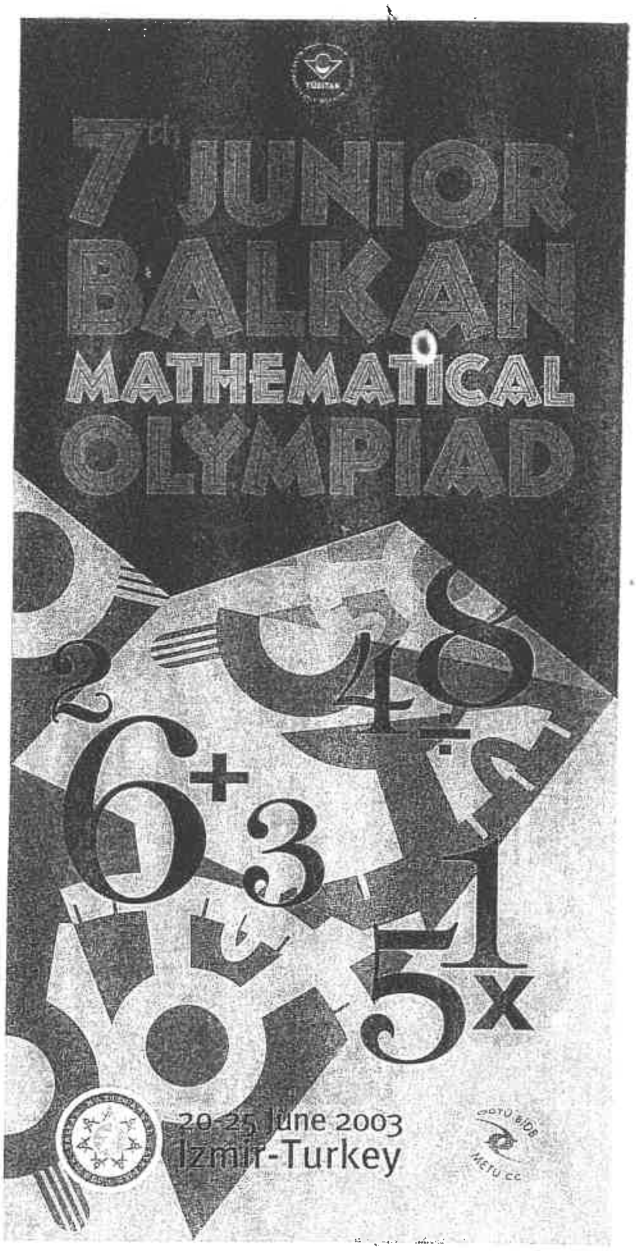
PROPOSED PROBLEMS

FOR

7^{th} JBMO



İZMİR-TURKEY

You will find in this booklet 20 problems in total, proposed by Bulgaria, Former Yugoslav Republic of Macedonia, Republic of Moldova, Romania and Yugoslavia (Serbia and Montenegro). We thank very much these countries for their proposals. The proposed problems and solutions presented here are essentially unedited with the exception of certain minor modifications that seemed necessary. Relatively more substantial corrections and suggestions of the Problem Committee appear under the heading of **Comments**.

The problems are classified into three categories: Algebra, Combinatorics and Geometry. There are eight problems in Algebra, five in Combinatorics and seven in Geometry. Each problem is listed under the category that seems to describe its content best. The problems in each category are listed to roughly reflect their order of difficulty based on the judgement of the Problem Committee.

Problem Committee:

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ALG 1. A number A is written with 2n digits, each of whish is 4, and a number B is written with n digits, each of which is 8. Prove that for each n, A+2B+4 is a total square.

Solution.

$$A = \underbrace{444...44}_{2n} = \underbrace{44...4}_{n}\underbrace{44...4}_{n} = \underbrace{44...4}_{n}\underbrace{00...0}_{n} - \underbrace{44...4}_{n} + \underbrace{88...8}_{n} = \underbrace{44...4}_{n} \cdot (10^n - 1) + B$$

$$= 4 \cdot \underbrace{11...1}_{n} \cdot \underbrace{99...9}_{n} + B = 2^2 \cdot \underbrace{11...1}_{n} \cdot 3^2 \cdot \underbrace{11...1}_{n} + B = \underbrace{66...6}_{n} \cdot \underbrace{66...6}_{n} + B = [3 \cdot \underbrace{22...2}_{n}]^2 + B$$

$$= [\tfrac{3}{4} \cdot \underbrace{88...8}_{n}]^2 + B = (\tfrac{3}{4} B)^2 + B.$$

So,

$$A + 2B + 4 = (\tfrac{3}{4} B)^2 + B + 2B + 4 = (\tfrac{3}{4} B)^2 + 2 \cdot \tfrac{3}{4} B \cdot 2 + 2^2 = (\tfrac{3}{4} B + 2)^2 = (\tfrac{3}{4} \cdot \underbrace{88...8}_{n} + 2)^2$$

$$= (3 \cdot \underbrace{22...2}_{n} + 2)^2 = \underbrace{66...6}_{n-1}8^2$$

ALG 2. Let a, b, c be lengths of triangle sides, $p = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ and $q = \frac{a}{c} + \frac{c}{b} + \frac{b}{a}$. Prove that $|p - q| < 1$.

Solution: One has

$$\begin{aligned} abc|p-q| &= abc\left|\frac{c-b}{a} + \frac{a-c}{b} + \frac{b-a}{c}\right| \\ &= |bc^2 - b^2c + a^2c - ac^2 + ab^2 - a^2b| = \\ &= |abc - ac^2 - a^2b + a^2c - b^2c + bc^2 + ab^2 - abc| = \\ &= |(b-c)(ac - a^2 - bc + ab)| = \\ &= |(b-c)(c-a)(a-b)|. \end{aligned}$$

Since $|b-c| < a$, $|c-a| < b$ and $|a-b| < c$ we infere

$$|(b-c)(c-a)(a-b)| < abc$$

and

$$|p-q| = \frac{|(b-c)(c-a)(a-b)|}{abc} < 1.$$

ALG 3: Let a, b, c be real numbers such that $a^2+b^2+c^2=1$. Prove that $P=ab+bc+ca-2(a+b+c)\geq -\frac{5}{2}$. Are there values of a, b, c, such that $P=-\frac{5}{2}$.

Solution: We have $ab+bc+ca=\frac{(a+b+c)^2-c^2-b^2-a^2}{2}=\frac{(a+b+c)^2-1}{2}$.

If put $t=a+b+c$ we obtain

$$P=\frac{t^2-1}{2}-2t=\frac{t^2-4t-1}{2}=\frac{(t-2)^2-5}{2}\geq -\frac{5}{2}.$$

Obviously $P=-\frac{5}{2}$ when $t=2$, i.e. $a+b+c=2$, or $c=2-a-b$. Substitute in $a^2+b^2+c^2=1$ and obtain $2a^2+2(b-2)a+2b^2-4b+3=0$. Since this quadratic equation has solutions it follows that $(b-2)^2-2(2b^2-3b+3)\geq 0$, from where

$$-3b^2+4b-6\geq 0$$

or

$$3b^2-4b+6\leq 0.$$

But $3b^2-4b+6=3\left(b-\frac{2}{3}\right)^2+\frac{14}{3}>0$. The contradiction shows that $P\neq -\frac{5}{2}$.

Comment: By the Cauchy Schwarz inequality $|t|\leq \sqrt{3}$, so the smallest value of P is attained at $t=\sqrt{3}$ and equals $1-2\sqrt{3}\approx -2.46$.

ALG 4.

Let a, b, c be rational numbers such that

$$\frac{1}{a+bc} + \frac{1}{b+ac} = \frac{1}{a+b}.$$

Prove that $\sqrt{\frac{c-3}{c+1}}$ is also a rational number.

Solution. By cancelling the denominators

$$(a+b)^2(1+c) = ab + c(a^2+b^2) + abc^2$$

and

$$ab(c-1)^2 = (a+b)^2.$$

If $c = -1$, we obtain the contradiction

$$\frac{1}{a-b} + \frac{1}{b-a} = \frac{1}{a+b}.$$

Furthermore,

$$(c-3)(c+1) = (c-1)^2 - 4 = \frac{(a+b)^2}{ab} - 4$$
$$= \frac{(a-b)^2}{ab} = \left(\frac{(a-b)(c-1)}{a+b}\right)^2.$$

Thus

$$\sqrt{\frac{c-3}{c+1}} = \frac{\sqrt{(c-3)(c+1)}}{c+1} = \frac{|a-b||c-1|}{(c+1)|a+b|} \in Q,$$

as needed.

ALG 5. Let ABC be a scalene triangle with $BC = a$, $AC = b$ and $AB = c$, where a, b, c are positive integers. Prove that

$$|ab^2 + bc^2 + ca^2 - a^2b - b^2c - c^2a| \geq 2.$$

Solution. Denote $E = ab^2 + bc^2 + ca^2 - a^2b - b^2c - c^2a$. We have

$$E = (abc - c^2a) + (ca^2 - a^2b) + (bc^2 - b^2c) + (ab^2 - abc) =$$

$$(b - c)(ac - a^2 - bc + ab) = (b - c)(a - b)(c - a).$$

So, $|E| = |a - b| \cdot |b - c| \cdot |c - a|$. By hypothesis each factor from $|E|$ is a positive integer. We shall prove that at least one factor from $|E|$ is greater than 1. Suppose that $|a - b| = |b - c| = |c - a| = 1$. It follows that the numbers $a - b$, $b - c, c - a$ are odd. So, the number $0 = (a - b) + (b - c) + (c - a)$ is olso odd, a contradiction. Hence, $|E| \geq 1 \cdot 1 \cdot 2 = 2$.

ALG 6.

Let a, b, c be positive numbers such that $a^2b^2 + b^2c^2 + c^2a^2 = 3$. Prove that

$$a + b + c \geq abc + 2.$$

Solution. We can consider the case $a \geq b \geq c$ which implies $c \leq 1$. The given inequality writes

$$a + b - 2 \geq (ab - 1)c \geq (ab - 1)c^2 = (ab - 1)\frac{3 - a^2b^2}{a^2 + b^2} \qquad (1)$$

Put $x = \sqrt{ab}$. From the inequality $3a^2b^2 \geq a^2b^2 + b^2c^2 + c^2a^2 = 3$ we infer $x \geq 1$ and from $a^2b^2 < a^2b^2 + b^2c^2 + c^2a^2 = 3$ we find $x \leq \sqrt[4]{3}$. As $a + b \geq 2\sqrt{ab} = 2x$ and $a^2 + b^2 \geq 2ab = 2x^2$, to prove the inequality (1) it will sufice to show that

$$2(x - 1) \geq (x^2 - 1)\frac{3 - x^4}{2x^2}.$$

As $x - 1 \geq 0$, the last inequality is equivalent to

$$4x^2 \geq (x + 1)(3 - x^4)$$

which can be easily obtained by multplying the obvious ones $2x^2 \geq x + 1$ and $2 \geq 3 - x^4$.
Equality holds only in the case when $a = b = c = 1$.

Comment: As it is,the solution is incorrect, it only proves the weaker inequality $a + b - 2 \geq (ab - 1)c^2$, that is: $a + b + c^2 \geq abc^2 + 2$. The problem committee could not find a reasonable solution. Instead the problem could be slightly modified so that the method of the proposed solution applies. The modified problem is:

ALG 6'. Let a, b, c be positive numbers such that $ab + bc + ca = 3$. Prove that

$$a + b + c \geq abc + 2$$

Solution. Eliminating c gives

$$a + b + c - abc = a + b + (1 - ab)c = a + b + \frac{(1 - ab)(3 - ab)}{a + b}.$$

Put $x = \sqrt{ab}$. Then $a + b \geq 2x$, and since $1 < x^2 < 3$, $\frac{(1-ab)(3-ab)}{a+b} \geq \frac{(1-x^2)(3-x^2)}{2x}$.
It then suffices to prove that

$$2x + \frac{(1 - x^2)(3 - x^2)}{2x} \geq 2.$$

This last inequality follows from the arithmetic-geometric means inequality

$$2x + \frac{(1 - x^2)(3 - x^2)}{2x} = \frac{3 + x^4}{2x} = \frac{1}{2x} + \frac{1}{2x} + \frac{1}{2x} + \frac{x^3}{2} \geq 4\left(\frac{1}{16}\right)^{\frac{1}{4}} = 2.$$

ALG 7.

Let x, y, z be real numbers greater than -1. Prove that

$$\frac{1+x^2}{1+y+z^2}+\frac{1+y^2}{1+z+x^2}+\frac{1+z^2}{1+x+y^2}\geq 2.$$

Solution. We have $y\leq \frac{1+y^2}{2}$, hence

$$\frac{1+x^2}{1+y+z^2}\geq \frac{1+x^2}{1+z^2+\frac{1+y^2}{2}}$$

and the similar inequalities.

Setting $a=1+x^2, b=1+y^2, c=1+z^2$, it sufices to prove that

$$\frac{a}{2c+b}+\frac{b}{2a+c}+\frac{c}{2b+a}\geq 1 \qquad (1)$$

for all $a, b, c\geq 0$.

Put $A=2c+b, B=2a+c, C=2b+a$. Then

$$a=\frac{C+4B-2A}{9},\ b=\frac{A+4C-2B}{9},\ c=\frac{B+4A-2C}{9}$$

and (1) rewrites as

$$\frac{C+4B-2A}{A}+\frac{A+4C-2B}{B}+\frac{B+4A-2C}{C}\geq 9$$

and consequently

$$\frac{C}{A}+\frac{A}{B}+\frac{B}{C}+4\left(\frac{B}{A}+\frac{C}{B}+\frac{A}{C}\right)\geq 15.$$

As $A, B, C>0$, by $AM-GM$ inequality we have

$$\frac{A}{B}+\frac{B}{C}+\frac{C}{A}\geq 3\sqrt[3]{\frac{A}{B}\cdot\frac{B}{C}\cdot\frac{C}{A}}$$

and

$$\frac{B}{A}+\frac{C}{B}+\frac{A}{C}\geq 3,$$

and we are done.

Alternative solution for inequality (1).

By the Cauchy-Schwarz inequality,

$$\frac{a}{2c+b}+\frac{b}{2a+c}+\frac{c}{2b+a}=\frac{a^2}{2ac+ab}+\frac{b^2}{2ab+cb}+\frac{c^2}{2bc+ac}\geq\frac{(a+b+c)^2}{3(ab+bc+ca)}\geq 1.$$

The last inequality reduces immediately to the obvious $a^2+b^2+c^2\geq ab+bc+ca$.

ALG 8. Prove that there exist two sets $A = \{x, y, z\}$ and $B = \{m, n, p\}$ of positive integers greater than 2003 such that the sets have no common elements and the equalities $x + y + z = m + n + p$ and $x^2 + y^2 + z^2 = m^2 + n^2 + p^2$ hold.

Solution. Let ABC be a triangle with $BC = a$, $AC = b$ $AB = c$ and $a < b < c$. Denote by m_a, m_b, m_c the lengths of medianes drawing from the vertices A, B, C respectively. Using the formulas

$$4m_a^2 = 2(b^2 + c^2) - a^2,\ 4m_b^2 = 2(a^2 + c^2) - b^2,\ 4m_c^2 = 2(a^2 + b^2) - c^2$$

we obtain the relations

$$4m_a^2 + 4m_b^2 + 4m_c^2 = 3a^2 + 3b^2 + 3c^2,$$

$$(4m_a^2)^2 + (4m_b^2)^2 + (4m_c^2)^2 = (2b^2 + 2c^2 - a^2)^2 +$$
$$(2a^2 + 2c^2 - b^2)^2 + (2a^2 + 2b^2 - c^2)^2 = 9a^4 + 9b^4 + 9c^4 =$$
$$(3a^2)^2 + (3b^2)^2 + (3c^2)^2.$$

We put $A = \{4m_a^2, 4m_b^2, 4m_c^2\}$ and $B = \{3a^2, 3b^2, 3c^2\}$. Let $k \geq 1$ be a positive integer. Let $a = k + 1$, $b = k + 2$ and $c = k + 3$. Because

$$a + b = (k + 1) + (k + 2) = 2k + 3 > k + 3 = c,$$

a triangle with such length sides there exist. After the simple calculations we have

$$A = \{3(k + 1)^2 - 2, 3(k + 2)^2 + 4, 3(k + 3)^2 - 2\},$$

$$B = \{3(k + 1)^2, 3(k + 2)^2, 3(k + 3)^2\}.$$

It easy to prove that

$$x + y + z = m + n + p = 3[(k + 1)^2 + (k + 2)^2 + (k + 3)^2],$$

$$x^2 + y^2 + z^2 = m^2 + n^2 + p^2 = 9[(k + 1)^4 + (k + 2)^4 + (k + 3)^4].$$

>From the inequality $3(k + 1)^2 - 2 > 2003$ we obtain $k \geq 25$. For $k = 25$ we have an example of two sets

$$A = \{2026, 2191, 2350\}, \quad B = \{2028, 2187, 2352\}$$

with desired properties.

COM 1. In a group of 60 students: 40 speak English; 30 speak French; 8 speak all the three languages; the number of students that speak English and French but not German is equal to the sum of the number of students that speak English and German but not French plus the number of students that speak French and German but not English; and the number of students that speak at least 2 of those languages is 28. How many students speak:
a) German; **b)** only English; **c)** only German ?

Solution: We use the following notation.
E = # students that speak English, F = # students that speak French,
G = # students that speak German; m = # students that speak all the three languages,
x = # students that speak English and French but not German,
y = # students that speak German and French but not English,
z = # students that speak English and German but not French.
The conditions x+y=z and x+y+z+8=28, imply that z=x+y=10, i.e. 10 students speak German and French, but not English. Then: G + E – y – 8 + F – x – 8 – 10 = 60, implies that G + 70 – 36 = 60. Hence: **a)** G = 36; **b)** only English speak 40 – 10 – 8 = 22 students; **c)** the information given is not enough to find the number of students that speak only German. This number can be any one from 8 to 18.

Comment: There are some mistakes in the solution. The corrections are as follows:
1. The given condition is x=y+z (**not** x+y=z); thus x = y+z = 10.
2. From G + 70 – 36 = 60 one gets G = 26 (**not** G = 36).
3. One gets "only German speakers" as G - y - z - 8 = 8.
4. "Only English speakers" are E - x - z - 8 = 22 - z, so this number can not be determined.

COM 2 Natural numbers 1, 2, 3, ..., 2003 are written in an arbitrary sequence $a_1, a_2, a_3, \ldots a_{2003}$. Let $b_1 = 1a_1$, $b_2 = 2a_2$, $b_3 = 3a_3$, ..., $b_{2003} = 2003a_{2003}$, and B be the maximum of the numbers $b_1, b_2, b_3, \ldots, b_{2003}$.

a) If $a_1 = 2003$, $a_2 = 2002$, $a_3 = 2001$, ..., $a_{2002} = 2$, $a_{2003} = 1$, find the value of B.

b) Prove that $B \geq 1002^2$.

Solution: a) Using the inequality between the arithmetical and geometrical mean, we obtain that $b_n = n(2004-n) \leq \left(\frac{n+(2004-n)}{2}\right)^2 = 1002^2$ for $n = 1, 2, 3, \ldots, 2003$. The equality holds if and only if $n = 2004 - n$, i.e. $n = 1002$. Therefore, $B = b_{1002} = 1002 \times (2004 - 1002) = 1002^2$.

b) Let $a_1, a_2, a_3, \ldots a_{2003}$ be an arbitrary order of the numbers 1, 2, 3, ..., 2003. First, we will show that numbers 1002, 1003, 1004, ... , 2003 cannot occupy the places numbered 1, 2, 3, ... , 1001 only. Indeed, we have $(2003-1002)+1 = 1002$ numbers and 1002 places. This means that at least one of the numbers 1002, 1003, 1004, ..., 2003, say a_m, lies on a place which number m is greater than 1001. Therefore, $B \geq ma \geq 1002 \times 1002 = 1002^2$.

COM 3. Prove that amongst any 29 natural numbers there are 15 such that sum of them is divisible by 15.

Solution: Amongst any 5 natural numbers there are 3 such that sum of them is divisible by 3. Amongst any 29 natural numbers we can choose 9 groups with 3 numbers such that sum of numbers in every group is divisible by 3. In that way we get 9 natural numbers such that all of them are divisible by 3. It is easy to see that amongst any 9 natural numbers there are 5 such that sum of them is divisible by 5. Since we have 9 numbers, all of them are divisible by 3, there are 5 such that sum of them is divisible by 15.

COM 4.

n points are given in a plane, not three of them colinear. One observes that no matter how we label the points from 1 to n, the broken line joining the points $1, 2, 3, \ldots, n$ (in this order) do not intersect itself.

Find the maximal value of n.

Solution. Notice that $n = 4$ satisfies the condition. Indeed, for a concave quadrilateral, this can be checked immediately.

Then, observe that for $n \geq 5$ one can choose four points A, B, C, D such that $ABCD$ is a convex quadrilateral. The diagonals AC and BD intersect at a point, hence labeling A, B, C, D with $1, 2, 3, 4$ we reach a contradiction.

Thus, it is sufficient to proove that from five points we can select four that are vertices of a convex quadrilateral. Consider the convex hull of the five points set. If this is not a triangle we are done. If it is a triangle, then draw the line through the two points inside the triangle. This line meet exactly two sides of the triangle. Let A be the common vertex of these sides. Then the four remaining points solve the claim.

COM 5. If m is a number from the set $\{1, 2, 3, 4\}$ and each point of the plane is painted in red or blue, prove that in the plane there exists at least an equilateral triangle with the vertices of the same colour and with length side m.

Solution. Suppose that in the plane there no exists an equilateral triangle with the vertices of the same colour and length side $m = 1, 2, 3, 4$.

First assertion: we shall prove that in the plane there no exists a segment with the length 2 such that the ends and the midpoint of this segment have the same colour. Suppose that the segment XY with length 2 have the midpoint T such that the points X, Y, T have the same colour (for example, red). We construct the equilateral triangle XYZ. Hence, the point Z is blue. Let U and V be the midpoints of the segments XZ and YZ respectively. So, the points U and V are blue. We obtain a contradiction, because the equilateral triangle UVZ have three blue vertices.

Second assertion: in the same way we prove that in the plane there no exists a segment with the length 4 such that the ends and the midpoint of this segment have the same colour.

Consider the equilateral triangle ABC with length side 4 and divide it into 16 equilateral triangles with length sides 1. Let D be the midpoint of the segment AB. The vertices A, B, C don't have the same colour. WLOG we suppose that A and B are red and C is blue. So, the point D is blue too. We shall investigate the following cases:

a) The midpoints E and F of the sides AC and, respectively, BC are red. From the first assertion it follows that the midpoints M and N of the segments AE and, respectively, BF are blue. Hence, the equilateral triangle MNC have three blue vertices, a contradiction.

b) Let E is red and F is blue. The second one position of E and F is simmetrical. If P, K, L are the midpoints of the segments CF, AD, BD respectively, then by first assertion P is red, M is blue and N is red. This imply that K and L are blue. So, the segment KL with length 2 has the blue ends and blue midpoint, a contradiction.

c) If E and F are blue, then the equilateral triangle EFC has three blue vertices, a contradiction.

Hence, in the plane there exists at least an equilateral triangle with the vertices of the same colour and with length side m, where $m \in \{1, 2, 3, 4\}$.

Comment: The formulation of the problem suggests that one has to find 4 triangles, one for each m from the set $\{1, 2, 3, 4\}$ whereas the solution is for one m. A better formulation is:

Each point of the plane is painted in red or blue. Prove that in the plane there exists at least an equilateral triangle with the vertices of the same colour and with length side m, where m is some number from the set $\{1, 2, 3, 4\}$.

GEO 1. Is there a convex quadrilateral, whose diagonals divide it into four triangles, such that their areas are four distinct prime integers.

Solution. No. Let the areas of those triangles be the prime numbers p, q, r and t. But for the areas of the triangles we have pq=rt, where the triangles with areas p and q have only a common vertex. This is not possible for distinct primes.

GEO 2. Is there a triangle whose area is $12cm^2$ and whose perimeter is 12cm.

Solution. No. Let r be the radius of the inscribed circle. Then 12 = 6r, i.e. r=2cm. But the area of the inscribed circle is $4\pi>12$, and it is known that the area of any triangle is bigger than the area of its inscribed circle.

GEO 3.

Let G be the centroid of the the triangle ABC. Reflect point A across C at A'. Prove that G, B, C, A' are on the same circle if and only if GA is perpendicular to GC.

Solution. Observe first that $GA \perp GC$ if and only if $5AC^2 = AB^2 + BC^2$. Indeed,

$$GA \perp GC \Leftrightarrow \frac{4}{9}m_a^2 + \frac{4}{9}m_c^2 = b^2 \Leftrightarrow 5b^2 = a^2 + c^2$$

Moreover,

$$GB^2 = \frac{4}{9}m_b^2 = \frac{2a^2+2c^2-b^2}{9} = \frac{9b^2}{9} = b^2$$

hence $GB = AC = CA'$ (1). Let C' be the intersection point of the lines GC and AB. Then CC' is the middle line of the triangle ABA', hence $GC||BA'$. Consequently, $GCA'B$ is a trapezoid. From (1) we find that $GCA'B$ is isosceles, thus cyclic, as needed.

Conversely, since $GCA'B$ is a cyclic trapezoid, then it is also isosceles. Thus $CA' = GB$, which leads to (1).

Comment: An alternate proof is as follows:

Let M be the midpoint of AC. Then the triangles MCG and $MA'B$ are similar. So GC is parallel to $A'B$.

$GA \perp GC$ if and only if $GM = MC$. By the above similarity, this happen if and only if $A'C = GB$; if and only if the trapezoid is cyclic.

GEO 4. Triangle ABC is inscribed in a circle k. Let D, E, and F be the midpoints of the arcs of k, $\widehat{BC}$, $\widehat{CA}$, and $\widehat{AB}$ respectively, $A \notin \widehat{BC}$, $B \notin \widehat{CA}$, and $C \notin \widehat{AB}$. Let segment DE meets CB and CA in points G and H respectively, and segment DF meet BC and BA in points I and J respectively. Let M and N be the midpoints of the segments GH and IJ respectively.

a) Find the angles of triangle DMN in terms of the angles $\alpha = \measuredangle BAC$, $\beta = \measuredangle CBA$, and $\gamma = \measuredangle ACB$.

b) If O is the circumcentre of triangle DMN and P is the intersection point of AD and EF, prove that the points O, P, M and N are concircular.

Comment: Because of a compatibility problem, the signs for arcs and angles appear as squares.

Solution: a) Since $\widehat{BD} = \widehat{DC} = \alpha$, $\widehat{CE} = \widehat{EA} = \beta$, and $\widehat{AF} = \widehat{FB} = \gamma$, it follows that $\measuredangle EDF = \frac{1}{2}(\beta + \gamma) = \frac{1}{2}(180^0 - \alpha) = 90^0 - \frac{\alpha}{2}$, $\measuredangle FED = 90^0 - \frac{\beta}{2}$, $\measuredangle DFE = 90^0 - \frac{\gamma}{2}$.

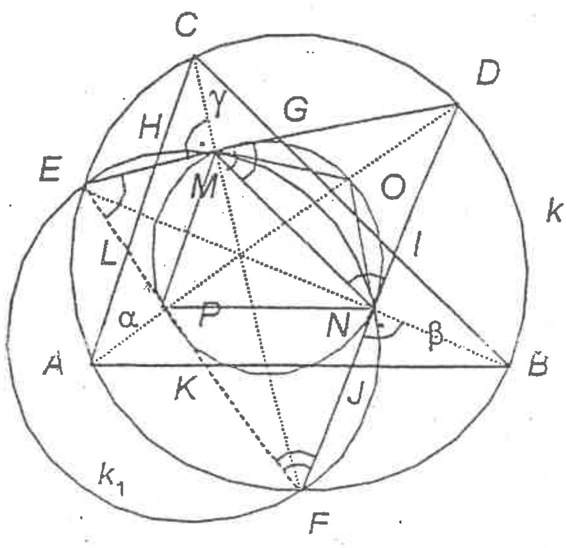
Using the properties of the angles whose vertices are inside a circle, we obtain that

$\measuredangle EHA = \frac{1}{2}\left(\widehat{AE} + \widehat{CD}\right) = \frac{1}{2}(\alpha + \beta) = \frac{1}{2}\left(\widehat{BD} + \widehat{CE}\right) = \measuredangle CGE$.

On the other hand, $\measuredangle EHA = \measuredangle CHG$. Therefore ΔCHG is isosceles ($CH=CG$). Since CF is an angle-bisector, the midpoint M of HG lies on CF. Also, CM is an altitude in ΔCHG. Therefore, $\measuredangle EMF = 90^0$. The same way we prove that $\measuredangle FNE = 90^0$. It follows that points E, F, N, and M lie on a circle k_1 (whose diameter is EF). Therefore,

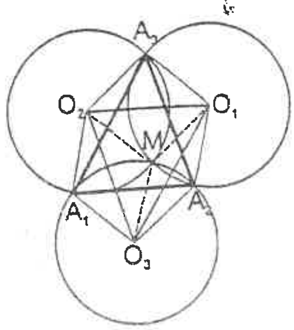
$\measuredangle DNM = \measuredangle DEF = 90^0 - \frac{\beta}{2}$, $\measuredangle DMN = \measuredangle DFE = 90^0 - \frac{\gamma}{2}$.

b) Let $AB \cap EF = K$ and $AC \cap EF = L$. As in a) we can prove that $\measuredangle APK = \measuredangle APL = 90^0$, $\measuredangle FPN = 90^0 - \frac{\alpha}{2}$ and $\measuredangle EPM = 90^0 - \frac{\alpha}{2}$. Since $\measuredangle AKP = \measuredangle ALP = 90^0 - \frac{\alpha}{2}$ we obtain $AB \parallel PN$ and $AC \parallel PM$. Hence $\measuredangle MPN = \measuredangle BAC = \alpha$. Since DMN is acute angle triangle (i.e. O is interior point) and $\measuredangle MDN = 90^0 - \frac{\alpha}{2}$ we have $\measuredangle MON = 180^0 - \alpha$. Therefore $\measuredangle MON + \measuredangle MPN = 180^0$ i.e. the points O, P, M and N are concircular.



GEO 5. Let three congruent circles intersect in one point M and A_1, A_2 and A_3 be the other intersection points for those circles. Prove that M is a orthocenter for a triangle $A_1A_2A_3$.

Solution: The quadrilaterals $O_3MO_2A_1$, $O_3MO_1A_2$ and $O_1MO_2A_3$ are rombes. Therefore, $O_2A_1 \parallel MO_3$ and $MO_3 \parallel O_1A_2$, which imply $O_2A_1 \parallel O_1A_2$. Because $O_2A_1 = O_3M = O_1A_2$ the quadrilateral $O_2A_1A_2O_1$ is parallelogram and then $A_1A_2 \parallel O_1O_2$ and $A_1A_2 = O_1O_2$. Similary, $A_2A_3 \parallel O_2O_3$ and $A_2A_3 = O_2O_3$; $A_3A_1 \parallel O_3O_1$ and $A_3A_1 = O_3O_1$. The triangles $A_1A_2A_3$ and $O_1O_2O_3$ are congruent.



Since $A_3M \perp O_1O_2$ and $O_1O_2 \parallel A_1A_2$ we infere $A_3M \perp A_1A_2$. Similary, $A_2M \perp A_1A_3$ and $A_1M \perp A_2A_3$. Thus, M is the orthocenter for the triangle $A_1A_2A_3$.

GEO-6.

Consider an isosceles triangle ABC with $AB = AC$. A semicircle of diameter EF, lying on the side BC, is tangent to the lines AB and AC at M and N, respectively. The line AE intersects again the semicircle at point P.

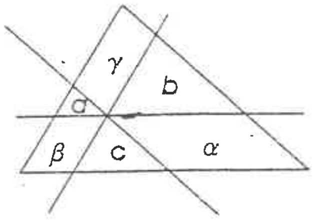
Prove that the line PF passes through the midpoint of the chord MN.

Solution. Let O be the center of the semicircle and let R be the midpoint of MN. It is obvious that MN is perpendicular to AO at point R. Since $\angle ANO$ is right, then from the leg theorem we have $AN^2 = AR \cdot AO$. From the power of a point theorem,

$$AP \cdot AE = AN^2 = AM^2 = AR \cdot AO.$$

Using the same theorem we infer that points P, R, O and E are concyclic, hence $\angle RPE$ is right. As $\angle FPE$ is also a right angle, the conclusion follows.

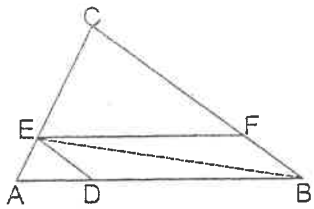
GEO 7. Through a interior point of a triangle, three lines parallel to the sides of the triangle are constructed. In that way the triangle is divided on six figures, areas equal $a, b, c, \alpha, \beta, \gamma$ (see the picture).



Prove that
$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma} \geqslant \frac{3}{2}.$$

Solution: We will prove the inequality in two steps. First one is the following

Lemma: Let ABC be a triangle, E arbitrary point on the side AC. Parallel lines to AB and BC, drown through E meet sides BC and AB in points F and D respectively. Then: $P_{BDEF} = 2\sqrt{P_{ADE} \cdot P_{EFC}}$ (P_X is area for the figure X).



The triangles ADE and EFC are similar. Then:
$$\frac{P_{BDEF}}{2P_{ADE}}=\frac{P_{BDE}}{P_{ADE}}=\frac{BD}{AD}=\frac{EF}{AD}=\frac{\sqrt{P_{EFC}}}{\sqrt{P_{ADE}}}.$$
Hence, $P_{BDEF} = 2\sqrt{P_{ADE} \cdot P_{EFC}}$.

Using this lemma one has $\alpha = 2\sqrt{bc}$, $\beta = 2\sqrt{ac}$, $\gamma = 2\sqrt{ab}$. The GM-AM mean inequality provides
$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma} \geqslant 3\sqrt[3]{\frac{abc}{\alpha\beta\gamma}} = 3\sqrt[3]{\frac{abc}{2^3\sqrt{a^2b^2c^2}}} = \frac{3}{2}.$$

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Deputy Leader:	Ivan Tonov
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	Tzvetelina Kirilova Tzeneva
	Vladislav Vladilenon Petkov
	Alexander Sotirov Bikov
	Deyan Stanislavov Simeonov
	Anton Sotirov Bikov

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	Yiannis Ioannides
	Anastasia Solea
	Nansia Drakou
	Michalis Rossides
	Domna Fanidou
Observer:	Myrianthi Savvidou

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	Faethontas Karagiannopoulos
	Stefanos Kasselakis
	Fragiskos Koufogiannis
	Efrosyni Sarla

REPUBLIC of MOLDOVA

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Deputy Leader:	Ana Costas
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	Andrei Frimu
	Mihaela Rusu
	Vladimir Vanovschi
	Dan Vieru
	Alexandru Zamorzaev

ROMANIA

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	Adrian Zahariuc
	Cristian Talau
	Beniamin Bogosel
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	Ahmet Kabakulak
	Türkü Çobanoğlu
	Burak Sağlam
	İbrahim Çimentepe
	Hale Nur Kazaçeşme

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Deputy Leader:	Marija Stanic
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	Jevremovic Marko
	Djoric Milos
	Lukic Dragan
	Andric Jelena
	Pajovic Jelena

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Deputy Leader:	Deniz Ahçıhoca
Contestants:	Havva Yeşildağlı
	Çağıl Şentip
	Buse Uslu
	Ali Yılmaz
	Demirhan Çetereisi
	Yakup Yıldırım

СЕДМА МАКЕДОНСКА
МАТЕМАТИЧКА ОЛИМПИЈАДА
ЗА УЧЕНИЦИ ДО 15,5 ГОДИНИ

MJMO 2003

СКОПЈЕ 2003

1.Prove that 7^n-1 is not divisible by 6^n-1 for any positive integer n.

2. 2003 denars were divided in several bags and the bags were placed in several pockets. The number of bags is greater than the number of denars in each pocket. Is it true that the number of pockets is greater than the number of denars in one of the bags?

3. In the triangle ABC, R and r are the radii of the circumcircle and the incircle, respectively; a is the longest side and h is the shortest altitude. Prove that $R/r > a/h$.

4. Prove that for all positive numbers x, y, z such that $x+y+z=1$ the following inequality holds

$$\frac{x^2}{1+y}+\frac{y^2}{1+z}+\frac{z^2}{1+x}\leq 1.$$

5.Is it possible to cover a 2003×2003 board with 1×2 dominoes placed horizontally and 1×3 threeminoes placed vertically?

THE 47-th MATHEMATICAL OLYMPIAD OF REPUBLIC OF MOLDOVA

Chişinău, March 9-12, 2003

7.1 Let $m > n$ be pozitive integers. For every positive integers k we define the number $a_k = (\sqrt{5}+2)^k + (\sqrt{5}-2)^k$. Show that $a_{m+n} + a_{m-n} = a_m \cdot a_n$.

7.2 Find all five digits numbers $\overline{abcde}$, written in decimal system, if it is known that $\overline{abcde} - \overline{ebcda} = 69993$, $\overline{bcd} - \overline{dcb} = 792$, $\overline{bc} - \overline{cb} = 72$.

7.3 In the triangle ABC with semiperemeter p the points M, N and P lie on the sides $(BC), (CA)$ and (AB) respectively. Show that $p < AM + BN + CP < 3p$.

7.4 Let a and b be positive integer such, that $a + b \le 10$. Find all pairs (a, b) such, that the fraction $(2n + a)/(5n + b)$ are irreducible for every natural number n.

7.5 A given rectangular table has at least one column and at least one line. He is full completed by the first positive integers, written consecutively from the left to the right and begining to the first line. It is known, that the number 170 is written on the middle line and in the same column with him on the last line is written the number 329. How much numbers are written in the given table.

7.6 Prove that for every positive integer n the number $a = (2n+1)^5 - 2n - 1$ is divisible by 240.

7.7 In the square $ABCD$ the point N is the middle point of the side $[AB]$ and the point M lies on the diagonal (AC) so that $AC = 4CM$. Prove that the angle DMN is right.

7.8 The real numbers $x_1, x_2, \ldots, x_{2003}$ satisfy the relations $x_1/1 = x_2/2 = x_3/3 = \ldots = x_{2003}/2003$ and $\sqrt{1^2 + 2^2 + \ldots + 2003^2} + \sqrt{x_1^2 + x_2^2 + \ldots + x_{2003}^2} = \sqrt{(1+x_1)^2 + (2+x_2)^2 + \ldots + (2003 + x_{2003})^2}$. Prove that $x_i \ge 0$ for every $i = 1, 2, \ldots, 2003$.

8.1 Calculate the sum

$$\frac{2^4+2^2+1}{2^7-2} + \frac{3^4+3^2+1}{3^7-3} + \ldots + \frac{2003^4+2003^2+1}{2003^7-2003} + \frac{1}{2\cdot 2003\cdot 2004}.$$

8.2 Let $[AB]$ be a segment and σ be one of the halfplane, determined by the straight line AB. The segments $[AP]$ and $[BQ]$ with integer lengths are situated in σ and are perpendicular to the straight line AB. The intersection point M of the straight lines AQ and BP is distanced at 8 units to the straight line AB. Find the lengths of the segments $[AP]$ and $[BQ]$, if it is known that the triangle BQM has a greatest area.

8.3 Let ABC be an acuteangled triangle such that $m(\angle ACB) \ne 45°$. The points M and N are the feets of the altitudes, drawn from the vertices A and B respectively. The points P and Q lyes on the halfstraight lines $(MA$ and $(NB$ respectively so that $MP = MB$ and $NQ = NA$. Prove that the straight lines PQ and MN are parallel.

8.4 The equation $x^{13} - x^{11} + x^9 - x^7 + x^5 - x^3 + x - 2 = 0$ has a real solution x_0. Show that $[x_0^{14}] = 3$, where $[a]$ is a integral part of the real number a.

8.5 Prove that every positive integer number $n \ge 3$ can be written as a sum of at least two consecutive positive integers if and only if he is not a power of the number 2.

8.6 The prime number p has the following property: the remainder r of the division of p by 210 is a composite number which can be represented us a sum of two perfect squares. Find the number r.

8.7 Through the arbitrary point of the triangle ABC construct (explain the steps of the construction) a straight line which divides the triangle ABC in two parts so that the ratio of they areas is equal to 3/4.

8.8 Let x be a real number. Find the smallest value of the expression $\sqrt{x^2+2x+4} + \sqrt{x^2 - \sqrt{3}x + 1}$.

9.1 In the space a geometrical configuration, which include n $(n \ge 3)$ distinct points, is given. A point A of this configuration has the following properties: if A is excluded from the configuration, then among the remaining points there are no colinear points; after the elimination of A from the configuration the number of all straight lines, that were constructed through any 2 points of the configuration, is lowered by 1/15 part. Find the value of n.

9.2 Let $x^2 + bx + c = 0$ be the equation, where b and c are two consecutive triangular numbers and $c > b \ge 10$. Prove that this equation has two irrational solutions. (The number m is triangular, if $m = n(n-1)/2$ for a certain positive integer $n \ge 1$).

9.3 The distinct points M and N lie on the hypotenuse (AC) of the right isosceles triangle ABC so that $M \in (AN)$ and $MN^2 = AM^2 + CN^2$. Prove that $m(\angle MBN) = 45°$.

9.4 Find all the functions $f : N^* \to N^*$ which verify the relation $f(2x+3y) = 2f(x) + 3f(y) + 4$ for every positive integers $x, y \ge 1$.

9.5 The numbers $a_1, a_2, \ldots, a_n$ are the first n positive integers with the property that the number $8a_k + 1$ is a perfect square for every $k = 1, 2, \ldots, n$. Find the sum $S_n = a_1 + a_2 + \ldots + a_n$.

9.6 Find all real solutions of the equation $x^4 + 7x^3 + 6x^2 + 5\sqrt{2003}\,x - 2003 = 0$.

9.7 The side lengths of the triangle ABC satisfy the relations $a > b \geq 2c$. Prove that the altitudes of the triangle ABC can not be the sides of any triangle.

9.8 The base of a pyramid is a convex polygon with 9 sides. All the lateral edges of the pyramid and all the diagonals of the base are coloured in a random way in red or blue. Prove that there exist at least three vertices of the pyramid which belong to a triangle with the sides coloured in the same colour.

10.1 Find all prime numbers a, b and c for which the equality $(a-2)! + 2b! = 22c - 1$ holds.

10.2 Solve the system $x + y + z + t = 6$, $\sqrt{1-x^2} + \sqrt{4-y^2} + \sqrt{9-z^2} + \sqrt{16-t^2} = 8$.

10.3 In the scalen triangle ABC the points A_1 and B_1 are the bissectrices feets, drawing from the vertices A and B respectively. The straight line A_1B_1 intersect the line AB at the point D. Prove that one of the angles $\angle ACD$ or $\angle BCD$ is obtuze and $m(\angle ACD) + m(\angle BCD) = 180°$.

10.4 Let $a > 1$ be not integer number and $a \neq \sqrt[p]{q}$ for every positive integers $p \geq 2$ and $q \geq 1$, $k = [\log_a n] \geq 1$, where $[x]$ is the integral part of the real number x. Prove that for every positive integer $n \geq 1$ the equality

$$[\log_a 2] + [\log_a 3] + \ldots + [\log_a n] + [a] + [a^2] + \ldots + [a^k] = nk,$$

holds.

10.5 The rational numbers p, q, r satisfy the relation $pq + pr + qr = 1$. Prove that the number $(1+p^2)(1+q^2)(1+r^2)$ is a square of any rational number.

10.6 Let $n \geq 1$ be a positive integer. For every $k = 1, 2, \ldots, n$ the functions $f_k : R \to R$, $f_k(x) = a_kx^2 + b_kx + c_k$ with $a_k \neq 0$ are given. Find the greatest possible number of parts of the rectangular plane xOy which can be obtained by the intersection of the graphs of the functions f_k $(k = 1, 2, \ldots, n)$.

10.7 The circle with the center O is tangent to the sides $[AB]$, $[BC]$, $[CD]$ and $[DA]$ of the convex quadrilateral $ABCD$ at the points M, N, K and L respectively. The straight lines MN and AC are parallel and the straight line MK intersect the line LN at the point P. Prove that the points A, M, P, O and L are concyclic.

10.8 Find all integers n for which the number $\log_{2n-1}(n^2+2)$ is rational.

11.1 Let $a, b, c, d \geq 1$ be arbitrary positive numbers. Prove that the equations system $ax - yz = c$, $bx - yt = -d$ has at least a solution (x, y, z, t) in positive integers.

11.2 The sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ satisfy the conditions $(1+\sqrt{3})^{2n+1} = a_n + b_n\sqrt{3}$ and $a_n, b_n \in Z$. Find the recurrent relation for each of the sequences (a_n) and (b_n).

11.3 The triangle ABC is rightangled in A, $AC = b$, $AB = c$ and $BC = a$. The halfstraight line $(Az$ is perpendicular to the plane (ABC), $M \in (Az$ so that α, β, γ are the mesures of the angles, formed by the edges MB, MC and the plane (MBC) with the plane (ABC) respectively. In the set of the triangular pyramids $MABC$ on consider the pyramids with the volumes V_1 and V_2 which satisfy the relations $\alpha + \beta + \gamma = \pi$ and $\alpha + \beta + \gamma = \pi/2$ respectively. Prove the equality $(V_1/V_2)^2 = (a+b+c)(1/a + 1/b + 1/c)$.

11.4 Find all the functions $f : [0; +\infty) \to [0; +\infty)$ which satisfy the conditions: : $f(xf(y)) \cdot f(y) = f(x+y)$ for every $x, y \in [0; +\infty)$; $f(2) = 0$; $f(x) \neq 0$ for every $x \in [0; 2)$.

11.5 Let $0 < a < b$ be real positive numbers. Prove that the equation $[(a+b)/2]^{x+y} = a^x b^y$ has at least a solution in the set $(a; b) \times (a; b)$.

11.6 Each of the plane angles of the vertex V of the tetrahedron $VABC$ has the measure equal to $60°$. Prove that $VA + VB + VC \leq AB + BC + CA$. When the equality holds?

11.7 The plane α is tangent in the points A_1, A_2 and A_3 to three spheres with different radii R_1, R_2 and R_3 respectively, situated in the same halfspace two by two exteriorly. The plane β is parallel to the plane α and intersect all three spheres so that the circles D_1, D_2 and D_3 are obtained. Find the distance between the planes α and β so that the sum of the volumes V_1, V_2 and V_3 of the cones with the bases D_1, D_2, D_3 and the vertices A_1, A_2, A_3 respectively, will be the greatest.

11.8 For every positive integer $n \geq 1$ we define the matrix $A_n = (a_{ij})_{1\leq i,j\leq n}$, where $a_{ij} = \max(i,j)/\min(i,j)$, $1 \leq i, j \leq n$. Calculate the determinant of the matrix A_n.

12.1 Prove that $\lim_{n\to+\infty} \ln(1 + 2e + 4e^4 + 6e^9 + \ldots + 2ne^{n^2})/n^2 = 1$.

12.2 For every positive integer $n \geq 2$ the affirmation P_n : "If the derivative $P'(X)$ of a polynomial $P(X)$ of degree n with real coefficients has $n-1$ real distinct roots, then there exists a real constant C such that the equation $P(x) = C$ has n real distinct solutions" is considered. Prove that P_4 is true. Is the affirmation P_5 true? Prove the answer.

12.3 In the circle with radius R the distinct chords $[AB]$ and $[CD]$ are concurrent and form an acute angle with mesure α. Prove that $AB + CD > 2R\sin\alpha$.

12.4 The real numbers α, β, γ satisfy the relations $\sin\alpha + \sin\beta + \sin\gamma = 0$ and $\cos\alpha + \cos\beta + \cos\gamma = 0$. Find all positive integers $n \geq 0$ for which $\sin(n\alpha + \pi/4) + \sin(n\beta + \pi/4) + \sin(n\gamma + \pi/4) = 0$.

12.5 For every positive integer $n \geq 1$ we define the polynomial $P(X) = X^{2n} - X^{2n-1} + \ldots - X + 1$, Find the remainder of the division of the polynomial $P(X^{2n+1})$ by the polynomial $P(X)$.

12.6 Fie $n \in N$. Find all the primitives of the function

$$f : R \to R, \quad f(x) = \frac{x^3 - 9x^2 + 29x - 33}{(x^2 - 6x + 10)^n}.$$

12.7 In a rectangular system xOy the graph of the function $f : R \to R$, $f(x) = x^2$ is drawn. The ordered triple B, A, C has distinct points on the parabola, the point $D \in (BC)$ such that the straight line AD is parallel to the axis Oy and the triangles BAD and CAD have the areas s_1 and s_2 respectively. Find the length of the segment $[AD]$.

12.8 Let $(F_n)_{n \in N^*}$ be the Fibonacci sequence so that: $F_1 = 1$, $F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ for every positive integer $n \geq 2$. Shown that $F_n < 3^{n/2}$ and calculate the limit $\lim_{n\to\infty}(F_1/2 + F_2/2^2 + \ldots + F_n/2^n)$.

The first selection test for IMO 2003 and BMO 2003, March 12, 2003

B1. Each side of the arbitrary triangle is divided into 2002 congruent segments. After that each interior division point of the side is joined with opposite vertex. Prove that the number of obtained regions of the triangle is divisible by 6.

B2. The positive real numbers x, y and z satisfy the relation $x + y + z \geq 1$. Prove the inequality

$$\frac{x\sqrt{x}}{y+z} + \frac{y\sqrt{y}}{x+z} + \frac{z\sqrt{z}}{x+y} \geq \frac{\sqrt{3}}{2}.$$

B3. The quadrilateral $ABCD$ is inscribed in the circle with center O, the points M and N are the middle points of the diagonals $[AC]$ and $[BD]$ respectively and P is the intersection point of the diagonals. It is known that the points O, M, N şi P are distinct. Prove that the points O, M, B and D are concyclic if and only if the points O, N, A and C are concyclic.

B4. Prove that the equation $1/a + 1/b + 1/c + 1/(abc) = 12/(a + b + c)$ has many solutions (a, b, c) in strictly positive integers.

The second selection test for IMO 2003, March 22, 2003

B5. Let $n \geq 1$ be positive integer. Find all polynomials of degree $2n$ with real coefficients

$$P(X) = X^{2n} + (2n - 10)X^{2n-1} + a_2X^{2n-2} + \ldots + a_{2n-2}X^2 + (2n - 10)X + 1,$$

if it is known that they have positive real roots.

B6. The triangle ABC has the semiperimeter p, the circumradius R, the inradius r and l_a, l_b, l_c are the lengths of internal bissecticies, drawing from the vertices A, B and C respectively. Prove the inequality $l_al_b + l_bl_c + l_cl_a \leq p\sqrt{3r^2 + 12Rr}$.

B7. The points M and N are the tangent points of the sides $[AB]$ and $[AC]$ of the triangle ABC to the incircle with the center I. The internal bissectrices, drawn from the vertices B and C, intersect the straight line MN at points P and Q respectively. If F is the intersection point of the swtraight lines CP and BQ, then prove that the straight lines FI and BC are perpendicular.

B8. Let $n \geq 4$ be the positive integer. On the checkmate table with dimensions $n \times n$ we put the coins. One consider the diagonal of the table each diagonal with at least two unit squares. What is the smallest number of coins put on the table so that on the each horizontal, each vertical and each diagonal there exists at least one coin. Prove the answer.

The third selection test for IMO 2003, March 23, 2003

B9. Let $n \geq 1$ be positive integer. A permutation $(a_1, a_2, \ldots, a_n)$ of the numbers $(1, 2, \ldots, n)$ is called quadratique if among the numbers a_1, $a_1 + a_2$, $\ldots, a_1 + a_2 + \ldots + a_n$ there exist at least a perfect square. Find the greatest number n, which is less than 2003, such that every permutation of the numbers $(1, 2, \ldots, n)$ will be quadratique.

B10. The real numbers $a_1, a_2, \ldots, a_{2003}$ satisfy simultaneously the relations: $a_i \geq 0$ for all $i = 1, 2, \ldots, 2003$; $a_1 + a_2 + \ldots + a_{2003} = 2$; $a_1a_2 + a_2a_3 + \ldots + a_{2003}a_1 = 1$. Find the smallest value of the sum $a_1^2 + a_2^2 + \ldots + a_{2003}^2$.

B11. The arbitrary point M on the plane of the triangle ABC does not belong on the straight lines AB, BC and AC. If S_1, S_2 and S_3 are the areas of the triangles AMB, BMC and AMC respectively, find the geometrical locus of the points M which satisfy the relation $(MA^2+MB^2+MC^2)^2 = 16(S_1^2+S_2^2+S_3^2)$.

B12. Let $n \geq 1$ be a positive integer. A square table of dimensions $n \times n$ is full arbitrarly completed by the numbers $1, 2, \ldots, n^2$ so that every number appear exactly once in the table. From each line one select the smallest number and the greatest of them is denote by x. From each column one select the greatest number and the smallest of them is denote by y. The table is called equilibrated if $x = y$. How match equilibrated tables there exist?

The first selection test for JBMO 2003, April 12, 2003

JB1. Let $n \geq 2003$ be a positive integer such that the number $1 + 2003n$ is a perfect square. Prove that the number $n + 1$ is equal to the sum of 2003 positive perfect squares.

JB2. The positive real numbers a, b, c satisfy the relation $a^2 + b^2 + c^2 = 3abc$. Prove the inequality

$$\frac{a}{b^2c^2} + \frac{b}{c^2a^2} + \frac{c}{a^2b^2} \geq \frac{9}{a+b+c}.$$

JB3. The quadrilateral $ABCD$ with perpendicular diagonals is inscribed in the circle with center O, the points M and N are the middle points of the sides $[BC]$ and $[CD]$ respectively. Find the value of the ratio of areas of the figures $OMCN$ and $ABCD$.

JB4. Let m and n be the arbitrary digits of the decimal system and a, b, c be the positive distinct integers of the form $2^m \cdot 5^n$. Find the number of the equations $ax^2 - 2bx + c = 0$, if it is known that each equation has a single real solution.

The second selection test for JMBO 2003, April 13, 2003

JB5. Prove that each positive integer is equal to a difference of two positive integers with the same number of the prime divisors.

JB6. The real numbers x and y satisfy the equalities

$$\sqrt{3x}\left(1 + \frac{1}{x+y}\right) = 2, \qquad \sqrt{7y}\left(1 - \frac{1}{x+y}\right) = 4\sqrt{2}.$$

Find the numerical value of the ratio y/x.

JB7. The triangle ABC is isosceles with $AB = BC$. The point F on the side $[BC]$ and the point D on the side $[AC]$ are the feets of the internal bissectrix drawn from A and altitude drawn from B respectively so that $AF = 2BD$. Find the measure of the angle ABC.

JB8. In the rectangular coordinate system every point with integer coordinates is called laticeal point. Let $P_n(n, n+5)$ be a laticeal point and denote by $f(n)$ the number of laticeal points on the open segment (OP_n), where the point $O(0,0)$ is the coordinates system origine. Calculate the number $f(1) + f(2) + f(3) + \ldots + f(2002) + f(2003)$.

7th Junior Balkan Mathematical Olympiad
20 - 25 June, 2003 İzmir - Turkey

English Version

1. Let n be a positive integer. A number A consists of $2n$ digits, each of which is 4; and a number B consists of n digits, each of which is 8. Prove that $A+2B+4$ is a perfect square.

2. Suppose there are n points in a plane no three of which are collinear with the following property:

   If we label these points as $A_1, A_2, \ldots, A_n$ in any way whatsoever, the broken line $A_1 A_2 \ldots A_n$ does not intersect itself.

   Find the maximal value that n can have.

3. Let k be the circumcircle of the triangle ABC. Consider the arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CA}$ such that $C \notin \overset{\frown}{AB}$, $A \notin \overset{\frown}{BC}$, $B \notin \overset{\frown}{CA}$. Let D, E and F be the midpoints of the arcs $\overset{\frown}{BC}$, $\overset{\frown}{CA}$, $\overset{\frown}{AB}$, respectively. Let G and H be the points of intersection of DE with CB and CA; let I and J be the points of intersection of DF with BC and BA, respectively. Denote the midpoints of GH and IJ by M and N, respectively.

   a) Find the angles of the triangle DMN in terms of the angles of the triangle ABC.

   b) If O is the circumcentre of the triangle DMN and P is the intersection point of AD and EF, prove that O, P, M and N lie on the same circle.

4. Let x, y, z be real numbers greater than -1. Prove that

$$\frac{1+x^2}{1+y+z^2}+\frac{1+y^2}{1+z+x^2}+\frac{1+z^2}{1+x+y^2}\geq 2.$$

Time allowed: 4 ½ hours.
Each question is worth 10 points.

7th Junior Balkan Mathematical Olympiad
20 - 25 June, 2003 İzmir - Turkey

Romanian Version

A1

1. Fie n un număr natural nenul.. Un număr A conţine $2n$ cifre, fiecare fiind 4; şi un număr B conţine n cifre, fiecare fiind 8. Demonstraţi că $A+2B+4$ este un pătrat perfect.

Macedonia
Slavica Grkovska.

C4

2. Fie n puncte în plan, oricare trei necoliniare, cu proprietatea:

   oricum am numerota aceste puncte $A_1, A_2, \ldots, A_n$, linia frântă $A_1 A_2 \ldots A_n$ nu se autointersectează.

   Găsiţi valoarea maximă a lui n.

România – Şerbănescu

3. Fie k cercul circumscris triunghiului ABC. Fie arcele $\widehat{AB}, \widehat{BC}, \widehat{CA}$ astfel încât $C \notin \widehat{AB}$, $A \notin \widehat{BC}$, $B \notin \widehat{CA}$ şi D, E, F mijloacele acestor arce. Fie G, H punctele de intersecţie ale lui DE cu CB, CA; fie I, J punctele de intersecţie ale lui DF cu BC, BA. Notăm mijloacele lui GH, IJ cu M, respectiv N.

   a) Găsiţi unghiurile triunghiului DMN în funcţie de unghiurile triunghiului ABC.

   b) Dacă O este circumcentrul triunghiului DMN şi P este intersecţia lui AD cu EF, arătaţi că O, P, M şi N aparţin unui acelaşi cerc.

Bulgaria - Ch. Lozanov

4. Fie x, y, z numere reale mai mari decât -1. Demonstraţi că:

$$\frac{1+x^2}{1+y+z^2}+\frac{1+y^2}{1+z+x^2}+\frac{1+z^2}{1+x+y^2} \geq 2.$$

România – Panaitopol

Timp de lucru: 4 ore şi jumătate.
Fiecare problemă este notată cu 10 puncte

Question 1

I. To do a special case $n \geq 2$.

II. To assert that $A + 2B + 4 = (\underbrace{6...6}_{n-1}8)^2$.

III. To observe that $A = 4 \times \frac{10^{2n} - 1}{9}$ and $B = 8 \times \frac{10^n - 1}{9}$.

IV. To observe that $A = 3^2 \times (\underbrace{2...2}_{n})^2 + 4 \times (\underbrace{2...2}_{n})$ or $A = \left(\frac{3B}{4}\right)^2 + B$.

I $\rightarrow$ 1 point
I + II $\rightarrow$ 2 points
III $\rightarrow$ 4 points **or** **IV** $\rightarrow$ 5 points

Question 2

I. To claim $n = 4$ with example for $n = 4$.

II. To show impossibility of the case when the set of points includes 4 points that form a convex quadrilateral.

III. To show that every set of $n \geq 5$ points contains 4 points forming a convex quadrilateral.

I $\rightarrow$ 2 points
II $\rightarrow$ 1 point
III $\rightarrow$ 4 points
II + III $\rightarrow$ 7 points

Question 3

Part a

I. Computing the angles of the triangle DEF.

II. Observing that the lines $CF \perp DE$ and that $BE \perp DF$.

I $\rightarrow$ 1 point
I + II $\rightarrow$ 3 points
Only Part a $\rightarrow$ 6 points

Part b

III. Completing the figure by drawing EF.

Part a + III $\rightarrow$ 7 points
Only Part b $\rightarrow$ 6 points

Question 4

I. To observe that $y \leq \frac{y^2 + 1}{2}$.

II. To observe that $1 + y + z^2 > 0$ and to obtain $\frac{1 + x^2}{1 + y + z^2} \geq \frac{1 + x^2}{1 + z^2 + \frac{1 + y^2}{2}}$.

III. To reduce to $\frac{C + 4B - 2A}{A} + \frac{A + 4C - 2B}{B} + \frac{B + 4A - 2C}{C} \geq 9$.

I $\rightarrow$ 1 point
I + II $\rightarrow$ 3 points
I + II + III $\rightarrow$ 5 points

Coordination Schedule

	Q1	Q2	Q3	Q4
09:30	MCD	ROM	BUL	MOL
10:00	YUG	CYP	HEL	TUR B
10:30	BUL	MCD	CYP	TUR*
11:00	TUR B	HEL	ROM	YUG
11:30	HEL	YUG	TUR B	CYP
14:00	MOL	TUR*	MCD	BUL
14:30	ROM	MOL	TUR	MCD
15:00	CYP	TUR B	YUG	HEL
15:30	TUR	BUL	MOL	ROM

*: Coordination of the home country

SCORES

1	ROM-6	Adrian Zahariuc	40	First Prize
2	ROM-3	Dragos Michnea	40	First Prize
3	MOL-6	Alexandru Zamorzaev	39	First Prize
4	MOL-1	Iurie Boreico	38	First Prize
5	ROM-5	Lucian Turea	38	First Prize
6	ROM-4	Cristian Talau	37	First Prize
7	BUL-4	Vladislav Vladilenon Petkov	33	Second Prize
8	HEL-1	Theodosios Douvropoulos	32	Second Prize
9	BUL-1	Alexander Sotirov Bikov	31	Second Prize
10	BUL-2	Anton Sotirov Bikov	31	Second Prize
11	TUR-4	Hale Nur Kazaçeşme	31	Second Prize
12	TUR-6	Sait Tunç	31	Second Prize
13	BUL-5	Deyan Stanislavov Simeonov	30	Second Prize
14	HEL-3	Faethontas Karagiannopoulos	30	Second Prize
15	MCD-5	Maja Tasevska	29	Second Prize
16	ROM-2	Sebastian Dumitrescu	29	Second Prize
17	BUL-6	Tzvetelina Kirilova Tzeneva	29	Second Prize
18	BUL-3	Asparuh Vladislavov Hriston	28	Second Prize
19	TUR-5	Burak Sağlam	24	Third Prize
20	TUR-1	İbrahim Çimentepe	23	Third Prize
21	YUG-4	Jevremovic Marko	22	Third Prize
22	YUG-1	Lukic Dragan	22	Third Prize
23	ROM-1	Beniamin Bogosel	21	Third Prize
24	YUG-5	Djoric Milos	21	Third Prize

25	MOL-4	Vladimir Vanovschi	21	Third Prize
26	YUG-2	Andric Jelena	19	Third Prize
27	YUG-6	Radojevic Mladen	19	Third Prize
28	MCD-4	Viktor Simjanovski	17	Third Prize
29	HEL-6	Efrosyni Sarla	16	Third Prize
30	TUR-2	Türkü Çobanoğlu	13	Third Prize
31	YUG-3	Pajovic Jelena	12	Third Prize
32	MCD-2	Aleksandar Iliovski	11	Third Prize
33	MCD-6	Tanja Velkova	11	Third Prize
34	MOL-2	Andrei Frimu	10	Honorary Mention
35	MOL-5	Dan Vieru	10	Honorary Mention
36	MCD-3	Oliver Metodijev	10	Honorary Mention
37	HEL-4	Stefanos Kasselakis	9	
38	HEL-5	Fragiskos Koufogiannis	8	
39	MCD-1	Matej Dobrevski	8	
40	HEL-2	Marina Iliopoulou	4	
41	MOL-3	Mihaela Rusu	4	
42	CYP-1	Nansia Drakou	4	
43	CYP-6	Anastasia Solea	3	
44	TUR-3	Ahmet Kabakulak	2	
45	CYP-4	Marina Kouyiali	2	
46	CYP-5	Michalis Rossides	2	
47	CYP-2	Domna Fanidou	1	
48	CYP-3	Yiannis Ioannides	0	

Cea de a 7-a Olimpiadă Balcanică de Matematică pentru juniori s-a desfășurat în perioada 20-25 iunie în Turcia în stațiunea Kusadasi (circa 90 km la sud de Izmir, pe malul mării Egee). Echipa României a fost condusă de Prof. dr. Dan Brânzei, asistat de Prof. grad I Dinu Șerbănescu. In clasamentul neoficial pe națiuni România ocupă primul loc urmată de Bulgaria, Turcia, Republica Moldova, Serbia, Macedonia, Grecia, Cipru. Componenții echipei României au obținut următoarele punctaje și medalii:

Dragoș Michnea (Satu Mare) – 40p – Aur
Adrian Zahariuc (Bacău) – 40p – Aur
Lucian Țurea (București) – 38p – Aur
Cristian Talău (Craiova) – 37p – Aur
Sebastian Dumitrescu (București) – 29p – Argint
Beniamin Bogoșel (Arad) – 21p – Argint.

Menționăm că primii doi sunt singurii care au realizat punctajul total.

Inainte de a se deplasa în Turcia, echipa României a fost găzduită trei zile la București. Exclusiv în scop de antrenament, în această perioadă, juniorii au participat la al 5-lea și al 6-lea test test de selecție pentru OIM. Prestația juniorilor la aceste teste a fost excelentă

Olimpiada Naţională de Matematică

Al cincilea test de selecţie pentru OIM – 19 iunie 2003

Subiectul 1

Un parlament are n deputaţi. Aceştia fac parte din 10 partide şi din 10 comisii parlamentare. Fiecare deputat face parte dintr-un singur partid şi dintr-o singură comisie.

Determinaţi valoarea minimă a lui n pentru care indiferent de componenţa numerică a partidelor şi indiferent de repartizarea în comisii, să existe o numerotare cu toate numerele $1, 2, \ldots, 10$ atât a partidelor cât şi a comisiilor, astfel încât cel puţin 11 deputaţi să facă parte dintr-un partid şi o comisie cu număr identic.

Subiectul 2

Se dă un romb $ABCD$ cu latura 1. Pe laturile (BC) şi (CD) există punctele M, respectiv N, astfel încât $MC + CN + NM = 2$ şi $\angle MAN = \frac{1}{2}\angle BAD$.

Să se afle unghiurile rombului.

Subiectul 3

Într-un plan înzestrat cu un sistem de coordonate XOY se numeşte *punct laticial* un punct $A(x, y)$ în care ambele coordonate sunt numere întregi. Un punct laticial A se numeşte *invizibil* dacă pe segmentul deschis OA există cel puţin un punct laticial.

Să se arate că pentru orice număr natural n, $n > 0$, există un pătrat de latură n în care toate punctele laticiale interioare, de pe laturi sau din vârfuri, sunt invizibile.

Timp de lucru: 4 ore

Olimpiada Naţională de Matematică 2003

Al şaselea test de selecţie pentru OIM – 20 iunie 2003

Problema 1.

Fie $ABCDEF$ un hexagon convex. Notăm cu A', B', C', D', E', F' mijloacele laturilor AB, BC, CD, DE, EF, FA respectiv. Se cunosc ariile triunghiurilor ABC', BCD', CDE', DEF', EFA', FAB'.

Să se afle aria hexagonului $ABCDEF$.

Problema 2.

O permutare $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ se numeşte *strânsă* dacă pentru orice $k = 1, 2, \ldots, n-1$ avem

$$|\sigma(k) - \sigma(k+1)| \leq 2.$$

Să se găsească cel mai mic număr natural n pentru care există cel puţin 2003 permutări strânse.

Problema 3.

Pentru orice număr natural n notăm cu $C(n)$ suma cifrelor sale în baza 10. Arătaţi că oricare ar fi numărul natural k există un număr natural m astfel încât ecuaţia $x + C(x) = m$ are cel puţin k soluţii.

Timp de lucru 4 ore

Proposed Problem #72

== Valentin Vornicu ==

June 20, 2003

Problem: A permutation $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is called *straight* if and only if for each integer k, $1 \leq k \leq n-1$ the following inequality is fulfilled

$$|\sigma(k) - \sigma(k+1)| \leq 2.$$

Find the smallest positive integer n for which there exist at least 2003 straight permutations.

Solution: The main trick is to look where n is positioned. In that idea let us denote by x_n the number of all the straight permutations and by a_n the number of straight permutations having n on the first or on the last position, i.e. $\sigma(1) = n$ or $\sigma(n) = n$. Also let us denote by b_n the difference $x_n - a_n$ and by a'_n the number of permutations having n on the first position, and by a''_n the number of permutations having n on the last position. From symmetry we have that $2a'_n = 2a''_n = a'_n + a''_n = a_n$, for all n-s. Therefore finding a recurrence relationship for $\{a_n\}_n$ is equivalent with finding one for $\{a'_n\}_n$.

One can simply compute: $a'_2 = 1$, $a'_3 = 2$, $a'_4 = 4$. Suppose that $n \geq 5$. We have two possibilities for the second position: if $\sigma(2) = n-1$ then we must complete the remaining positions with $3, 4, \ldots, n$ thus the number of ways in which we can do that is a'_{n-1} (because the permutation $\sigma' : \{1, 2, \ldots, n-1\} \to \{1, 2, \ldots, n-1\}$, $\sigma'(k) = \sigma(k+1)$, for all k, $1 \leq k \leq n-1$, is also a straight permutation).

If on the second position we have $n-2$, $\sigma(2) = n-2$, then $n-1$ can only be in the last position of the permutation or on the third position, i.e. $\sigma(3) = n-1$ or $\sigma(n) = n-1$. If $\sigma(n) = n-1$, then we can only have $\sigma(n-1) = n-3$ thus $\sigma(3) = n-4$ and so on, thus there is only one permutation of this kind. On the other hand, if $\sigma(3) = n-1$ then it follows that $\sigma(4) = n-3$ and now we can complete the permutation in a'_{n-3} ways (because the permutation $\sigma' : \{1, 2, \ldots, n-3\} \to \{1, 2, \ldots, n-3\}$, $\sigma'(k) = \sigma(k+3)$, for all k, $1 \leq k \leq n-3$, is also a straight permutation).

Summing all up we get the recurrence:

$$a'_n = a'_{n-1} + 1 + a'_{n-3} \Rightarrow a_n = a_{n-1} + a_{n-3} + 2, \ \forall\, n \geq 5. \tag{1}$$

The recurrence relationship for $\{b_n\}$ can be obtained by observing that for each straight permutation $\tau : \{1, 2, \ldots, n+1\} \to \{1, 2, \ldots, n+1\}$ for which $2 \leq \tau^{-1}(n+1) \leq n$ we can obtain a straight permutation $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ by removing $n+1$. Indeed $n+1$ is "surrounded" by n and $n-1$, so by removing it, n and $n-1$ become neighbors, and thus the newly formed permutation is indeed straight. Now, if $\tau^{-1}(n) \in \{1, n+1\}$ then the newly formed permutation σ was counted as one of the a_n-s, minus the two special cases in which n and $n-1$ are on the first and last positions. If $\tau^{-1}(n) \notin \{1, n+1\}$ then certainly σ was counted with the b_n-s. Also, from any straight permutation of n elements, not having n and $n-1$ in the first and last position, thus n certainly being neighbor with $n-1$, we can make a straight $n+1$-element permutation by inserting $n+1$ between n and $n-1$.

Therefore we have obtained the following relationship:

$$b_{n+1} = a_n - 2 + b_n = x_n - 2, \ \forall\, n \geq 4. \tag{2}$$

From (1) and (2) we get that

$$x_n = x_{n-1} + a_{n-1} + a_{n-3}, \ \forall\, n \geq 5.$$

It is obvious that $\{x_n\}_n$ is a "fast" increasing sequence, so we will compute the first terms using the relationships obtained above, which will prove that the number that we are looking for is $n = 16$:

$a_2 = 2$	$x_2 = 2$	$a_9 = 62$	$x_9 = 164$
$a_3 = 4$	$x_3 = 6$	$a_{10} = 92$	$x_{10} = 254$
$a_4 = 8$	$x_4 = 12$	$a_{11} = 136$	$x_{11} = 388$
$a_5 = 12$	$x_5 = 22$	$a_{12} = 200$	$x_{12} = 586$
$a_6 = 18$	$x_6 = 38$	$a_{13} = 294$	$x_{13} = 878$
$a_7 = 28$	$x_7 = 64$	$a_{14} = 432$	$x_{14} = 1308$
$a_8 = 42$	$x_8 = 104$	$a_{15} = 634$	$x_{15} = 1940$

$$x_{16} = 2868$$

Enunțurile problemelor din atenția juriului la cea de a 7-a JBMO (Kusadasi, Turcia, 20-25 iunie 2003)

A.1. Un număr A este scris cu 2n cifre, fiecare dintre acestea fiind 4; un număr B este scris cu n cifre, fiecare dintre acestea fiind 8. Demonstrați că, pentru orice n, A+2B+4 este pătrat perfect.

A.2. Fie a, b, c lungimile laturilor unui triunghi, $p=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$, $q=\frac{a}{c}+\frac{c}{b}+\frac{b}{a}$. Demonstrați că $|p-q|<1$.

A.3. Fie a, b, c numere reale astfel încât $a^2+b^2+c^2=1$. Demonstrați că $P=ab+bc+ca-2(a+b+c)\geq -5/2$. Există valori pentru a, b, c încât $P=-5/2$?

A.4. Fie a, b, c numere raționale astfel încât $\frac{1}{a+bc}+\frac{1}{b+ac}=\frac{1}{a+b}$. Demonstrați că $\sqrt{\frac{c-3}{c+1}}$ este de asemenea număr rațional.

A.5. Fie ABC triunghi neisoscel cu lungimile a, b, c ale laturilor numere naturale. Dempnstrați că $|ab^2+bc^2+ca^2-a^2b-b^2c-c^2a|\geq 2$.

A.6. Fie a, b, c numere pozitive astfel ca $a^2b^2+b^2c^2+c^2a^2=3$. Demonstrați că $a+b+c\geq abc+2$.

A.6'. Fie a, b, c numere pozitive astfel ca $ab+bc+ca=3$. Demonstrați că $a+b+c\geq abc+2$.

A.7. Fie x, y, z numere mai mari ca -1. Demonstrați că $\frac{1+x^2}{1+y+z^2}+\frac{1+y^2}{1+z+x^2}+\frac{1+z^2}{1+x+y^2}\geq 2$.

A.8. Demonstrați că există mulțimi disjuncte $A=\{x,y,z\}$ și $B=\{m,n,p\}$ de numere naturale mai mari ca 2003 astfel ca $x+y+z=m+n+p$ și $x^2+y^2+z^2=m^2+n^2+p^2$.

C.1. Într-un grup de 60 studenți: 40 vorbesc engleza, 30 vorbesc franceza, 8 vorbesc toate cele trei limbi. Numărul celor ce vorbesc doar engleza și franceza este egal cu suma celor care vorbesc doar germana și franceza cu a celor ce vorbesc doar engleza și germana. Numărul celor ce vorbesc cel puțin două dintre aceste limbi este 28. Cât de mulți studenți vorbesc: a) germana; b) numai engleza; c) numai germana.

C.2. Numerele 1, 2, 3, ..., 2003 sunt scrise într-un șir $a_1, a_2, a_3, \ldots, a_{2003}$. Fie $b_1 = 1\exists a_1$, $b_2 = 2\exists a_2$, $b_3 = 3\exists a_3$, ..., $b_{2003} = 2003\exists a_{2003}$ și B maximul numerelor $b_1, b_2, b_3, \ldots, b_{2003}$.

a) Dacă $a_1=2003, a_2=2002, a_3=2001, \ldots, a_{2003}=1$, găsiți valoarea lui B.

b) Demonstrați că $B\geq 1002^2$.

C.3. Demonstrați că într-o mulțime de 29 numere naturale există 15 a căror sumă este divizibilă cu 15.

C.4. Fie n puncte în plan, oricare trei necoliniare, cu proprietatea că oricum le-am numerota $A_1, A_2, \ldots, A_n$, linia frântă $A_1A_2\ldots A_n$ nu se autointersectează. Găsiți valoarea maximă a lui n.

C.5. Fie mulțimea $M=\{1,2,3,4\}$. Fiecare punct al planului este colorat în roșu sau albastru. Demonstrați că există cel puțin un triunghi echilateral cu latura $m\in M$ cu vârfurile de aceeași culoare.

G.1. Există un patrulater convex pe care diagonalele să-l împartă în patru triunghiuri cu ariile numere prime distincte?

G.2. Există un triunghi cu aria 12cm^2 și perimetrul 12?

G.3. Fie G centrul de greutate al triunghiului ABC și A' simetricul lui A față de C. Demonstrați că punctele G, B, C, A' sunt conciclice dacă și numai dacă GA ζ GC.

G.4. Fie *k* cercul circumscris triunghiului *ABC*. Fie arcele *AB, BC, CA* astfel încât

$C \notin \overset{\frown}{AB}, A \notin \overset{\frown}{BC}, B \notin \overset{\frown}{CA}$ şi F, D, E mijloacele acestor arce. Fie G, H punctele de intersecţie ale lui DE cu CB, CA; fie I, J punctele de intersecţie ale lui DF cu BC, BA. Notăm mijloacele lui GH, IJ cu M, respectiv N.

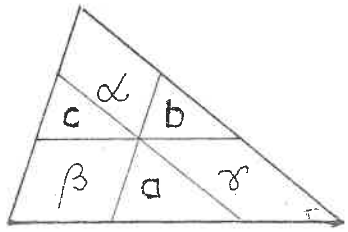
a) Găsiţi unghiurile triunghiului DMN în funcţie de unghiurile triunghiului ABC.

b) Dacă O este circumcentrul triunghiului DMN şi P este intersecţia lui AD cu EF, arătaţi că O, P, M şi N aparţin unui acelaşi cerc.

G.5. Trei cercuri egale au în comun un punct M şi se intersectează câte două în puncte A, B, C. Demonstraţi că M este ortocentrul triunghiului ABC. [1]

G.6. Fie ABC un triunghi isoscel cu AB = AC. Un semicerc de diametru EF situat pe baza BC este tangent laturilor AB, AC în M, N. AE retaie semicercul în P. Demonstraţi că dreapta PF trece prin mijlocul corzii MN.

G.7. Paralelele la laturile unui triunghi duse printr-un punct interior împart interiorul triunghiului în şase părţi cu ariile notate ca în figură.



Demonstraţi că $\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}\geq\frac{3}{2}$.

A1 - Mac ; A2 - Yug ; A3 - Bul ; A4 - Rom ; A5 - Mold ; A6, A7 - Rom ; A8 - Mold

C1 - Mac ; C2 - Bul ; C3 - Yug ; C4 - Rom ; C5 - Mold.

G1, G2 - Mac ; G3 - Rom ; G4 - Bul ; G5 - Yug ; G6 - Rom. ; G7 - Yug.

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[1] Identificată drept problema piesei de 5 lei a lui Ţiţeica.