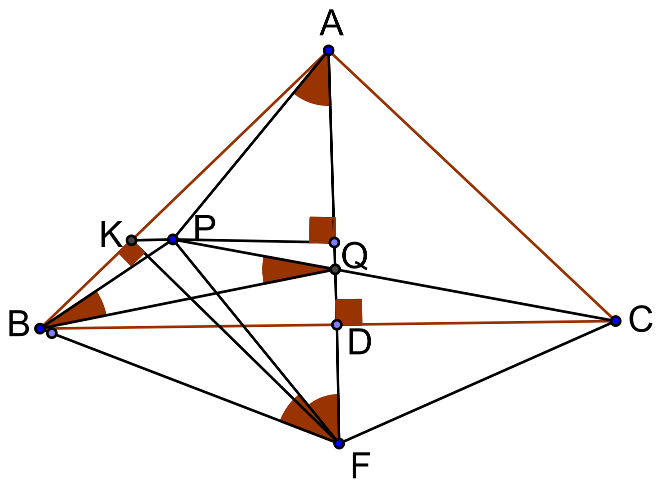
Because AD is the perpendicular bisector of the segment BC we obtain two pairs of congruent triangles: BDP and CDP, ABP and ACP. If we denote $\angle PBC = \angle PCB = \alpha$, then $\angle PBA = \angle PCA = 45^\circ - \alpha$ and $\angle PAB = \angle PAC = 45^\circ$. We have

$$\angle PAB + \angle PBC + \angle PCA = 45^\circ + \alpha + 45^\circ - \alpha = 90^\circ.$$

$'' \Leftarrow''$ Suppose that the point P lie inside of the triangle ABC such, that

$$\angle PAB + \angle PBC + \angle PCA = 90^\circ.$$

Let D be the midpoint of the side BC. It follows that the straight lines AD and BC are perpendicular. If AP is not perpendicular to BC, without loss of generality, assume that P is inside of the triangle ABD (see figure).



Let F be the point on AD with $|PA| = |PF|$ and Q is the intersection point of the segments PC and AD. Denote $\angle PAF = \angle PFA = \frac{x}{2}$ and $\angle QBC = \angle QCB = \frac{45^\circ + y}{2}$. We obtain that $\angle PAB = 45^\circ - \frac{x}{2}$, $\angle PCA = \frac{45^\circ - y}{2}$.. From the given angle equality, it is easy to see that $\angle PBQ = \frac{x}{2}$. Because $\angle PBQ = \angle PFQ = \frac{x}{2}$, it follows that the quadrilateral BPQF is cyclic and $\angle PQB = \angle PFB = 45^\circ + y$. We have

$$\angle FBP = \angle FPB = \frac{135^\circ - y}{2}, \qquad \angle FCP = \angle FPC = \frac{135^\circ - x - y}{2}.$$

From the isosceles triangles BPF and FPC we obtain the equalities $|FB| = |FP| = |FC|$. This implies that F is the circumcenter of triangle PBC.

On the other hand, let K be the point on AB such that the line KP and AF are perpendicular. Since $\angle AKF = 90^\circ$, we get $|FB| > |FK| > |FP|$, which leads to a contradiction. As a result, we conclude that the lines AP and BC are perpendicular. $\square$

Solution 2. " $\Rightarrow$ " The same proof as in the Solution 1.

$''\Leftarrow''$ We apply the trigonometrical version of Ceva concurrence theorem: The lines AP, BP and CP are concurrent in P if and only if

$$\frac{\sin\angle PAC}{\sin\angle PAB}\cdot\frac{\sin\angle PBA}{\sin\angle PBC}\cdot\frac{\sin\angle PCB}{\sin\angle PCA} = 1.$$

By using the angles notations from Solution 1 we obtain the equality

$$\frac{\sin(45^\circ + \frac{x}{2})}{\sin(45^\circ - \frac{x}{2})}\cdot\frac{\sin(\frac{45^\circ}{2} - (\frac{x}{2} + \frac{y}{2}))}{\sin(\frac{45^\circ}{2} + (\frac{x}{2} + \frac{y}{2}))}\cdot\frac{\sin(\frac{45^\circ}{2} + \frac{y}{2})}{\sin(\frac{45^\circ}{2} - \frac{y}{2})} = 1.$$

Let $m = \tan\frac{45^\circ}{2}$, $n = \tan\frac{x}{2}$, $p = \tan\frac{y}{2}$. The numbers m, n, p satisfy the relations $0 < m, n, p < 1$. By applying trigonometrical calculus we obtain the equalities

$$\frac{\sin(45^\circ + \frac{x}{2})}{\sin(45^\circ - \frac{x}{2})} = \frac{1+n}{1-n}, \quad \frac{\sin(\frac{45^\circ}{2} + \frac{y}{2})}{\sin(\frac{45^\circ}{2} - \frac{y}{2})} = \frac{m+p}{m-p},$$

$$\frac{\sin(\frac{45^\circ}{2} - (\frac{x}{2} + \frac{y}{2}))}{\sin(\frac{45^\circ}{2} + (\frac{x}{2} + \frac{y}{2}))} = \frac{m-n-p-mnp}{m+n+p-mnp}.$$

The Ceva relation has the following form:

$$\frac{1+n}{1-n}\cdot\frac{m-n-p-mnp}{m+n+p-mnp}\cdot\frac{m+p}{m-p} = 1.$$

From the last equality we obtain the factorization

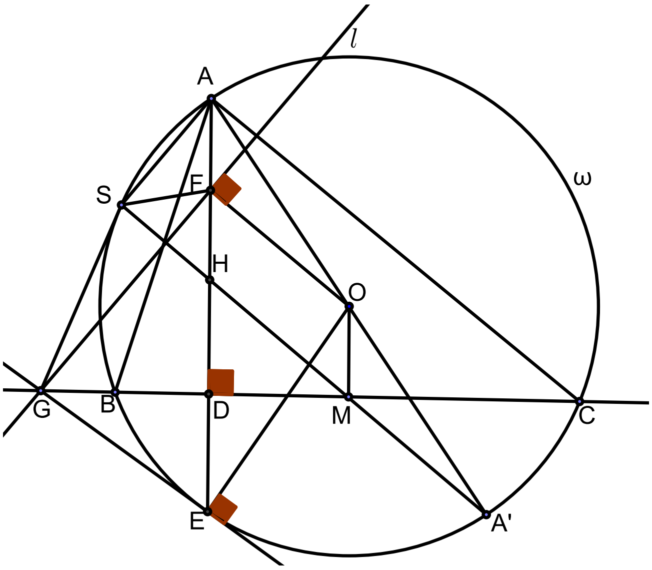
$$2n[m(1-m) + (1+m)p^2 + (1+m^2)np] = 0.$$

Because $m(1-m) + (1+m)p^2 + (1+m^2)np > 0$, then $n = tan\frac{x}{2} = 0$. So, $x = 0$. It follows that the point P lie on the altitude AD and the lines AP and BC are perpendicular. $\square$

G3. Let ABC be an acute triangle with circumcircle ω and circumcenter O. The perpendicular from A to BC intersects BC and ω at D and E, respectively. Let F be a point on the segment AD, such that $2 \cdot FD = AE$. Let l be the perpendicular to OF through F. Prove that l, the tangent to ω at E and the line BC are concurrent.

(Proposed by Stefan Lozanovski, North Macedonia)

Solution 1. Let $l \cap BC = G$. We will prove that GE is tangent to ω. Let H be the orthocenter of ABC. It is well-known that $HD = DE$. From $2\overline{FD} = \overline{AE}$ we get that F is the midpoint of AH.



Let M be the midpoint of BC. It is well known that MH passes through A' - the antipode of A in ω. If S is the second intersection of MH and ω, then $\angle HSA \equiv \angle A'SA = 90^\circ$, so S lies on the circle with diameter AH, which is centered at F. Therefore, $FS = FH$ ... (1)

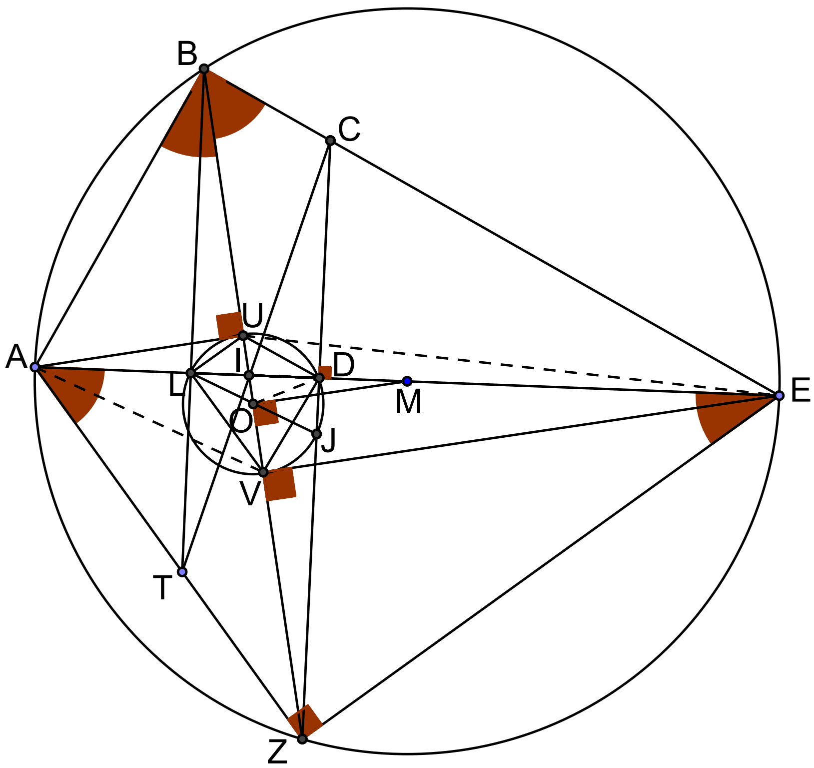
Since $\angle MSA \equiv \angle A'SA = 90^\circ$ we have $MH \perp AS$. But since O and F are centers of ABC and (ASH) and AS is their common chord, we have $OF \perp AS$. Therefore, $MH \parallel OF$ ... (2)

Since FH and OM are both perpendicular to BC, we get $FH \parallel OM$. Using (2), we get that $OFHM$ is a parallelogram. Therefore $FH = OM$. (This can alternatively be proven by the well-known fact that $AH = 2OM$). Using (1), we get that $FS = OM$. Using (2) again, we get that $OFSM$ is an isosceles trapezoid and therefore it's cyclic. Using that $OFGM$ is also cyclic ($\angle OFG = 90^\circ = \angle OMG$), we get that $OFSGM$ is cyclic and therefore $\angle OSG = \angle OFG = 90^\circ$. We have $HS \parallel OF$ and $OF \perp FG$, so $HS \perp FG$. Using (1), we get that FG is the side bisector of SH, so $GS = GH$. Since $HD = DE$, we also get $GH = GE$. Therefore $GS = GH = GE$. Finally, we get that $\triangle OSG \cong \triangle OEG$ (by SSS), so $\angle OEG = \angle OSG = 90^\circ$, i.e. GE is tangent to ω. $\square$

Solution 2. Let H be the orthocenter of $\triangle ABC$ and M the midpoint of BC (thus $OM \perp BC$). Since $2FD = AE$, we get that F is the midpoint of AH. It is known that $HD = DE$ and $AH = 2OM$. Since $FH = \frac{1}{2}AH = OM$ and $FH \parallel OM$, we obtain that $FHMO$ is a parallelogram, hence $FO = HM$. Since H and E are symmetric with respect to the line BC, we obtain that $EM = HM = FO$, which shows that $FOME$ is an isoscelles trapezium, hence cyclic. Let G be the intersection point of BC with the tangent to ω at E. Then$\angle GEO = 90^\circ = \angle GMO$, hence OG is the diameter of the circumcircle of $FOME$, and therefore we must have $\angle GFO = 90^\circ$. $\square$

G4. Let $ABCD$ be a convex quadrilateral with $\angle B = \angle D = 90^\circ$. Let E be the point of intersection of BC with AD and let M be the midppoint of AE. On the extension of CD, beyond the point D, we pick a pint Z such that $MZ = \frac{AE}{2}$. Let U and V be the projections of A and E respectively on BZ. The circumcircle of the triangle DUV meets again AE at the point L. If I is the point of intersection of BZ with AE, prove that the lines BL and CI intersect on the line AZ.

Solution.



Since $MZ = \frac{AE}{2} = AM = ME$ then $\angle AZE = 90^\circ$ and A, B, E, Z belong to a circle with center M.

Let O be the projection of M on BZ. Then $BO = OZ$. Since $AUEV$ is a trapezium, and M is the midpoint of the diagonal AE, then O is the midpoint of UV. Thus $UO = OV$. Since also $BU = ZV$, then $BO = OZ$.

Let J be the point of intersection of CD with the circumcircle of the triangle DUV. Since $\angle LDJ = 90^\circ$, then L and J are antidiametric points of the circle and so the points L, O, J are collinear with $OL = OJ$. Since also $\angle ZOJ = \angle LOB$ then the triangles ZOJ and BOL are equal.

We deduce that $\angle OZJ = \angle LBO$ and therefore ZD is parallel to BL. Since ZD is perpendicular to AE, it follows that BL is also perpendicular to AE.

Let T be the point of intersection of BL with AZ. Since the triangles ABL and CED

are similar, then

$$\frac{BL}{ED} = \frac{AL}{CD} \tag{1}$$

Since the triangles ALT and ZDE are similar, then

$$\frac{LT}{ED} = \frac{AL}{ZD} \tag{2}$$

From (1) and (2) we deduce that

$$\frac{BL}{LT} = \frac{ZD}{CD}$$

from which it follows that CI passes through T.

Alternative Approach: We can also show that BL is perpendicular to AE as follows.

Since $\angle ZVE = 90^\circ = \angle ZDE$, then D, E, Z, V are concyclic. Then $\angle IDV = \angle IZE$ and $\angle IVD = \angle IEZ$. So the triangles ZIE and IVD are similar.

Since A, B, E, Z are concyclic, then the triangles ZIE and AIB are similar. Since D, U, L, V are concylic, then the triangles IVD and IUL are concyclic.

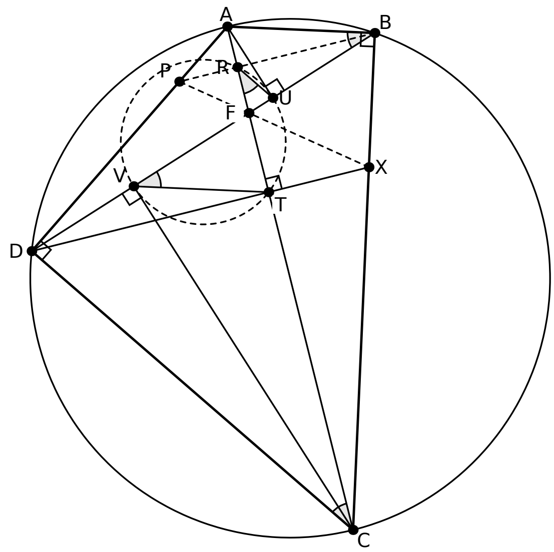
From the above, it follows that the triangles AIB and IUL are similar. Then $\angle ILU = \angle ABI$, and so A, B, U, L are concyclic. Thus $\angle ALB = \angle AUB = 90^\circ$, i.e. BL is perpendicular to AE. $\square$

Comment[19]**.** The initial problem statement is not correct for the case when A lies between B and E. Furthermore, the addition of point Z and the framing of the problem in terms of quadrilateral $ABCD$ is artificial and the first step used to get rid of Z is too simple and omitting it doesnt reduce at all from the complexity of the problem. In the first solution, it is not proven that O is the circumcenter of circle (DUV), and proving this would replicate the same ideas presented in the second solution. The second solution is based on angle chasing, yet the similarity arguments seem extraneous.

[19] All comments are made by PSC.

G4B[20]. Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC = 90°$. Let U and V be the projections of A and C onto BD. Suppose that the perpendicular from D to AC intersects AC and BC at T and X, respectively. The circumcircle of triangle TUV intersects AC again at R. If F is the intersection of AC and BD, prove that the lines XF, BR and AD are concurrent.

Solution.



Since $\angle DVC = \angle DTC = 90°$, $DVTC$ is cyclic. We have

$$\begin{aligned} \angle DBA &= \angle DCA \quad &&(\text{since } ABCD \text{ cyclic}) \\ &= \angle TVB \quad &&(\text{since } DVTC \text{ cyclic}) \\ &= \angle TRU \quad &&(\text{since } VTUR \text{ cyclic}) \end{aligned}$$

and therefore $\angle DBA = \angle TRU$, which shows that $ABUR$ is cyclic. In particular, $\angle ARB = \angle AUB = 90°$, which shows that BR is perpendicular to AC, hence BR is parallel to XD.

Let P be the point of intersection of BR with AD. Since triangles ABR and XCT are similar, we have

$$\frac{BR}{CT} = \frac{AR}{XT}. \tag{1}$$

Since triangles ARP and DTC are similar, we have

$$\frac{RP}{CT} = \frac{AR}{DT}. \tag{2}$$

From (1) and (2) we deduce that

$$\frac{BR}{RP} = \frac{DT}{XT} \Rightarrow \frac{RP}{XT} = \frac{BR}{DT} = \frac{RF}{FT}.$$

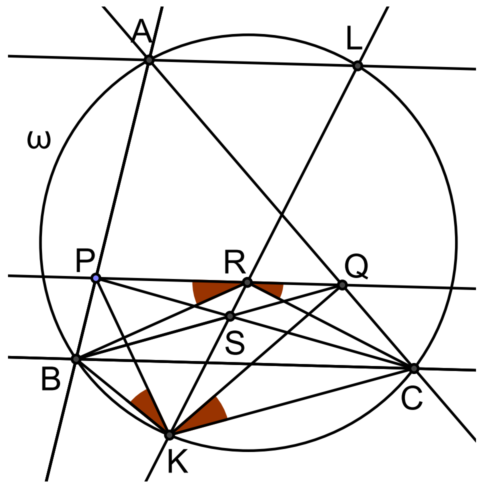
Hence, $\triangle PRF \sim \triangle XTF$, from which it follows that P, X and F are collinear. $\square$

[20]Alternative formulation and solution proposed by PSC.

G5. Let ABC be an acute scalene triangle with circumcircle ω. A line parallel to BC cuts the sides AB and AC at points P and Q respectively. Let L ($L \neq A$) be a point on ω such that $AL \parallel BC$. Denote by S the intersection of segments BQ and CP. If K is the second intersection point of LS with ω, prove that $\angle BKP = \angle CKQ$.

(Proposed by Ervin Macić, Bosnia and Herzegovina)

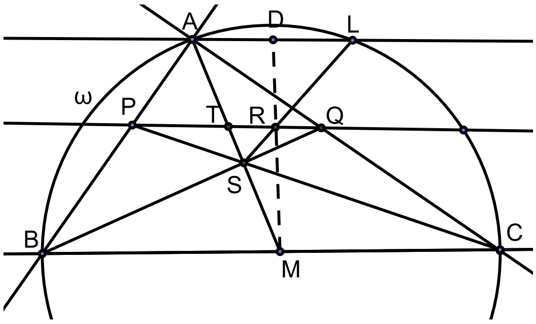
Solution 1.



Denote the intersection of SL with PQ as R. We prove $BKRP$ and $CKRQ$ are cyclic (this is a direct consequence of Reim's theorem): $\angle ALR = \angle ALK = 180°\angle ABK = 180\angle PBK$ but also $\angle ALR = \angle LRQ = \angle PRK$. Now obviously $\angle KRQ = 180°\angle KCQ$. Now notice that we need to prove $\angle BRP = \angle CRQ$ since $\angle BRP = \angle BKP$ and $\angle CKQ = \angle CRQ$. This is equivalent to proving $BR = CR$ ($\angle BRP = \angle RBC$ and $\angle CRQ = \angle RCB$). Notice that we would need PR and RM to be the interior and exterior angle bisectors of ARS. This would mean AR and LR are symmetric w.r.t. bisector of BC, which is sufficient. From here we can proceed in multiple ways:

Let U and V be the intersections of circles $CKRQ$ and $BKRP$ with CP and BQ respectively. Since S lies on the radical axis of these two circles, we must have $US \cdot SC = VS \cdot SB$ so quadrilateral $BUVC$ is cyclic and thus $PUVQ$ is cyclic ($\angle UVB = \angle UCB = \angle CPQ$). Since $PUVQ$ is cyclic we get $\angle QUC = \angle PVB = \angle PRB = \angle CRQ$, and we're done. $\square$

Solution 2.



Let M be the midpoint of BC and D the midpoint of AL. The intersection of AS with PQ is T. We have A, S, M are collinear by Ceva or similar triangles (Thales). We need to prove M, R, D are collinear. This is equivalent to proving $\frac{MT}{MA} = \frac{TR}{AD} = \frac{2TR}{AL}$ but $\frac{TR}{AL} = \frac{ST}{SA}$ so it suffices to prove $\frac{MT}{MA} = \frac{2ST}{SA}$ and here we are basically done since A, T, S, M lie on one line:

$$\frac{SA}{MA} = \frac{2ST}{MT} \Leftrightarrow 1 - \frac{SA}{MA} = 1 - \frac{2ST}{MT}$$
$$\Leftrightarrow \frac{MS}{MA} = \frac{MS - ST}{MT} \Leftrightarrow \frac{MT}{MA} = \frac{MS - ST}{MS} = 1 - \frac{ST}{MS}$$
$$\Leftrightarrow \frac{ST}{MS} = \frac{AT}{MA}$$

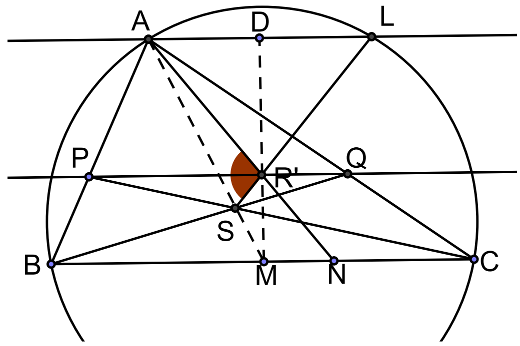
which is true since $\frac{AQ}{AC} = \frac{PQ}{BC}$. □

Solution 3. Same notations as Proof 2: We need to prove that M, R, D are collinear. By Menelaus it's enough to prove

$$\frac{AD}{DL} \cdot \frac{LR}{RS} \cdot \frac{MS}{MA} = 1 \Leftrightarrow 1 \cdot \frac{AT}{TS} \cdot \frac{MS}{MA} = 1 \Leftrightarrow \frac{AT}{TS} = \frac{MA}{MS},$$

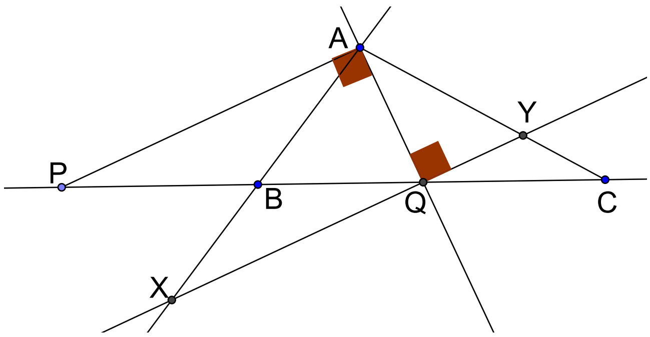
which is true since $\frac{AQ}{AC} = \frac{PQ}{BC}$. □

Solution 4.



Let R' be the point on PQ such that $BR' = CR'$. We'll prove S, R', L are collinear. Let M be the midpoint of BC. By Thales we know A, S, M are collinear. Denote by T the intersection of AS with PQ. We prove that $R'T$ is the interior angle bisector of $\angle AR'S$ by

Lemma. For a given triangle ABC, let $P, Q \in BC$, such that $Q \in (BC)$ and $B \in (PQ)$. If $\frac{PB}{PC} = \frac{BQ}{QC}$ and $\angle PAQ = 90^\circ$, then AP, AQ are the exterior and, respectively, interior angle bisectors of $\angle BAC$.



Proof. Let X and Y be the intersections of the line through Q perpendicular to AQ with AB and AC respectively. By similar triangles ($\triangle BXQ \sim \triangle APB$ and $\triangle QYC \sim \triangle PAC$. We get these since XY is parallel to AP) we have $QX = BQ \cdot \frac{AP}{PB}$ and $QY = QC \cdot \frac{AP}{PC}$. Now obviously $\frac{QX}{QY} = 1$, by the condition, hence by SAS congruence ($\triangle AXQ$ and $\triangle AYQ$) the conclusion follows.

Apply this lemma on triangle $AR'S$ and the segments RT and RM. From here the collinearity is obvious since $\angle AR'P = \angle LR'Q = \angle PR'S$. (The condition $\frac{AT}{AM} = \frac{TS}{MS}$ can be checked to be true by simple similar triangles or Thales). $\square$

NUMBER THEORY. SOLUTIONS

NT1. Find all positive integers a, b, c such that $ab+1$, $bc+1$ and $ca+1$ are all equal to factorials of some positive integers.

(Proposed by Nikola Velov, North Macedonia)

Solution. Because of symmetry, we can assume that $a \geq b \geq c$. In particular, this means that $ab+1 \geq ac+1$. Now if $ac+1 = x!$ and $ab+1 = y!$, we have

$$x! = ac+1 \leq ab+1 = y!$$

So $x! \mid y!$ or $ac+1 \mid ab+1$. From here we obtain:

$$ac+1 \mid (ab+1)-(ac+1) = a(b-c).$$

Using that $ac+1$ and a are relatively prime, this means that $ac+1 \mid b-c$. We conclude that either $b = c$ or $ac+1 \leq b-c$. However, we also see that $0 \leq b-c < b \leq a < ac+1$, which means that we must have $b = c$.

Let $bc+1 = z!$ for some positive integer z. Because $b = c$, this means that $b^2+1 = z!$, but this is only possible if $z = 2$ and $b = 1$. Indeed, if $z \geq 3$, then $3 \mid z!$, so $3 \mid b^2+1$. However, this would mean that $b^2 \equiv 2 \pmod 3$, which is not possible. Checking $z = 1$ directly we see that this is not possible as well, so $b = c = 1$.

Finally, $a+1 = ac+1 = x!$, so $a = x!-1$ for some positive integer x. Because a is also a positive integer, we must have $x \geq 2$. Taking symmetry into consideration we obtain that all solutions (a, b, c) are of the form $(x!-1, 1, 1)$, $(1, x!-1, 1)$ or $(1, 1, x!-1)$ for some positive integer $x \geq 2$. We can easily see that all such triples indeed satisfy the condition of the problem. $\square$

NT2. The real numbers x, y and z are such that $x^2 + y^2 + z^2 = 1$.

a) Determine the smallest and the largest possible values of $xy + yz - xz$.

b) Prove that there does not exist a triple (x, y, z) of rational numbers, which attains any of the two values in a).

(Proposed by Bulgaria)

Solution. a) We have $xy+yz-xz \geq -(x^2+y^2+z^2) = -1 \Leftrightarrow (x+y)^2+(y+z)^2+(x-z)^2 \geq 0$ and equality holds only for $(\pm\frac{1}{\sqrt{3}}, \mp\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$. On the other hand, $(x + z - y)^2 \geq 0 \Leftrightarrow xy + yz - xz \leq \frac{1}{2}$ and equality holds for example when $(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

b) The minimum case is ruled out since $\sqrt{3}$ is irrational. For the maximum case, it is enough to consider $x^2+y^2+z^2 = 1$ with $y = x+z$ – that is, the equation $x^2+z^2+(x+z)^2 = 1$. Suppose the latter has a rational solution and write it as $x = \frac{p}{r}$, $z = \frac{q}{r}$, where $p, q, r \neq 0$ are integers. Then $p^2 + q^2 + (p + q)^2 = r^2 \Leftrightarrow (2p + q)^2 + 3q^2 = 2r^2$. Now modulo 3 gives that $2p + q$ and r are divisible by 3, whence q (and thus p) is divisible by 3. Writing $p = 3p_1$, $q = 3q_1$, $r = 3r_1$, we reach $(2p_1 + q_1)^2 + 3q_1^2 = 2r_1^2$, which is the same equation as the above one for p, q and r. Finally, if the integer s is such that 3^{s+1} does not divide r, then performing the above s more times will yield an equation of the form $(2p' + q')^2 + 3q'^2 = 2r'^2$, with the right-hand side not divisible by 3, which is impossible.□

NT3. For a given set $A = \{x_1, x_2, x_3, x_4, x_5\}$ of five positive integers denote the sum of its elements by S_A. Let T_A denote the number of triples (i, j, k) with $1 \leqslant i < j < k \leqslant 5$ for which $x_i + x_j + x_k$ divides S_A. Among all sets of five positive integers find, with proof, the maximum value that T_A can attain.

(Proposed by Boris Stanković, Bosnia and Herzegovina)

Solution. We will prove that maximum value T_A can attain is 4. Let $A = \{x_1, x_2, x_3, x_4, x_5\}$ be set of five distinct positive integers such that $x_1 < x_2 < x_3 < x_4 < x_5$. Call triple (i, j, k) with $1 \leqslant i < j < k \leqslant 5$ *good* if $x_i + x_j + x_k$ divides S_A. Obviously by size argument triplets (3,4,5), (2,4,5),(1,4,5),(2,3,5), (1,3,5) aren't good because for example

$$x_5 + x_3 + x_1 \mid S_A \Longleftrightarrow x_5 + x_3 + x_1 \mid x_2 + x_4$$

which is impossible since $x_5 > x_4$ and $x_3 > x_2$. Analogously we can show that any triple of form $(x, y, 5)$ where $y > 2$ isn't good.

Because of that number of good triplets is at most 5 and only triplets (1,2,5), (2,3,4), (1,3,4), (1,2,4), (1,2,3) can be good. But if triplets (1,2,5) and (2,3,4) are simultaneously good we have that:

$$x_1 + x_2 + x_5 \mid x_3 + x_4 \Rightarrow x_5 < x_3 + x_4 \tag{1}$$

and

$$x_2 + x_3 + x_4 \mid x_1 + x_5 \Rightarrow x_2 + x_3 + x_4 \leqslant x_1 + x_5 \overset{(1)}{<} x_1 + x_3 + x_4 < x_2 + x_3 + x_4,$$

which is impossible. Therefore, $T_A \leqslant 4$.

To show that $T_A = 4$ is possible consider numbers $1, 2, 3, 4, 494$. This works because $6 \mid 498$, $7 \mid 497$, $8 \mid 496$, and $9 \mid 495$. □

Remark. Motivation for construction is to realize that if we choose x_1, x_2, x_3, x_4 we can get all the conditions x_5 must satisfy. Let $S = x_1 + x_2 + x_3 + x_4$. Now only restrictions for our choice of x_5 is that the following must be true: $S - x_i \mid x_i + x_5 \longleftarrow x_5 \equiv -x_i \mod (S - x_i) \forall i \in \{1, 2, 3, 4\}$. If someone is familiar with the Chinese Remainder Theorem it is obvious that if all numbers $S - x_1, S - x_2, S - x_3, S - x_4$ are pairwise coprime, such x_5 must exist. To make all these numbers pairwise coprime it's natural to take x_1, x_2, x_3, x_4 to be all odd and then solve mod 3 issues. Fortunately it can be seen that $1, 5, 7, 11$ easily works because $13, 17, 19, 23$ are pairwise coprime.

However, even without the knowledge of this theorem it makes sense intuitively that this system must have a solution for some x_1, x_2, x_3, x_4. By taking $(x_1, x_2, x_3, x_4) = (1, 2, 3, 4)$ we get pretty simple system which can be solved by hand rather easily.

There are many other possible constructions and they don't really require knowledge of CRT but they all require some kind of insight which often very important for senior number theory problems. Therefore we can consider this problem a great transition from junior to senior olympiad problems because it has some bounding ideas which are pretty standard for junior problems while finding example requires more senior approach.

NT4. Dragoş, the early ruler of Moldavia, and Maria the Oracle play the following game. Firstly, Maria chooses a set S of prime numbers. Then Dragoş gives an infinite sequence $x_1, x_2, \ldots$ of distinct positive integers. Then Maria picks a positive integer M and a prime number p from her set S. Finally, Dragoş picks a positive integer N and the game ends. Dragoş wins if and only if for all integers $n \geq N$ the number x_n is divisible by p^M; otherwise, Maria wins. Who has a winning strategy if the set S must be: a) finite b) infinite?

(Proposed by Bulgaria)

Solution. We show that in both cases Dragoş can win.

Suppose firstly that Maria chooses the finite set $S = \{p_1, p_2, \ldots, p_k\}$, where $p_1 < p_2 < \ldots < p_k$. Then Dragoş can use the sequence $x_n = (p_1 p_2 \cdots p_k)^n$ (which is increasing and hence consists of distinct terms). Now, no matter what M and p Maria picks, Dragoş can give $N = M$ in order to win – indeed, x_n is divisible by p^n for each p in S and hence by p^M for all $n \geq N$.

Now consider the case when Maria chooses the infinite set $S = \{p_1, p_2, \ldots\}$, where $p_1 < p_2 < \ldots$. Then Dragoş can use the sequence $x_n = (p_1 p_2 \cdots p_n)^n$ (which is increasing and hence consists of distinct terms). Now, no matter what M and p_k Maria picks, Dragoş can give $N = \max(M, k)$ in order to win – indeed, x_k is divisible by p_k^n for each p_k in S whenever $n \geq k$ and hence by p^M for all $n \geq \max(M, k)$. $\square$

NT5. Find all pairs of integers (x, y) such that $x^2 + 5y^2 = 2021y$.

(Proposed by Bulgaria)

Answer. $(0,0)$, $(\pm 390, 100)$, $(\pm 408, 289)$

Solution. The cases $x = 0$ and $y = 0$ are immediate; we can without loss of generality treat $x > 0$ and hence $y > 0$. Clearly, $x^2 = y(2021 - 5y)$ and the greatest common divisor of the multipliers on the right divides $2021 = 43 \cdot 47$. Since $2021 - 5y > 0$, i.e. $y \leq 404$, we have the following possibilities: $43 \mid y$, $47 \mid y$ and $\mathrm{GCD}(y, 2021 - 5y) = 1$. If $y = 43z$, then $x = 43t$ and we get $t^2 = z(47 - 5z)$, $z \leq 9$ and a direct verification shows there is no solution. If $y = 47z$, then $x = 47z$ and we get $t^2 = z(43 - 5z)$, $z \leq 8$ and a direct verification shows there is no solution. In the last case we necessarily have $y = m^2$ and $2021 - 5y = n^2$ for positive integers m and n, i.e. $n^2 + 5m^2 = 2021$. Now we get $5m^2 < 2021$, i.e. $m \leq 20$; moreover m is not divisible by 3 (else $n^2 \equiv 2 \pmod 3$) and does not give remainder 0, 2 or 5 when divided by 7 (else $n^2 \equiv 5, 6 \pmod 7$). The remaining ones are $m = 1, 4, 8, 10, 11, 13, 17, 20$ and for $2021 - 5m^2$ we calculate $44^2 < 1941 < 2016 < 45^2$ (so $m = 1$ and $m = 4$ do not work), $41^2 < 1701 < 42^2$ (so $m = 8$ does not work), $1521 = 39^2$ (respectively $y = 100$), $37^2 < 1376 < 1416 < 38^2$ (so $m = 11$ and $m = 13$ do not work), $576 = 24^2$ (respectively $y = 289$) and $4^2 < 21 < 5^2$ (so $m = 20$ does not work). $\square$

NT6. Given a positive integer $n \geqslant 2$, we define $f(n)$ to be the sum of all remainders obtained by dividing n by all positive integers less than n. For example dividing 5 with $1, 2, 3$ and 4 we have remainders equal to $0, 1, 2$ and 1 respectively. Therefore $f(5) = 0 + 1 + 2 + 1 = 4$.

Find all positive integers $n \geqslant 3$ such that $f(n) = f(n-1) + (n-2)$.

(Proposed by Cyprus)

Solution. Given any $d < n$ we write a_d for the remainder when $n-1$ is divided by d and b_d for the remainder when n is divided by d.

If $d|n$ then we have $a_d = d-1$ and $b_d = 0$. If $d \nmid n$ then we have $b_d - a_b = 1$. Thus

$$\begin{aligned} f(n) - f(n-1) &= \sum_{d \nmid n} (b_d - a_d) + \sum_{d|n} (b_d - a_d) \\ &= \sum_{d \nmid n} 1 - \sum_{d|n} (d-1) \\ &= \sum_{d} 1 - \sum_{d|n} d \\ &= (n-1) - [\sigma(n) - n] \\ &= 2n - 1 - \sigma(n) \end{aligned}$$

Here, all sums are over all integers $d \in \{1, 2, \ldots, n-1\}$ satisfying the claimed properties, $\sigma(n)$ is the sum of all positive divisors of n (including n) and $d(n)$ is the total number of positive divisors of n (including n).

So $f(n) = f(n-1) + (n-2)$ if and only if $\sigma(n) = n+1$. But since $1, n$ are divisors of n, then $\sigma(n) \geqslant n+1$ with equality if and only if n is a prime number.

Note: Several other modifications are possible. E.g. one can ask to find all n for which $f(n) > f(n-1) + (n - \sqrt{n} - 2)$. This results to finding all n for which $\sigma(n) < n + 1 + \sqrt{n}$ which again gives that n must be a prime number. (Otherwise it has $1, n$ and another number greater or equal to $\sqrt{n}$ as divisors.)

Another modification is to ask to find all n such that $f(n) = f(n-1) - 1$. This happens exactly when n is a perfect number. $\square$

NT7. Alice chooses a prime number $p > 2$ and then Bob chooses a positive integer n_0. Alice, in the first move, chooses an integer $n_1 > n_0$ and calculates the expression $s_1 = n_0^{n_1} + n_1^{n_0}$; then Bob, in the second move, chooses an integer $n_2 > n_1$ and calculates the expression $s_2 = n_1^{n_2} + n_2^{n_1}$; etc. one by one. (Each player knows the numbers chosen by the other in the previous moves.) The winner is the one who first chooses the number n_k such that p divides $s_k(s_1 + 2s_2 + \cdots + ks_k)$. Who has a winning strategy?

(Proposed by Borche Joshevski, North Macedonia)

Solution. We will prove that for any prime $p > 2$, Bob can win by choosing $n_0 = (p-1)^2$. Then

$$s_1 = n_0^{n_1} + n_1^{n_0} \equiv \begin{cases} 1 & \text{if } p \mid n_1 \\ 2 & \text{if } p \nmid n_1 \end{cases} \not\equiv 0 \pmod p \text{ for any } n_1 > n_0.$$

We will use the following two properties:

1) If $p \mid n_k$ then it is obvious that for $n_{k+1} = pn_k > n_k$ we have $p|s_{k+1}$.

2) If $p \nmid n_k$ and n_k is odd then for $n_{k+1} = (p-1)(pn_k + 1) > n_k$ we have

$$n_{k+1} \equiv -1 \pmod p, \text{ and } s_{k+1} = n_k^{n_{k+1}} + n_{k+1}^{n_k} \equiv 1 + (-1)^{n_k} \equiv 0 \pmod p.$$

From 1) and 2) it follows that: If one of the players chooses n_k to be odd or even number divisible by p but does not win in his current move then the other can win in the next move. If the player cannot chose n_k for which he wins, then it is clear that the only possibility not to lose is to choose n_k to be an even and $p \nmid n_k$.

Let $n_{2j} = (p-1)^{2m_{2j}}$ for $j = 1, , \frac{p-1}{2}$, where m_{2j} is chosen so that $n_{2j} > n_{2j-1}$.

Then

$$s_j = n_{j-1}^{n_j} + n_j^{n_{j-1}} = \begin{cases} 1 & \text{if } p \nmid n_{j-1}n_j \\ 2 & \text{if } p \nmid n_{j-1}n_j \end{cases} \not\equiv 0 \pmod p$$

and for $k < p-1$ we have $s_1 + 2s_2 + \ + ks_k \equiv$

$$\equiv \begin{cases} 2(1+2+\ +k-1)+k & \text{if } p \mid n_k \\ 2(1+2+\ +k) & \text{if } p \nmid n_k \end{cases} = \begin{cases} k^2 & \text{if } p \mid n_k \\ k(k+1) & \text{if } p \nmid n_k \end{cases} \not\equiv 0 \pmod p$$

and $s_1 + 2s_2 + \ + (p-1)s_{p-1} \equiv 2(1+2+...+(p-1)) \equiv 0 \pmod p$.

As $p-1$ is even, we get that the winning move belongs to Bob. $\square$

COMBINATORICS. SOLUTIONS

C1+NT. In Mathcity, there are infinitely many buses and infinitely many stations. The stations are indexed by the powers of 2: $1, 2, 4, 8, 16, \ldots$ Each bus stops by finitely many (distinct) stations, and the bus number is the sum of indices of the stations it stops by. The bus numbers form an arithmetic progression with common difference r and whose first term is the favourite number of the mayor.

For which positive integers r is it certain that, no matter the favourite number of the mayor, there is a bus stopping by any given m stations (and possibly, other stations)?

(Proposed by Savinien Kreczman and Martin Rakovsky, France)

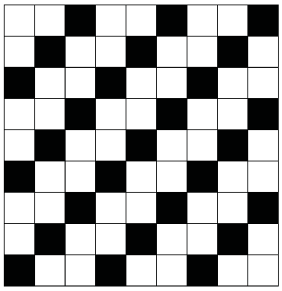
Solution. If r is even and the favourite number of the mayor is 2, no bus will ever go by the station number 1. Thus, even numbers r do not satisfy the problem requirement.

If r is odd, consider m bus stations with numbers $2^{a_1}, \ldots, 2^{a_m}$, such that $a_1 < a_2 < \ldots < a_m$, and let f be the favourite number of the mayor. Since r is odd, it is coprime with 2^{a_m+1}, and thus there exists a positive integer q such that $rq \equiv 1 \pmod{2^{a_m+1}}$. Then, let s be a positive integer such that $s \equiv -1 - f \pmod{2^{a_m+1}}$. According to the problem statement, there is a bus with number $f + r(qs)$.

Since $f + r(qs) \equiv f + s \equiv -1 \equiv 2^0 + 2^1 + \ldots + 2^{a_m} \pmod{2^{a_m+1}}$, this bus will go by each of the stations $2^0, 2^1, \ldots, 2^{a_m}$. In particular, it goes by each of our initial m stations, and thus r satisfies the problem statement.

In conclusion, the solutions of the problem are the odd integers r. $\square$

C2. Let n be a positive integer. We are given a $3n \times 3n$ board whose unit squares are colored in black and white in such way that starting with the top left square, every third diagonal is colored in black and the rest of the board is in white. In one move, one can take a 2×2 square and change the color of all its squares in such way that white squares become orange, orange ones become black and black ones become white. Find all n for which, using a finite number of moves, we can make all the squares which were initially black white, and all squares which were initially white black.



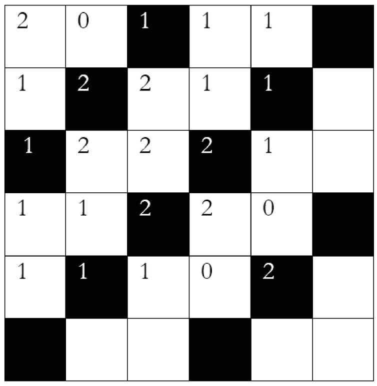
(Proposed by Boris Stankovć and Marko Dimitrić, Bosnia and Herzegovina)

Solution 1. (by proposers) Firstly, observe that if we change the color of one square 3 times, it goes back to its original color. As the final configuration does not depend on the order of moves and placing a move on a square 3 times does not change the configuration, we can suppose that a move is placed on any square exactly 0, 1 or 2 times. This further implies that a white square must change its colour 2 (modulo 3) times and that a black square must be changed 1 (modulo 3) times.

Suppose we can reverse the colours of the squares, as required in the problem statement, for some $3n \times 3n$ board.

Claim: n cannot be odd.

Proof. Let us associate each 2×2 square with its top left unit square. Let's take a look at the first column. Its top square is white and is included in only one 2×2 square, so that square has to be placed 2 times. The second square in this column is also white and is included in its 2×2 square as well as in the one associated with the first unit square in the column. This square has already been turned black by taking the 2×2 square of the first unit square in this column twice, so its own 2×2 square has to be taken 0 times.
We use similar arguments to show that next four 2×2 squares have to be called 1,1,1 and 0 times. After that we have the white square included only in its own 2×2 square and in the one above it, which has been taken 0 times, so we have the same situation we had in the beginning. That means that in the first column numbers of times successive 2×2 squares have to be taken make a cycle (2,0,1,1,1,0). Now take a look at the last square in this column. It is obviously black. Also, its 2×2 square is not included in the board, so this unit square is included only in the 2×2 square of a unit square above it, which has been taken 0 times if $6 \mid 3n + 3$ and 1 time if $6 \mid 3n$. We know that this square must change its color once and that is only possible if $6 \mid 3n$, which means that $2 \mid n$. Here is the construction for $n = 2$:



Here the number inside the unit square denotes the number of times the 2×2 square with that unit square as its top left square is taken. Now for $n = 2k$, $k > 1$, we can partition the board into several disjoint 6×6 boards, which we can solve separately, which will solve the entire board (it is trivial that those 6×6 boards are colored the same way the board in case $n = 2$ is colored). $\square$

Solution 2. (by Milica Vugdelić, Serbia) As in the first solution, we can assume that a move is placed on every square 0, 1 or 2 times. Also, use the same fact that white squares must be change colour 2 (modulo 3) times and that black squares must change colour 1 (modulo 3) times. Use the same construction for even n. Suppose that we can obtain the required board for some n.

Claim 1: The number of moves is divisible by 3.

Proof. Fix a sequence of moves and assign to each unit square the number of times it has changed colour in the sequence. Then the sum of all assigned values is equal to $2W + B$ modulo 3, where B and W are the number of black and white squares on the initial board, respectively. It is easy to see that $3 \mid W$ and $3 \mid B$ so $3 \mid 2W + B$. On the other hand, the sum of all assigned values is just 4 times the number of moves (every change of colours of a 2×2 square adds 4 to our sum). So the number of moves is divisible by three. $\square$

Claim 2: n cannot be odd.

Proof. Consider the set of every other square in every other row of the board (if we set the coordinate sysem with the top left square as $(1,1)$ and the bottom right as $(3n, 3n)$, we see that these squares are just the ones with both coordinates even).Observe that every move changes the colour of exactly one of these squares, so the sum of assigned values of these squares must be equal to the number of moves. If $2 \nmid n$ ($n = 2k+1, k \geqslant 0$) then we have $\frac{3n}{2}^2 = (3k+1)^2$ squares with both coordinates even, of which $b = \frac{2}{3}(3k)^2 + 3 = 6k+3$ are initially black and the rest are white. But the sum of assigned values of these squares will be $2((3k+1)^2 - b) + b$ modulo 3 and since $3 \mid b$ we see that this is not divisible by three. But then the number of moves is not divisible by three, a contradiction. $\square$

Solution 3. (by Mateja Vukelić, Serbia) As in the first two solutions, we can assume that a move is placed on every square 0, 1 or 2 times. Also, use the same fact that white squares must be change color 2 (modulo 3) times and that black squares must change color 1 (modulo 3) times. Use the same construction for even n. Let *power* of each square be 0 if it is white, 1 if it is orange and 2 if it is black. Now color the board as a chessboard in blue and red, and let the top left square be red. Define the $PowerOfRed$ as the sum of *powers* of all red squares.We define the $PowerOfBlue$ the same way. Let $S = PowerOfRed - PowerOfBlue$.

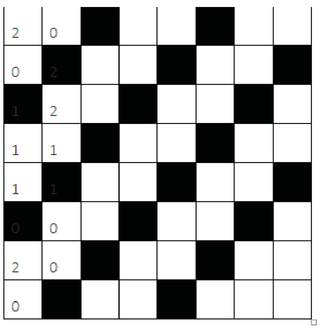
Claim 1: After each move, S stays the same modulo 3.

Proof. It is obvious that each move affects exactly 2 red squares and 2 blue ones. Notice that, when looking at *powers* of squares modulo 3, in each move we add 1 to the *power* of each square. That means that each move increases both the $PowerOfBlue$ and $PowerOfRed$ by 2 each, so S increases by $2 - 2 = 0$, which means it stays the same. □

Claim 2: n cannot be odd.

Proof. Let $n = 2k + 1$. The number of red squares is $R = (3k + 1)(6k + 3) + 3k + 2$ and the number of blue squares is $B = (3k + 1)(6k + 3) + 3k + 1$. Notice that, at the beginning if some square blue square is black, its entire diagonal is black and it contains $3m$ squares, so the number of all black blue squares is divisible by 3, which means that the $PowerOfBlue$ is divisible by 3. We use the same argument to show that, at the beginning, the $PowerOfRed$ is divisible by 3 as well. This means that the S is divisible by 3. Now let's take a look at our desired endboard. Now the number of black blue squares is the same as the number of white blue squares on the starting board, which is B minus some number divisible by 3. This equals $3k + 1$ modulo 3, so now $PowerOfBlue \equiv 2 * (3k + 1)(mod\ 3)$. We use the same argument to show that now $PowerOfRed \equiv 2 * (3k + 2)(mod\ 3)$, so $S \equiv 2 * (3k + 2 - 3k - 1) \equiv 2(mod\ 3)$. However, we already proved that after each move S stays the same modulo 3, so there is no way to apply some sequence of moves which will turn our starting board into desired endboard. □

Remark. The problem can be solved for every $n \times n$ board, not just $3n \times 3n$, but for choices of n not divisible by 3 the board looks rather ugly, so we chose to focus on this subproblem. However, if one wishes to solve the general problem, our period/cycle argument can quickly show that for every positive integer k, $n = 6k + 1$,$n = 6k + 4$ and $n = 6k + 5$ doesnot work. This leaves us with $n = 6k + 2$.



Here we have 8×8 board. Again, the number inside the unit square denotes the number of times the 2×2 square with that unit square as its top left square is taken. We can get that this must be part of correct strategy if one exists simply by starting from the top left square, seeing how many times its square must be changed so that it turns black, and going down the column and repeating the process for the second column. However we now get that the bottom square in the second column has changed its color 2 times, but since the board is $(6k+2) \times (6k+2)$, we know that this square is black, so it has to change its color 1 time. This is enough to say that $8x8$ board does not work. For other $(6k+2) \times (6k+2)$ boards we get the completely same situation because numbers in each column form a cycle with the length 6 Thus, we proved that even in general case, only boards which work are $6k \times 6k$ ones. $\square$

C3. We have a set of 343 closed jars, each containing blue, yellow and red marbles with the number of marbles from each color being at least 1 and at most 7. No two jars have exactly the same contents. Initially all jars are with the caps up. To flip a jar will mean to change its position from cap-up to cap-down or vice versa. It is allowed to choose a triple of positive integers (b, y, r) with $b, y, r \in \{1, 2, ..., 7\}$, and flip all the jars whose number of blue, yellow and red marbles differ by not more than 1 from b, y, r, respectively. After n moves all the jars turned out to be with the caps down. Find all possible values of n.

(Proposed by Bulgaria)

Solution. Call a jar *important* if each of the quantities of blue, yellow and red marbles in it is 1, 4 or 7. There are $3 \cdot 3 \cdot 3 = 27$ important jars. It is easy to check that any move flips exactly one important jar. Thus, after an even number of moves, the number of flipped jars is even. Therefore, the required outcome can appear only after an odd number of moves which is no less than 27. In order to achieve any such number, make a move for each of the 27 important jars, followed by any number of *parasite* pairs of moves with (say) $(b, y, r) = (1, 1, 1)$. Thus n can be any odd number which is at least 27. $\square$

C4. Alice and Bob play a game together as a team on a 100×100 board with all unit squares initially white. Alice sets up the game by coloring exactly k of the unit squares red at the beginning. After that, a legal move for Bob is to choose a row or column with at least 10 red squares and color all of the remaining squares in it red. What is the smallest k such that Alice can set up a game in such a way that Bob can color the entire board red after finitely many moves.

(Proposed by Nikola Velov, North Macedonia)

Solution. We will show by induction on $m+n$ the following:

Fix integers $m, n \geq 1$ and $0 \leq a < n$, $0 \leq b < m$. Assume that the game is being played on a board with m rows and n columns, but assume that a row is a legal move for Bob if it has at least a red squares and a column is a legal move for Bob if it has at least b red squares. Then if it is possible to win the game, there must be at least ab squares initially red (colored by Alice at the beginning of the game).

The base case $m = n = 1$ and $a = b = 0$ is trivial.

Assume the contrary. This means that there are strictly less than ab red squares initially. If $a = 0$ or $b = 0$ this is trivially not possible. Otherwise, provided the game can be won by the children, there is either a row with at least a red squares or a column with at least b red squares (if this is not the case then there are no legal moves). Assume without loss of generality that Bob's first turn is a row. Now make a new board by cutting out this row (which is now completely red).

We obtain a new board with $a_1 = a$ and $b_1 = b - 1$, because all columns from the first board now have at least one red square (from the row we cut out). The total number of red squares on the new board is less than $ab - a = a(b-1) = a_1 b_1$, while the board is $(m-1) \times n$. We also have $0 \leq a_1 < n$ and $0 \leq b_1 < m - 1$. But this board cannot be won by induction (we have $(m-1) + n < m + n$), so the original board cannot be won as well. We conclude that we must have at least ab squares which are initially red.

Now taking $n = m = 100$ and $a = b = 10$ we see that we need at least 10×10 red squares, or in other words, $k \geq 100$. To show that this is indeed possible, notice that Alice can color the upper left corner 10×10 subsquare red. Now Bob first makes moves on the top 10 rows, then on all the columns, in this order. This makes the entire board red, so $k = 100$. $\square$

C5. Let A be a subset of $\{1, 2, 3, \ldots, 2021\}$ such that whenever a, b, c are three not necessarily distinct elements of A, then $|a + b - c| > 10$.

What is the largest possible size of A?

(Proposed by Cyprus)

Solution. The set $A = \{1016, 1017, \ldots, 2021\}$ has the necessary property as $a, b, c \in A$ implies that $a + b - c \geqslant 1016 + 1016 - 2021 = 11$. Notice that this set has 1006 elements. We will show that this is optimal.

Let k be the minimal element of A. Then $k = |k + k - k| \geqslant 10$. For every m, at least one of $m, m + k - 10$ *does not belong* to A, since $k + m - (m + k - 10) = 10$.

Claim 1: A contains at most $k - 11$ out of any $2k - 22$ consecutive integers.

Proof: We can partition the set $\{m + 1, m + 2, \ldots, m + 2k - 22\}$ into $k - 11$ pairs as follows:

$$\{m + 1, m + k - 10\}, \{m + 2, m + k - 9\}, \ldots, \{m + k - 11, m + k - 22\},$$

It remains to note that A can contain at most one element of each pair. $\square$

Claim 2: A contains at most $(t + k - 11)/2$ out of any t consecutive integers.

Proof: Write $t = q(2k - 22) + r$ with $r \in \{0, 1, 2 \ldots, 2k - 21\}$. From the set of the first $q(2k - 22)$ integers, by Claim 1 at most $q(k - 11) = \frac{t-r}{2}$ can belong to A. From the last r integers, at most $\min\{r, k - 11\}$ can belong to A. Theorefore, at most

$$\frac{t - r}{2} + \min\{r, k - 11\} \leq \frac{t - r}{2} + \frac{r + k - 11}{2} = \frac{t + k - 11}{2}$$

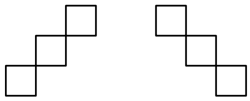
of the t consecutive integers can belong to A, as claimed. $\square$

By Claim 2, amongst $k + 1, k + 2, \ldots, 2021$ at most

$$\frac{(2021 - k) + (k - 11)}{2} = 1005 \text{ integers belong to } A.$$

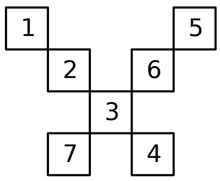
Since amongst $\{1, 2, \ldots, k\}$ only k belongs to A, then A has at most 1006 elements as required. $\square$

C6. Given an $m \times n$ table consisting of mn unit cells. Alice and Bob play the following game: Alice goes first and the one who makes move colors one of the empty cells with one of the given three colors. Alice wins if there exist 3 cells placed diagonally (as in one of the figures below), having three different colors. Otherwise Bob is the winner. Determine the winner for all cases of m and n where $m, n \geq 3$.



(Proposed by Toghrul Abbasov, Azerbaijan)

Solution. For the sake of simplicity, we will label colors as a, b, c. Assume that $m \geq 5, n \geq 4$ or $m \geq 4, n \geq 5$. Without loss of generality, consider second case (for other case, we may rotate table). Since $m \geq 4, n \geq 5$, we can take the following subfigure from the table:



Assume Alice colors 3 as a. After Bob makes move, at least one of triples $\{1, 2, 4\}$ and $\{5, 6, 7\}$ will remain uncolored. Suppose it is $\{1, 2, 4\}$. Then, Alice colors 2 as b. After next move of Bob, again at least one of 1 and 4 is still empty. Alice will choose uncolored one and color it with c. Thus, Alice wins.

Now, consider the case $m = n = 4$: Let's split the squares of the 4×4 table in pairs as follows:

1	8	3	5
7	1	5	4
3	6	2	8
5	4	6	2

Whenever Alice chooses a cell, Bob will color its pair using same color as Alice. Thus, Bob wins (since for any diagonal of 3 squares, 2 of the squares will be paired).

Now, suppose $m = 3, n = 6$ (the symmetric case $m = 6, n = 3$ is treated similarly). Similar to above, we split the cells of the table into pairs as follows:

1	9	2	7	3	6
5	1	8	4	6	5
2	8	3	9	4	7

Whenever Alice chooses cell, Bob will color its pair using same color as Alice. Thus, Bob wins.

Likewise, for $m = 3, n = 4$ (and $n = 3, m = 4$) the pairing

1	5	2	4
6	1	4	3
2	6	3	5

shows that Bob has winning strategy.
Consider now the case $m = n = 3$.

1	2	3
4	5	6
7	8	9

Look at the following pairs: (4,2) (8,6). Whenever Alice chooses a cell numbered 2, 4, 6, or 8, Bob will color its pair with same color as Alice. If Alice colors 1 (3, 7, and 9 cases are just rotation) with color a, Bob will color 5 with the color a. Then Bob will color 3 or 7 (which one is empty) with color a after Alices move. If Alice colors 5 with color a, then Bob will color at least one cell from pairs (1,9) and (3,7) with color a in his next 2 moves. So Bob wins.

Suppose $m = 3, n \geq 7$ (the case $m \geq 7, n = 3$ is similar). Consider the 3×7 subtable from the top left.

1	2	3	4	5	6	7	...	...
8	9	10	11	12	13	14	...	...
15	16	17	18	19	20	21	...	...

Let Alice color 1 as a.

1. Assume Bob doesnt select 9,17,15,3,11,5,19 in his next move. Then, Alice colors 9 as b. Bob must color 17 as one of a, b (suppose it is a). Then, Alice chooses 11 and color it as b. Again, Bob must color 5. Finally, Alice will choose 3 and color it as a. Now, Bob need to make move on both 15 and 19 simultaneously. Alice wins.

2. Assume Bobs next move is either 11 or 5. At least one color is unused. Then, Alice colors 17 with any color among remaining ones. Now, Bob should make move on 9 and uncolored cell between 5 and 11. Thus, Alice wins.

3. Assume Bobs next move is 3. At least one color is unused. Then, Alice colors 9 with any color among remaining ones. Now, Bob should make move on 15 and 17. Thus, Alice wins.

4. Assume Bobs next move is 15. At least one color is unused. Then, Alice colors 9 with any color among remaining ones. Now, Bob should make move on 3 and 17. Thus, Alice wins.

5. Assume Bobs next move is 19. Alice colors 9 as b. Then, Bob must color 17 as one of a, b (suppose it is a). Then, Alice colors 11 as b. Again, Bob must choose 5 and color it as one of a, b. Now, there is at least one unused color among 5 and 19. Alice colors 13 with any color among remaining ones. Bob should make move on 7 and 21. Alice wins.

6. Assume Bobs next move is 17. Obviously, Bob should color it as a. Otherwise, Alice may choose remaining color for 9 and win. So, suppose A colors 11 as b. Then, Bob must color 5 as one of a, b (suppose it is a). Then, Alice colors 13 as b. Bobs next move is 21. Alice colors 19 as any of unused color among 11 and 13 (this clearly exists). Now, Bob should make move on 3 and 7. Alice wins.

7. Assume Bobs next move is 9. Obviously he should color it as a. Alice colors 15 as b. Then, Bob must colors 3 as one of a, b (suppose it is a). Alice colors 11 as b. Again, Bob must color 19 as one of a, b. Alice colors 13 as different colors from 19 so that Bob makes his next move on 7. Since there is at least one unused color among 11 and 13, Alice may choose any of remaining ones, and force Bob to color 17 and 21 simultaneously. Alice wins.

Finally, consider the case $m = 3, n = 5$ (the case $n = 3, m = 5$ is similar).

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15

Alice will color one of the even-numbered cells in the first move. Since there are 7 (i.e. an odd number of) even-numbered cells in total, Alice can force Bob to start coloring odd-numbered cells.

1. If Bob colors cell 1 with a, Alice will color cell 7 with b. Then, Bob will have to color cell 13 with a or b. Next, Alice will color cell 9 with c. So, Bob will need to color cell 5, and Alice will color 3 with a. Bobs subsequent move will guarantee to have at least 11 or 15 to be uncolored. If 11 is remained uncolored, then, Alice will color it with c, and in the end, 3, 7, 11 will have different colors. Alice wins. Otherwise, if 15 remained uncolored, Alice will color it with b, and 3, 9, 15 will have different colors. Alice wins.

2. If Bob colors 5,11 or 15, the case is similar (by symmetry) to 1).

3. If Bob colors 7 with a, Alice will color 1 with b, and then the process will continue as in case 1). Alice wins.

4. If Bob colors 9, the case is similar (by symmetry) to 3). Alice wins.

5. Assume Bob colors cell 3 with a. Alice will then color 11 with b. Since Bob will color 7 either a or b, Alice will color 1 with c. Therefore, Bob will choose one of the colors of 1 or 7 to paint 13. As there will be at most two different colors used for 3 and 13, Alice paints 9 with one of the remaining colors. Since Bob will paint one of the cells 5 or 15 in the next move, Alice will paint with different color and win.

6. If Bob colors 13, the case is similar (by symmetry) to 5). Alice wins.

Remark. The above solution deals with all cases of $m, n \geq 3$. However, it is suggested to ask for $mn \geq 16, m, n \geq 3$ to make problem easier since the case 3×5 is not easy as other cases.