5TH European Mathematical Cup

3rd December 2016–11th December 2016

Junior Category

Problems and Solutions

Problem 1. A grasshopper is jumping along the number line. Initially it is situated at zero. In k-th step, the length of his jump is k.

a) If the jump length is even, then it jumps to the left, otherwise it jumps to the right (for example, firstly it jumps one step to the right, then two steps to the left, then three steps to the right, then four steps to the left...). Will it visit on every integer at least once?

b) If the jump length is divisible by three, then it jumps to the left, otherwise it jumps to the right (for example, firstly it jumps one step to the right, then two steps to the right, then three steps to the left, then four steps to the right...). Will it visit every integer at least once?

(Matko Ljulj)

Solution. Let us denote with x_k position in the k-th step.

a) For even $n = 2k$ we have

$$x_{2k} = 1 - 2 + 3 + \ldots + (2k-1) - 2k =$$
$$(1 + 2 + 3 + \ldots + 2k) - 2(2 + 4 + 6 + \ldots + 2k) =$$
$$\frac{3k(3k+1)}{2} - 4\frac{k(k+1)}{2} = -k.$$

For odd $n = 2k + 1$ we have

$$x_{2k+1} = x_{2k} + (2k+1) = k + 1.$$

Hence we see that all integers occur exactly once in sequence $(x_k)_k$: positive integer n occur in $(2n-1)$-th place, negative integer $-n$ (for some $n > 0$) occurs in $(2n)$-th place.

b) For $n = 3k$ we have

$$x_{3k} = 1 + 2 - 3 + \ldots + (3k-2) + (3k-1) - 3k =$$
$$(1 + 2 + 3 + \ldots + 3k) - 2(3 + 6 + 9 + \ldots + 3k) =$$
$$\frac{2k(2k+1)}{2} - 6\frac{k(k+1)}{2} = \frac{3k(k-1)}{2}.$$

For $k = 0, 1$ we have that $x_{3k} = 0$. For all other k we have $x_{3k} > 0$ since it is a product of positive numbers. For $n = 3k+1, n = 3k+2$ we have

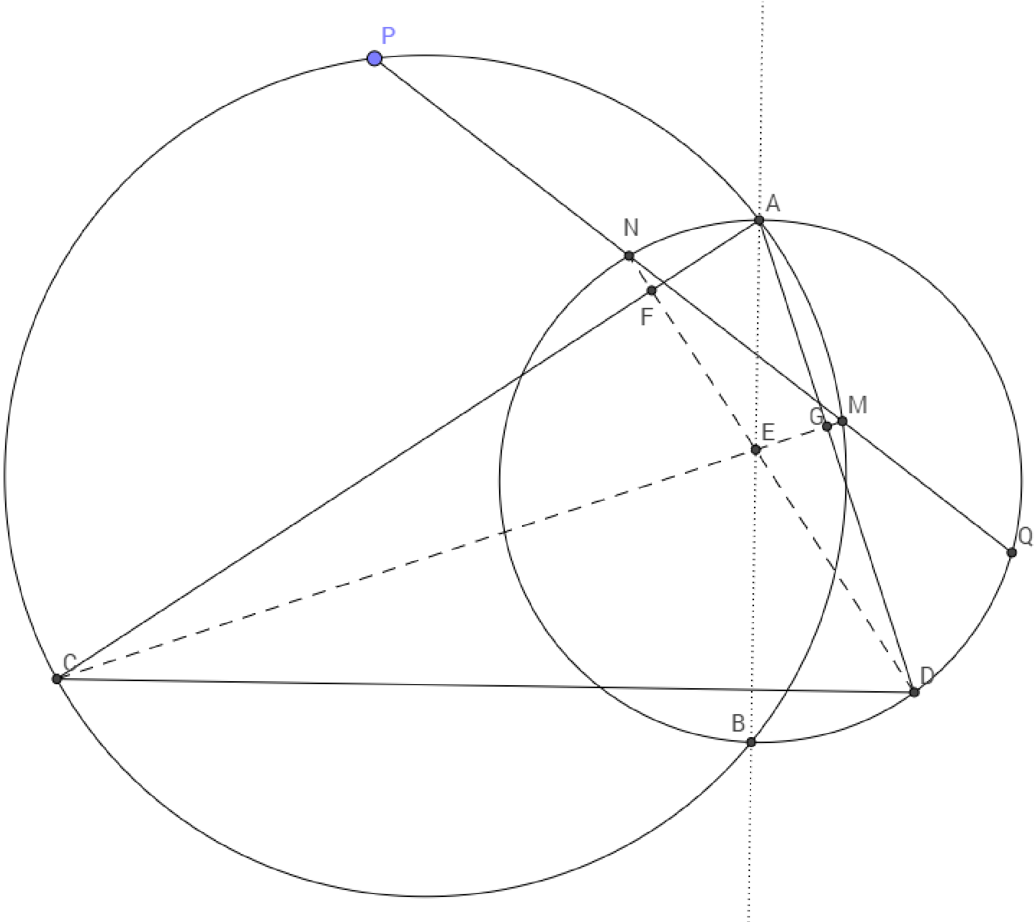
$$x_{3k+1} = x_{3k} + (3k+1) > 0, x_{3k+2} = x_{3k} + (3k+1) + (3k+2) > 0.$$

Thus, all x_k are non-negative, and grasshopper will not reach any negative integer.

Problem 2. Two circles C_1 and C_2 intersect at points A and B. Let P, Q be points on circles C_1, C_2 respectively, such that $|AP| = |AQ|$. The segment $\overline{PQ}$ intersects circles C_1 and C_2 in points M, N respectively. Let C be the center of the arc BP of C_1 which does not contain point A and let D be the center of arc BQ of C_2 which does not contain point A. Let E be the intersection of CM and DN. Prove that AE is perpendicular to CD.

(Steve Dinh)

First Solution. We present the following sketch:



As $AP = AQ$ the triangle APQ is isosceles, which implies $\angle APQ = \angle AQP$.
Angles over the same chord AM of C_1 imply $\angle ACM = \angle APM$.
As C is the midpoint of the chord BP, we have $\angle PAC = \angle CAB$, analogously $\angle DAQ = \angle BAD$.
This implies that $2\angle CAD = \angle PAQ$.
Combining the results above we get as sum of the angles in triangle APQ that $2\angle CAD + 2\angle APQ = 180^\circ$ which in turn implies $\angle ACN + \angle DAC = 90^\circ$ and in particular $AD \perp CM$. Analogously we conclude $DN \perp AC$.
We now conclude that this implies E is the orthocenter of the triangle ACD implying $AE \perp CD$ completing the proof.

Second Solution. As $AP = AQ$ the triangle APQ is isosceles, which implies $\angle APQ = \angle AQP$.
Angles over the same chord AM of C_1 imply $\angle MBA = \angle APM, analogously this implies \angle ABM = \angle APQ$
Combining the above we conclude $\angle MBA = \angle NBA$ so in particular AB is angle bisector of $\angle MBN$.
As C is the midpoint of the arc BP we have $\angle PMC = \angle BMC$.
We note this implies E lies on 2 angle bisectors of the triangle BNM, so is its incenter.
This implies that A, E, B are collinear.
We are now able to remove M, N, E from the picture and it is enough to show $CD \perp AB$. Let $\alpha = \angle CAB$ and $\beta = \angle BAD$. Then this is equivalent to $AC \cdot \cos\alpha = AD \cdot \cos\beta$.
Ptolomey's theorem for cyclic quadrilateral $APCB$ implies that

$$AC = \frac{BC \cdot AP + AB \cdot CP}{BP} = \frac{BC(AP + AB)}{2\cos\alpha \cdot BC}$$

After simplifying and taking an analogous equality for C_2 and cyclic quadrilateral $ABDC$ gives

$$AC\cos\alpha = \frac{AP + AB}{2} = \frac{AQ + AB}{2} = AD\cos\beta$$

completing the proof.

Remark: Note that we are using only the very basic trigonometry, namely for a right angled triangle ($BP = 2\cos\alpha \cdot BC$ follows by taking the midpoint of BP and considering 2 right-angled triangles this creates.) This can be alltogether avoided using similar triangles.

Problem 3. Prove that for all positive integers n there exist n distinct, positive rational numbers with sum of their squares equal to n.

(Daniyar Aubekerov)

First Solution. We will prove this claim by induction. For basis, we find solutions for $n = 1, 2, 3$:

$$1^2 = 1,\ \left(\frac{1}{5}\right)^2 + \left(\frac{7}{5}\right)^2 = 2,\ 1,\ 1^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{7}{5}\right)^2 = 3.$$

Now, let us assume that for all integers less than n the claim is true. Let us prove the claim for n. If $n = 4k$ for some integer k, then, by induction hypothesis, there exist rationals $x_1, \ldots, x_k$ such that

$$x_1^2 + \ldots + x_k^2 = k.$$

$$\implies (2x_1)^2 + \ldots + (2x_k)^2 = 4k.$$

Let a be the smallest rational number from the left hand side of the above equation. We will replace this number with numbers

$$\frac{3}{5}a,\ \frac{4}{5}a.$$

By this, we get one more summand on the left hand side, but the equality still holds. Since a was the smallest and $\frac{3}{5}a < \frac{4}{5}a < a$, all rationals are still distinct. We will continue this procedure until we get $n = 4k$ rationals.

Before we continue, notice the following: let those $n = 4k$ rationals denote with

$$\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n},$$

where $GCD(p_i, q_i) = 1$, for all $1 \leqslant i \leqslant n$. Then, all $p_1, \ldots, p_n$ are even numbers. That is because of multiplying first k rationals with 4, and because of the fact that multiplying rationals with $\frac{4}{5}$ and $\frac{3}{5}$ cannot turn even numerator to the odd numerator.

Now, we observe the case $n \neq 4k$. We will use a combination of solution for $n = 4k$ and for $n = 1, 2, 3$:

$$n = 4k+1:\ \left(\frac{p_1}{q_1}\right)^2 + \ldots + \left(\frac{p_n}{q_n}\right)^2 + 1^2 = n,$$

$$n = 4k+2:\ \left(\frac{p_1}{q_1}\right)^2 + \ldots + \left(\frac{p_n}{q_n}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{7}{5}\right)^2 = n,$$

$$n = 4k+3:\ \left(\frac{p_1}{q_1}\right)^2 + \ldots + \left(\frac{p_n}{q_n}\right)^2 + 1^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{7}{5}\right)^2 = n.$$

All numbers are still distinct because first $4k$ numbers have even numerators, while the others do not have. This concludes the induction and the proof of the problem.

Second Solution. Firstly, let us prove that there are infinitely many pairs of rationals such that

$$x^2 + y^2 = 2.$$

Let us take any Pythagorean triple (a, b, c), with $b > a$. Then we can take $x = \frac{b-a}{c}, y = \frac{b+a}{c}$.

Now, we take any number n. If it is even, then we will take $n/2$ pairs of rationals with sum of squares equal 2. If it is odd, we will take $(n-1)/2$ of such pairs, and one number 1.

To be sure that all numbers are distinct, we can take primitive Pythagorean triples such that all of them have unique third member c of the triple.

It is clear that they are nonzero. Let us now prove that all rationals are distinct. Firstly, if $\frac{b-a}{c} = \frac{b+a}{c}$, that implies $a = 0$, which is impossible for a member of Pythagorean triple.

Let us now assume that two different primitive Pythagorean triples (a, b, c) and (a', b', c') (with $c \neq c'$) generate at least two same rational numbers. Since sum of squares of those rationals is the same, another pair of rationals must be equal as well. Thus we have to have either

$$\frac{b-a}{c} = \frac{b'-a'}{c'} \text{ and } \frac{b+a}{c} = \frac{b'+a'}{c'} \implies \frac{b-a}{b'-a'} = \frac{b+a}{b'+a'} = \frac{c}{c'} = \lambda \in \mathbb{Q}, \text{ or}$$

$$\frac{b-a}{c} = \frac{b'+a'}{c'} \text{ and } \frac{b+a}{c} = \frac{b'-a'}{c'} \implies \frac{b-a}{b'+a'} = \frac{b+a}{b'-a'} = \frac{c}{c'} = \lambda \in \mathbb{Q}.$$

In both cases we have $a^2 + b^2 = c^2 = \lambda^2(c')^2 = \lambda^2((a')^2 + (b')^2)$ and $b^2 - a^2 = \lambda^2((b')^2 - (a')^2)$. Hence $c^2 = \lambda^2(c')^2$, $a^2 = \lambda^2(a')^2$, $b^2 = \lambda^2(b')^2$. But then, if $\lambda = p/q$, then either $p \mid a, b, c$ or $q \mid a', b', c'$ or $\lambda = 1$, which contradicts the fact that our triples are primitive or that $c' \neq c$. All in all, we get contradiction, thus all rationals are distinct.

Problem 4. We will call a pair of positive integers (n, k) with $k > 1$ *a lovely couple* if there exists a table $n \times n$ consisting of ones and zeros with following properties:

- In every row there are exactly k ones.
- For each two rows there is exactly one column such that on both intersections of that column with the mentioned rows, number one is written.

Solve the following subproblems:

a) Let $d \neq 1$ be a divisor of n. Determine all remainders that d can give when divided by 6.

b) Prove that there exist infinitely many lovely couples.

(Miroslav Marinov, Daniel Atanasov)

Solution. Let us firstly prove several lemmas. Before that, notice that changing two columns or two rows of the table will not change the properties of our table.

Lemma 1: In every column there are exactly k ones.
Proof: It is impossible that one column contains n ones. If we suppose the contrary, then on the rest of the table, consisting of $n-1$ columns, we would have to have $n(k-1) \geqslant n$ ones such that no two ones are in the same column, which is impossible.

Thus, every column contains at least one zero. Let us now suppose that there exists a column with more than k ones. Without loss of generality, let this column be the first column, where ones are written in the first $k+1$ rows, and at least one digit zero, which this column must contain, is written in last row. Again, without loss of generality, let the last row contain ones in the second, third, …, $(k+1)$-th column.

On the intersection of 2nd column and first $k+1$ rows there can be at most one digit one, because, in the contrary, some two of the first $k+1$ rows would have first and second column in common. Same argument holds for intersection of the 3rd column and first $k+1$ rows, …, $(k+1)$-th column and first $k+1$ rows. Hence, on the intersection of first $k+1$ rows, and 2nd, 3rd, …, $(k+1)$-th row there are at most k ones.

However, for the last row and for every row among the first $k+1$ rows, there must exist exactly one column such that both rows contain digit one in that column. This is only possible if those ones are on the intersection of first $k+1$ rows, and 2nd, 3rd, …, $(k+1)$-th row. Thus, in the mentioned zone there must be exactly $k+1$ ones, which leads to contradiction.

Thus we conclude that every column contains at most k digits one. Since the whole table consists of nk digits one, we have that every column contains exactly k digits one.

Lemma 2: We have $n = k^2 - k + 1$.
Proof: Let us count the pairs of ones in the same column. On the one hand, since there are n columns, every column contains k ones, there are

$$n\binom{k}{2}$$

pairs of ones in the same column. On the other hand, every pair of ones from the same column determine exactly one pair of rows, since each pair of rows has exactly one column in common. Thus, the number of pairs of ones from the same column is also equal to

$$\binom{n}{2}.$$

Identifying mentioned two expressions we get $n = k^2 - k + 1$.

Now, we will prove the problem.

Solution of a) part: When varying k, we see that $n \equiv 1$ (mod 6) or $n \equiv 3$ (mod 6). Both options are possible, see examples for $k = 2, k = 3$ below.

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Let q be a prime divisor of the number $n = k^2 - k + 1$. Since n is odd, q is odd as well, thus possible remainders modulo 6 are $1, 3, 5$. We will prove that remainder 5 is not possible. Let us suppose that $q = 6t + 5$. Then, since $q \mid k^3 + 1 = (k+1)(k^2 - k + 1)$ we have $k^3 \equiv -1\ (mod\ q)$. On the other hand, we have $k^{q-1} = k^{6t+4} \equiv 1\ (mod\ q)$. From those last two identities we get $k \equiv -1\ (mod\ q) \implies n \equiv 3 (mod\ q)$, i.e. $q \mid 3$, contradiction.

Let d be any divisor of the number n. From above, it is either 1 or a product of prime numbers of the form $6t + 1$ and 3. Anyhow, we have that remainder of d when divided by six is either 1 or 3.

Solution of b) part: We will prove that for any prime p, the pair of numbers $(p^2 + p + 1, p + 1)$ is a lovely couple. Let us denote with $A(i, j)$ the number in the table on the intersection of the i-th row and j-th column, with the convention that we count rows and columns from the zero in this part of the solution.

We define our table in the following way (example for $p = 5$ is at the end)

Rule a For $\alpha, \beta, \gamma \in \{0, \dots, p-1\}$ we have

$$A(\alpha p + \beta, \gamma p + \delta) = 1 \iff \delta \equiv \alpha\gamma + \beta\ (mod\ p),$$

Rule b $A(p^2 + \alpha, \alpha p + \beta) = 1$, for all $\alpha \in \{0, \dots, p\}$, $\beta \in \{0, \dots, p-1\}$,

Rule c $A(\alpha p + \beta, p^2 + \alpha) = 1$, for all $\alpha \in \{0, \dots, p\}$, $\beta \in \{0, \dots, p-1\}$,

Rule d $A(p^2 + p, p^2 + p) = 1$,

Rule e On all other unmentioned fields are zero.

Let us prove that this table has all properties. Firstly, let us prove that in every row there is exactly $p + 1$ ones.

Case 1: In i-th row, $i < p^2$: $i = \alpha p + \beta$, for some $0 \leqslant \alpha, \beta \leqslant p - 1$. Then for every $\gamma \in \{0, \dots, p-1\}$ there exists exactly one $\delta \in \{0, \dots, p-1\}$ such that $\delta \equiv \alpha\gamma + \beta\ (mod\ p) \implies$ there are exactly p digits one in first p^2 columns. Last digit k is in the column $p^2 + \alpha$, according to the Rule c.

Case 2: In i-th row, $i \geqslant p^2$: $i = p^2 + \alpha$, for some $0 \leqslant \alpha \leqslant p$. Those ones are written in the columns (according to the Rule b) $\alpha p + 0, \dots, \alpha p + p - 1$ and (according to the Rule c or d) in the last column.

In the same manner it can be proved that every column contains exactly $p + 1$ ones. Thus, it is sufficient to prove that every two rows have at least one column in common.

Case 1: $i, j \geqslant p^2$: $i = p^2 + \alpha_i, j = p^2 + \alpha_j$ for some $0 \leqslant \alpha_i, \alpha_j \leqslant p$. According to the Rule c or d: $A(i, p^2 + p) = A(j, p^2 + p) = 1$.

Case 2: $i < p^2, j = p^2 + p$: $i = \alpha_i p + \beta_i$ for some $0 \leqslant \alpha_i \leqslant p$, $0 \leqslant \alpha_j \leqslant p - 1$. According to the Rule c we have $A(i, p^2 + \alpha_i) = 1$, and according to the Rule b: $A(j, p^2 + \alpha_i) = 1$.

From now on, all mentioned variables $\alpha_i, \alpha_j, \beta_i, \beta_j, \gamma, \delta$ are from the set $\{0, \dots, p-1\}$.

Case 3: $i < p^2, p^2 \leqslant j < p^2 + p$: $i = \alpha_i p + \beta_i$, $j = p^2 + \alpha_j$. According to Rule a, there is exactly one δ such that $A(i, \alpha_j p + \delta)$. According to the Rule b: $A(j, \alpha_j p + \delta)$.

Case 4a: $i, j < p^2$: $i = \alpha_i p + \beta_i, j = \alpha_j + \beta_j$ with $\alpha_i = \alpha_j := \alpha$. According to the Rule c: $A(i, p^2 + \alpha) = A(j, p^2 + \alpha) = 1$.

Case 4b: $i, j < p^2$: $i = \alpha_i p + \beta_i, j = \alpha_j + \beta_j$ with $\alpha_i \neq \alpha_j := \alpha$. Let us define

$$\gamma = (\alpha_i - \alpha_j)^{-1}(\beta_j - \beta_i).$$

It is clear that then we have $\alpha_i\gamma + \beta_i = \alpha_j\gamma + \beta_j =: \delta$. According to the Rule a: $A(i, \gamma p + \delta) = A(j, \gamma p + \delta) = 1$.

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5^{TH} European Mathematical Cup

3^{rd} December 2016–11^{th} December 2016

Senior Category

MLADI NADARENI MATEMATIČARI
Marin Getaldić

Problems and Solutions

Problem 1. Is there a sequence $a_1, \dots, a_{2016}$ of positive integers, such that every sum

$$a_r + a_{r+1} + \ldots + a_{s-1} + a_s$$

(with $1 \leqslant r \leqslant s \leqslant 2016$) is a composite number, but

a) $GCD(a_i, a_{i+1}) = 1$ for all $i = 1, 2, \dots, 2015$;

b) $GCD(a_i, a_{i+1}) = 1$ for all $i = 1, 2, \dots, 2015$ and $GCD(a_i, a_{i+2}) = 1$ for all $i = 1, 2, \dots, 2014$?

$GCD(x, y)$ denotes the greatest common divisor of x, y.

(Matija Bucić)

First Solution. We will solve this problem for any length n of the sequence.

a) Yes, there is such sequence.
For this part, we will construct solution by taking n consecutive positive integers $a_i = m + i$, for some positive integer m. We will determine number m at the end of the proof.
Firstly, notice that two consecutive elements of the sequence are coprime, since they are consecutive numbers.
Every sum of consecutive members of sequence is of the form

$$(a+1) + (a+2) + \ldots + (b-1) + b = \frac{b(b+1)}{2} - \frac{a(a+1)}{2} = \frac{(b-a)(b+a+1)}{2}.$$

For $b \geqslant a + 3$, numerator of the expression above consists of two factors, each greater or equal to 3, and at least one of them is even, thus number is composite.
Thus, we have to choose m such that all sums of one and all sums of two consecutive members of sequence are composite. That is, following numbers need to be composite:

$$m+1, m+2, \dots, m+n, 2m+3, 2m+5, \dots 2m+(2n-1).$$

This is achieved for $m = (2n+1)! + 1$. Namely, numbers $(2n+1)! + k$ and $2 \cdot (2n+1)! + k$ are composite for all $2 \leqslant k \leqslant 2n+1$ since they are divisible by k, and greater than k.

b) Again, the answer is yes.
Similarly like in first part, we will take some n consecutive odd numbers: $a_i = 2m + (2i - 1)$, for some positive integer m.
It is clear that they are integers, and they are positive.
We will have $GCD(a_i, a_{i+1}) = GCD(a_i, a_{i+2}) = 1$ because differences of mentioned numbers are always 2 or 4. Since numbers are odd, they have to be coprime.
Every sum of consecutive members of sequence is of the form

$$(2a+1) + (2a+3) + \ldots + (2b-3) + (2b-1) = b^2 - a^2 = (b-a)(b+a).$$

For $b \geqslant a + 2$ number from above is composite because both factors are greater or equal to 2.
Thus, we have to choose m such that all numbers

$$2m+1, 2m+3, \dots, 2m+(2n-1)$$

are composite. This is achieved by taking $m = (2n)! + 1$, with similar arguments like in first part.

Second Solution. (For part b) only.)
We will show that there exists a sequence for b) part of the problem.
It is obvious that this will imply that the answer for the a) part of the solution is yes.
We will form the sequence by induction. For the basis, we will take $a_1 = 4$, $a_2 = 35$. Those numbers are composite, their sum is composite and they are coprime.
Let us assume that we have n positive integers with properties from the text of the problem. Let $p_1, p_2, \ldots, p_n, p_{n+1}$ be some prime numbers greater than

$$a_1 + a_2 + \ldots + a_n.$$

Notice that this immediately means that those primes are greater than any sum of consecutive numbers, and specially, that all those sums (including solely integers a_i) are coprime with mentioned primes.
We will get a_{n+1} by solving system of modular equations. Existence of such positive integer is provided by Chinese remainder theorem. The system is the following:

$$a_{n+1} \equiv 1\ (mod\ a_n,\ \ a_{n+1} \equiv 1\ (mod\ a_{n-1}),$$
$$a_{n+1} \equiv -(a_n + \ldots + a_{n-k})\ (mod\ p_k),\ \ k = 0, 1, 2, \ldots, n-1$$
$$a_{n+1} \equiv 0\ (mod\ p_{n+1}).$$

In first row we provided that $GCD(a_{n+1}, a_n) = GCD(a_{n+1}, a_{n-1}) = 1$.

In second two rows we provided that all sums of consecutive numbers including a_{n+1} are composite.

Chinese remainder theorem can be applied here since all primes are greater than a_n and a_{n-1}, and thus they are coprime.

Third Solution. (For part b) only.) As before, it is sufficient to prove the existence of the sequence for b) part only.
We will form recursion: $a_{-1} = 1, a_0 = 3, a_k = a_{k-1}^2 - a_{k-2}^2$, for $k \geqslant 1$. (Here values a_{-1} and a_0 are just auxiliary terms).
All numbers are positive integers, moreover we will prove that $a_k \geqslant a_{k-1} + 2$, which we get from induction:

$$a_k = a_{k-1}^2 - a_{k-2}^2 \geqslant a_{k-1}^2 - (a_{k-1} - 2)^2 = 4a_{k-1} - 4 \geqslant a_{k-1} + 2,$$

since $a_{k-1} \geqslant a_0 = 3$.
If there is some index k and some prime p such that p divides a_k and a_{k-1} or divides a_k and a_{k-2}, then from equation $a_k = a_{k-1}^2 - a_{k-2}^2$ we get that p divides a_{k-1} and a_{k-2}. In the same manner, p then divides a_{k-2} and a_{k-3}, it divides a_{k-3} and a_{k-4}, and so on, thus it divides a_{-1} and a_0, which is impossible.
Let us now prove that all sums of consecutive elements are composite:

$$a_r + \ldots + a_s = (a_{r-1}^2 - a_{r-2}^2) + \ldots + (a_{s-1}^2 - a_{s-2}^2) = a_{s-1}^2 - a_{r-2}^2 = (a_{s-1} - a_{r-2})(a_{s-1} + a_{r-2}).$$

First factor is greater than 1 since $a_{s-1} \geqslant a_{r-1} \geqslant a_{r-2} + 2$. Second factor is clearly greater than 1, hence the product is composite.

Fourth Solution. (For part a) only.)
The answer is yes.
Similarly like in the first solution, we will take sequence of consecutive third powers of positive integers: $a_i = (i+1)^3$.
Like in first solution, consecutive elements are coprime. It is clear that all numbers are positive integers.
All possible sums of consecutive elements are of the form

$$(a+1)^3 + (a+2)^3 + \ldots + (b-1)^3 + b^3 = \left(\frac{b(b+1)}{2}\right)^2 - \left(\frac{a(a+1)}{2}\right)^2 = \left(\frac{b(b+1)}{2} - \frac{a(a+1)}{2}\right)\left(\frac{b(b+1)}{2} + \frac{a(a+1)}{2}\right) =$$
$$= ((a+1) + (a+2) + \ldots + (b-1) + b)\left(\frac{b(b+1)}{2} + \frac{a(a+1)}{2}\right).$$

Second factor is greater or equal than first one. Second is greater than 1 if all elements of sequence are greater than 1. Since we chose numbers in that way, the number is composite.

Problem 2. For two positive integers a and b, Ivica and Marica play the following game: Given two piles of a and b cookies, on each turn a player takes $2n$ cookies from one of the piles, of which he eats n and puts n of them on the other pile. Number n is arbitrary in every move. Players take turns alternatively, with Ivica going first. The player who cannot make a move, loses. Assuming both players play perfectly, determine all pairs of numbers (a, b) for which Marica has a winning strategy.

(Petar Orlić)

Solution. Marica wins the game if $|a-b| \leqslant 1$, otherwise Ivica wins.

We will say that a player is in a losing position if it is his turn and $|a-b| \leqslant 1$, while calling all other positions winning positions. It is easy to see that the only positions in which one cannot make a move are $(0,0)$,$(0,1)$,$(1,0)$,$(1,1)$ and that they are all losing positions.

Claim 1. *If a player is in a losing position, then regardless of his move he must leave a winning position for the other player.*

Proof. If the piles are of sizes x and $x+1$, then after a move they will have sizes $x-2k$ i $x+k+1$ (their difference is $3k+1$) or $x+k$ i $x-2k+1$ (their difference is $3k-1$). In both cases, the difference is at least 2. If the piles have x and x cookies each, then after a move they will have $x-2k$ and $x+k$ cookies (there difference is $3k$, which is at least 3). Since the difference of the number of cookies is always bigger than 1, we have proven that this is a winning position. □

Claim 2. *A player who is in a winning position can always leave a losing position after his turn.*

Proof. If the piles are of sizes x and $x+3a$ (where $a \geqslant 0$), one can take $2a$ cookies from the second pile and and leave two piles containing $x+a$ and $x+a$ cookies. If the piles are of sizes x and $x+3a+1$ (where $a \geqslant 0$), one can take $2a$ cookies from the second pile and leave two piles containing $x+a$ and $x+a+1$ cookies.

If the piles are of sizes x and $x+3a-1$ (where $a \geqslant 0$), one can take $2a$ cookies from the second pile and and leave two piles containing $x+a$ and $x+a-1$ cookies. Since the difference in each case is less than 2, thus a player can always leave a loosing position if he is in a winning position. □

We have now proven that if Ivica is in a loosing position in the begging, Marica can always ensure that he is in a winning position and win. Similarly, if Ivica is in a winning position in the begging, he can always ensure that he is in a winning position and win. So, Marica wins only when Ivica is in a losing position in his first turn. This is true only when $|a-b| \leqslant 1$.

Problem 3. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that equality

$$f(x + y + yf(x)) = f(x) + f(y) + xf(y)$$

holds for all real numbers x, y.

(Athanasios Kontogeorgis)

Solution. We easily see that $f(x) = 0, x \in \mathbb{R}$ and $f(x) = x, x \in \mathbb{R}$ are solutions. Let us assume that f satisfies the given equation but is not a constant or identity.

Throughout the proof, we denote by (x_0, y_0) the initial equation with $x = x_0, y = y_0$.

Then, $(-1, y)$ implies $f(-1 + y(1 + f(-1))) = f(-1)$. Let us assume that $1 + f(-1) = c \neq 0$. Then, for any $cy - 1$ achieves all real numbers and hence $f(z) = f(-1)\ \forall z \in \mathbb{R}$ so f is a constant, a contradiction. Hence, $f(-1) = -1$.

Let us assume that there is some $\alpha \in \mathbb{R}$ such that $f(\alpha) = -1$, but $\alpha \neq -1$. Then, $(\alpha, y) : f(\alpha) = f(\alpha)+f(y)+\alpha f(y) \implies 0 = f(y)(1 + \alpha)$. Since $\alpha \neq -1$, we get $f(y) = 0, y \in \mathbb{R}$, a contradiction. Thus, we have shown

$$f(x) = -1 \iff x = -1 \tag{1}$$

$$(x, -1) : \ f(x - 1 - f(x)) = f(x) - 1 - x. \tag{2}$$

Since we assumed that f is not the identity, there exists a real number x_0 such that $f(x_0) \neq x_0$. We set $a := f(x_0)-x_0 \neq 0$. Putting $x = x_0$ in the above equation gives:

$$f(-1 - a) = a - 1. \tag{3}$$

We get from $(-1 - a, y)$: and equation (3)

$$f(-1 - a + ya) = a - 1 - af(y). \tag{4}$$

If we now put $y = 1$, we get $a(1 - f(1)) = 0$ so as $a \neq 0$ we get $f(1) = 1$.

Now $(1, 1)$ gives us $f(3) = 3$.

Putting $(1, y - 1)$ gives us

$$f(2y - 1) = 2f(y - 1) + 1. \tag{5}$$

Using $f(3) = 3$ in (3) with $y = 3$ we get $f(2a - 1) = -2a - 1$, while using $y = a$ in (5) we get $f(2a - 1) = 2f(a - 1) + 1$, combining the two gives us

$$f(a - 1) = -1 - a. \tag{6}$$

We get from $(a - 1, 2 - y)$:

$$f(-a - 1 + ay) = -1 - a + af(2 - y). \tag{7}$$

Combining this with (4) we get:

$$a(f(y) + f(2 - y) - 2) = 0. \tag{8}$$

So as $a \neq 0$, we get $f(y) + f(2 - y) = 2$ for all y.

Putting $y = 1 + 2x$ here, gives $f(1 + 2x) + f(1 - 2x) = 2$, which when combined with 5 with $y = x + 1$ gives, $f(1 - 2x) = 1 - 2f(x)$.

While (5) for $y = 1 - x$ gives $f(1 - 2y) = 1 + 2f(-y)$, which combined with the above implies $f(-x) = -f(x)$ for all x.

Let us put $(x, -y)$ in initial equation, and then subtract the original equation (for (x, y)). We obtain:

$$f(x + y(1 + f(x))) + f(x - y(1 + f(x))) = 2f(x). \tag{9}$$

We substitute y with

$$\frac{y}{1 + f(x)}$$

and get

$$f(x + y) + f(x - y) = 2f(x), \tag{10}$$

which is valid for all x, y, with $f(x) + 1 \neq 0 \iff x \neq -1$. But, from f being odd and (8), we see that this is valid for $x = -1$, as well. In (10) we put $x = y$ to obtain $f(2x) = 2f(x)$. In the same equation we put $\frac{x-y}{2}$, $\frac{x+y}{2}$ and obtain

$$f(x) + f(y) = f(x + y). \tag{11}$$

Using this additivity, we can simplify the original equation:

$$f(xf(y)) = yf(x) \tag{12}$$

In the last equation we can firstly put $(1, y) \implies f(f(y)) = y$ and secondly $f(y)$ instead of y: $f(xy) = f(x)f(y)$.

It is well known that from identities $f(1) = 1$, $f(xy) = f(x)f(y)$ and $f(x + y) = f(x) + f(y)$ we can conclude that $f(x) = x$. Which is a contradiction.

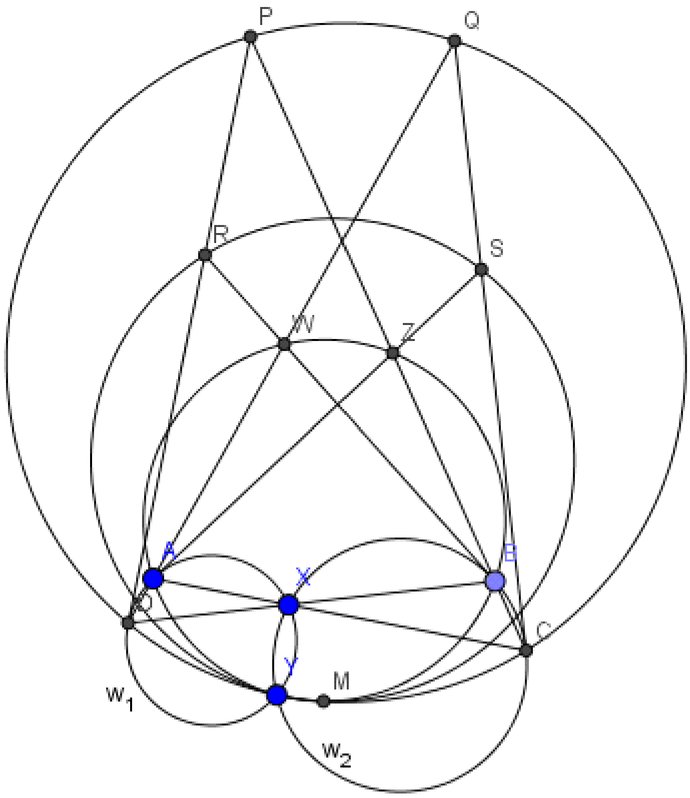
For the well known claim, we notice that $f(x^2) = f(x)^2$ implies $f(x) \geq 0$ for $x \geq 0$, which implies, combined with $f(x+y) = f(x)+f(y)$ that f is non-decreasing which in turn is enough to combine with the standard density of rationals argument to solve Cauchy's equation.

Hence, the functions presented at the start give all possible solutions.

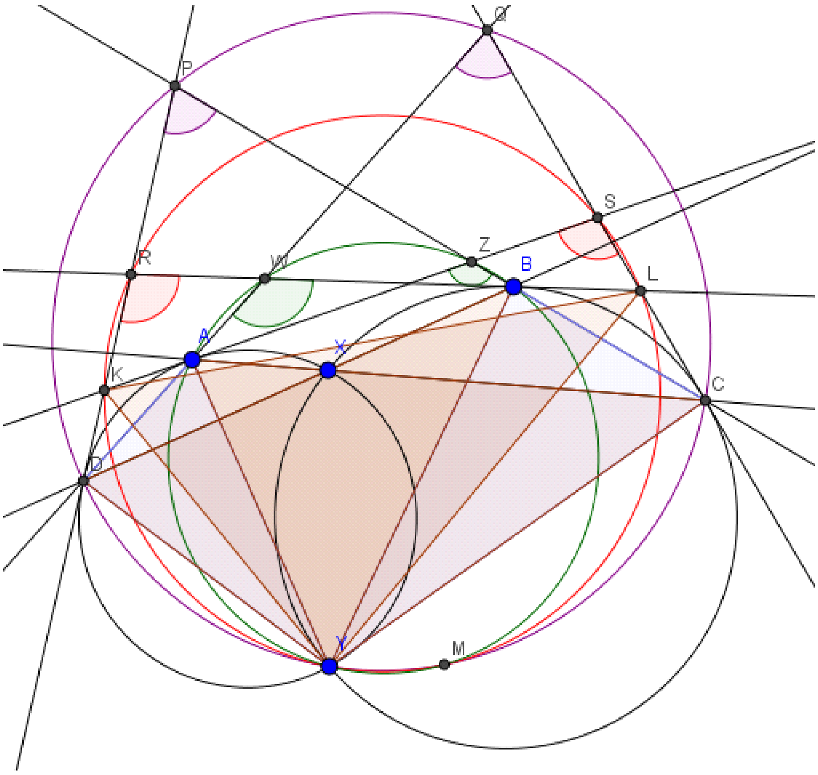
Problem 4. Let C_1, C_2 be circles intersecting in X, Y. Let A, D be points on C_1 and B, C on C_2 such that A, X, C are collinear and D, X, B are collinear. The tangent to circle C_1 at D intersects BC and the tangent to C_2 at B in P, R respectively. The tangent to C_2 at C intersects AD and tangent to C_1 at A, in Q, S respectively. Let W be the intersection of AD with the tangent to C_2 at B and Z the intersection of BC with the tangent to C_1 at A. Prove that the circumcircles of triangles YWZ, RSY and PQY have two points in common, or are tangent in the same point.

(Misiakos Panagiotis)

Solution. We present the following sketch:



Consider K, L the intersections of the pairs of tangents at (A, D) to C_1 and (B, C) to C_2 respectively.



Notice that $\angle WAZ = \angle KAD = \angle AXD = \angle BXC = \angle LBC = \angle ZBW$. So $WZBA$ is a cyclic quadrilateral. Furthermore, $\angle AZB = \angle AZC = 180^\circ - \angle ZAC - \angle ZCA = 180^\circ - \angle AYX - \angle BYX = 180^\circ - \angle AYB$. Thus the circle from A, Y, B passes from Z, and since W, Z, B, A are concyclic W, Z, B, Y, A belong to the same circle.

Analogous angle chase gives P, Q, C, Y, D concyclic.

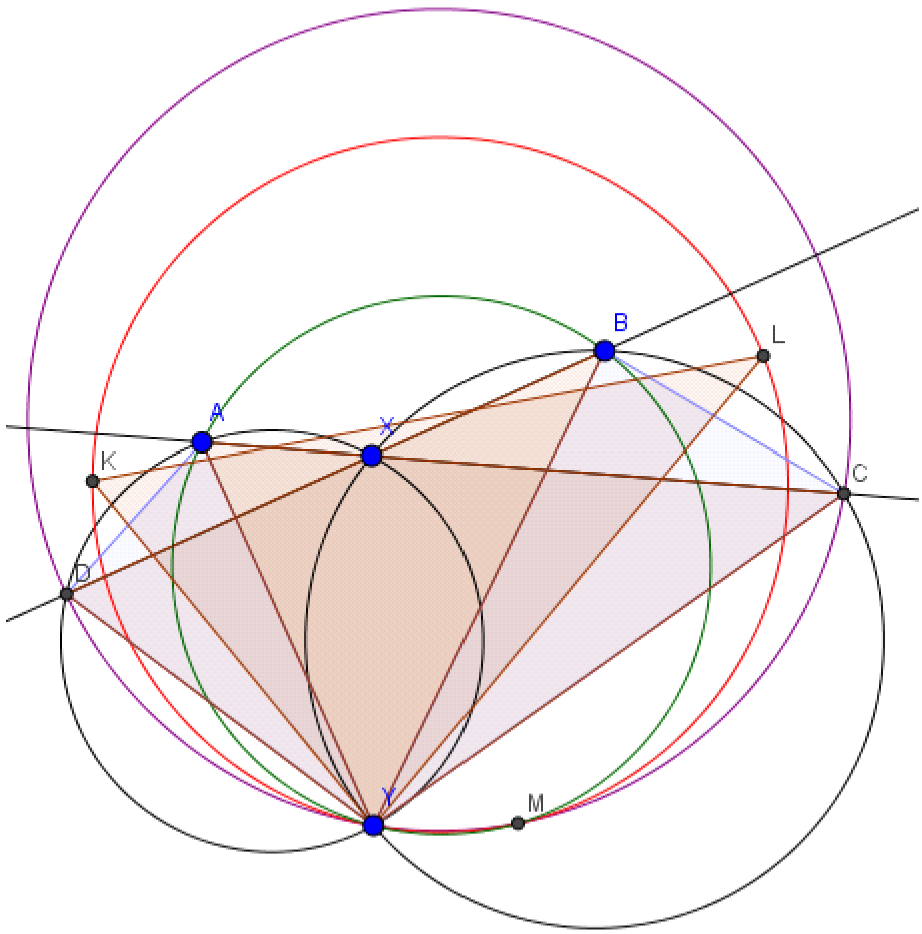
K, Y, L, S, R are also concyclic, this follows from $\angle ASC = 180^\circ - \angle SAC - \angle SCA = 180^\circ - \angle AYC$.

We have, $\angle XDY = \angle XAY$ and $\angle YBX = \angle YCX$ which implies $\triangle DYB \sim \triangle YAC$. This implies $\angle DYB = \angle AYC$.

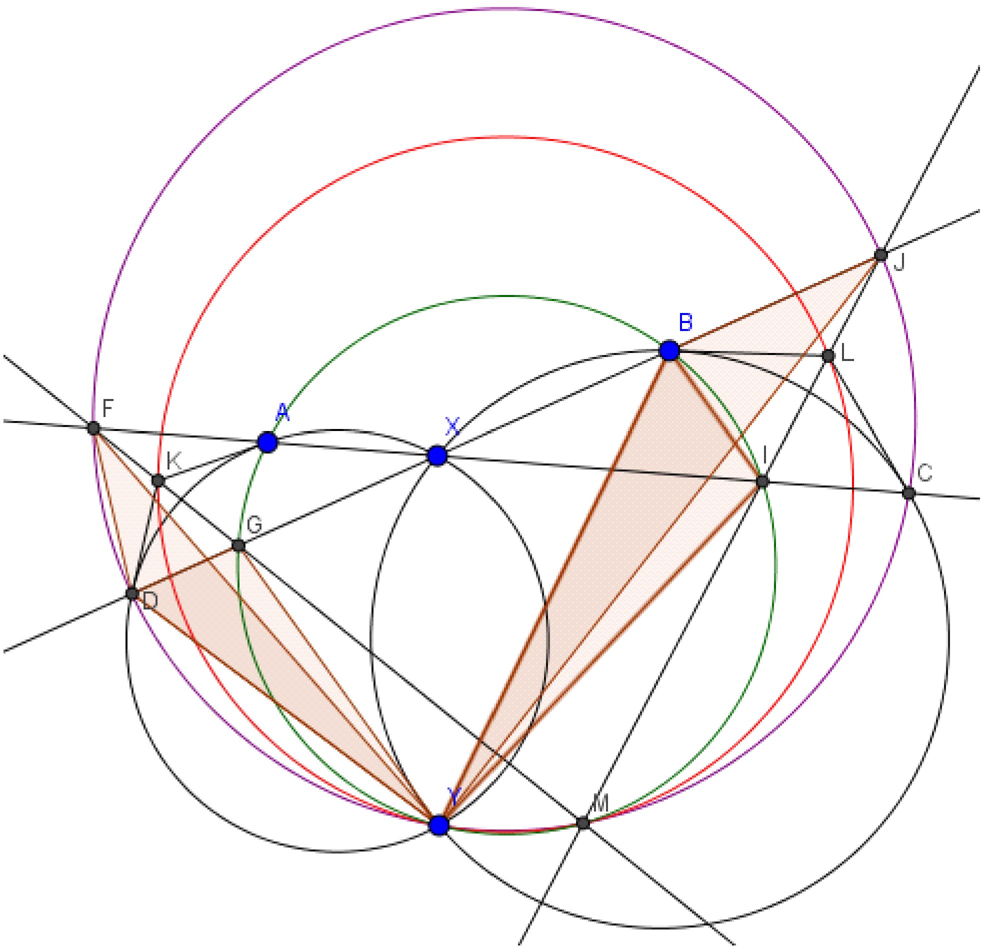
We have $\angle DRB = 180^\circ - \angle RDB - \angle DBR = 180^\circ - \angle DYX - \angle XYB = 180^\circ - \angle DYB = 180^\circ - \angle AYC = 180^\circ - \angle AYX - \angle XYC = 180^\circ - \angle XCS - \angle XAS = \angle ASC$. This implies $KRSL$ is cyclic.

$\triangle DYB \sim \triangle YAC$ also implies $\angle DYA = \angle BYC$, as well as $\frac{DY}{AY} = \frac{YB}{YC}$ which implies $\triangle DYA \sim \triangle BYC$. This further implies the isosceles triangles AKD and LBC have same angles so quadrilaterals $DYAK$ and $BYCL$ are also similar, in particular implying $\angle KYD = \angle LYB$. This in turn implies $180^\circ - \angle DRL = \angle DYB = \angle KYL$ which in turn implies Y is on the same circle as K, R, S, L.

We now proceed to show circumcircles of YKL, YDC, YAB have two common points.



Let F, J be the points of intersections of AC, BD with circle DYC respectively and G, I be the points of intersection of BD,AC with circle ABY.



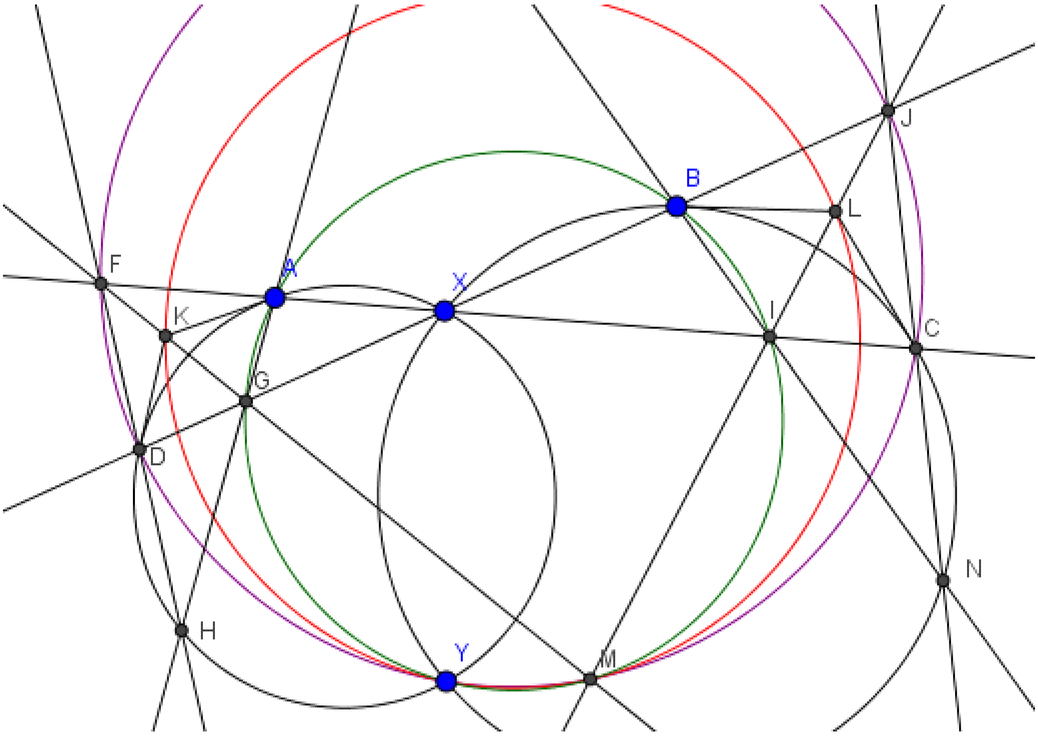
Now let M be the intersection of lines FG, JI. We will eventually prove that this will be a second common point for the three circles.

First we show that the FDY, BJY are similar. For this note that $\angle FXJ = \angle XJC + \angle XCJ = \angle FYJ + \angle DJC = \angle FYJ + 180^\circ - \angle DYC \implies \angle FYJ = \angle FXJ - 180^\circ + \angle DYC = \angle DYC - \angle AXD = \angle AYC = \angle DYB$. Thus $\angle FYJ = \angle DYB$ and so $\angle FYD \sim \angle JYB$. While $\angle DJY = \angle DFY$ showing $\triangle FDY \sim JBY$ as claimed.

Also $\triangle DGY \sim \triangle BIY$ are similar , since $\angle AXB = \angle XBI + \angle BIX = \angle XBI + \angle AYB = 180^\circ - \angle GYI + \angle AYB = 180^\circ - \angle AYG - \angle BYI \implies \angle AXD = \angle AYG + \angle IYB \implies \angle AYG + \angle GYD = \angle AYG + \angle IYB \implies \angle GYD = \angle IYB$. Also, $\angle BIY = \angle DGY$, $\angle YBI = \angle YAI = \angle YAX = \angle YDX = \angle YDG$ and we get our result.

Now we get that the spiral similarity that sends $D \to B$ and $F \to J$ also sends $G \to I$, so $\triangle FGY \sim \triangle JIY$, so $\angle YGM = \angle YIM$ and $\angle YFM = \angle YJM$, so M belongs to both of the circumcircles of FYJ and GIY, hence M is the (other than Y) common point of circumcircles of ABY and CDY.

Since $\angle FMJ = \angle FYJ = \angle DYB = \angle KYL$ it remains to show that K, L belong on the Lines FG, JI respectively (then circle KYL would pass through M.)



Let H denote the point of intersection of lines AG, FD. Then $\angle HDX = 180^\circ - \angle FDX = 180^\circ - \angle FMJ = \angle GAI = \angle GAX$, so H belongs to the circumcircle of triangle ADX.

Similarly denote N (the intersection of lines BI, JC) and it will for analogous reasons belong to the circumcircle of $\triangle BXC$.

Now from Pascal's theorem for the hexagons $AAXDDH$ and $BBXCCN$ we derive that F, K, G as well as J, L, I are collinear. The conclusion follows.