11^{th} Iranian Geometry Olympiad

October 18, 2024

Contest problems with solutions

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With special thanks to Mahdi Etesamifard and Hesam Rajabzadeh.

Contents

Elementary Level 3

Problems 3

Solutions 5

Intermediate Level 15

Problems 15

Solutions 17

Advanced Level 27

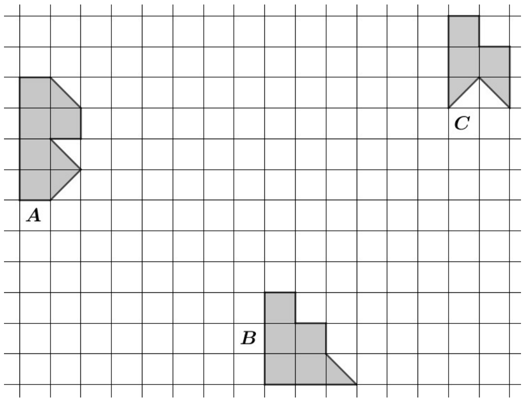
Problems 27

Solutions 29

Elementary Level

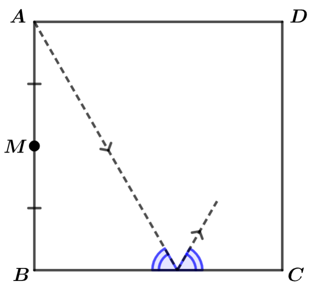
Problems

Problem 1. Reflect each of the shapes A, B over some lines l_A, l_B respectively and rotate the shape C such that a 4×4 square is obtained. Identify the lines l_A, l_B and the center of the rotation, and also draw the transformed versions of A, B and C under these operations.



(→ p.5)

Problem 2. $ABCD$ is a square with side length 20. A light beam is radiated from A and intersects sides BC, CD, DA respectively and reaches the midpoint of side AB. What is the length of the path that the beam has taken?



(→ p.6)

Problem 3. Inside a convex quadrilateral $ABCD$ with $BC > AD$, a point T is chosen. S lies on the segment AT such that $DT = BC, \angle TSD = 90°$. Prove that if $\angle DTA + \angle TAB + \angle ABC = 180°$, then $AB + ST \geqslant CD + AS$.

(→ p.8)

Problem 4. An inscribed n-gon ($n > 3$), is divided into $n - 2$ triangles by diagonals which meet only in vertices. What is the maximum possible number of congruent triangles obtained? (An inscribed n-gon is an n-gon where all its vertices lie on a circle)

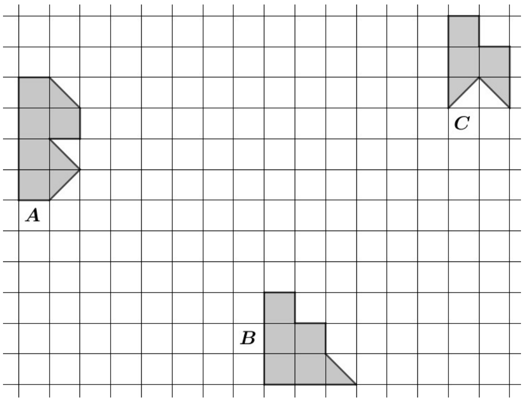
(→ p.9)

Problem 5. Points Y, Z lie on the smaller arc BC of the circumcircle of an acute triangle $\triangle ABC$ (Y lies on the smaller arc BZ). Let X be a point such that the triangles $\triangle ABC, \triangle XYZ$ are similar (in this exact order) with A, X lying on the same side of YZ. Lines XY, XZ intersect sides AB, AC at points E, F respectively. Let K be the intersection of lines BY, CZ. Prove that one of the intersections of the circumcircles of triangles $\triangle AEF, \triangle KBC$ lie on the line KX.

(→ p.10)

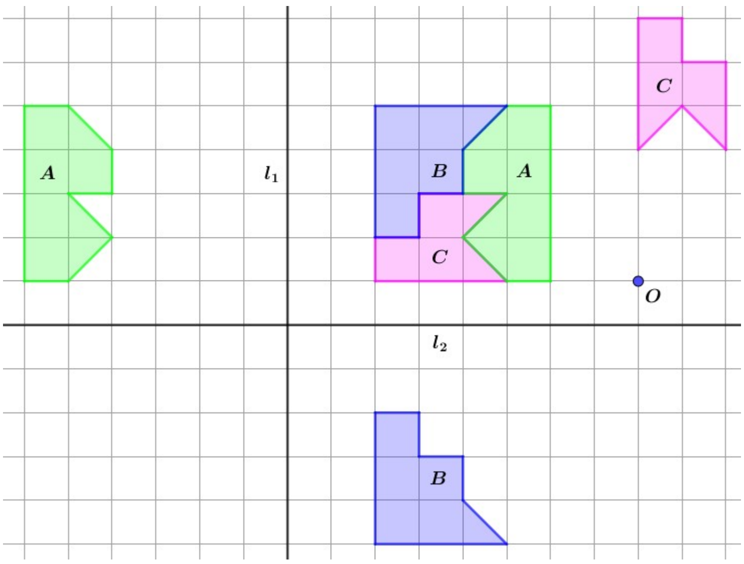
Solutions

Problem 1. Reflect each of the shapes A, B over some lines l_A, l_B respectively and rotate the shape C such that a 4×4 square is obtained. Identify the lines l_A, l_B and the center of the rotation, and also draw the transformed versions of A, B and C under these operations.

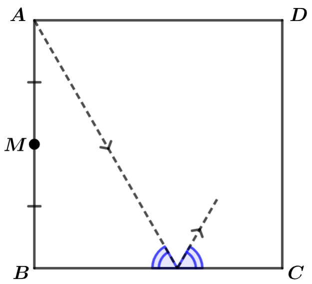


Proposed by Mahdi Etesamifard - Iran

Solution. As shown in the figure, we can create a good transformation in this manner. Note that the rotation of shape C is 90 degrees counterclockwise around point O.
Two lines l_1, l_2 are considered vertical and horizontal, respectively, to translate shapes A and B.



Problem 2. $ABCD$ is a square with side length 20. A light beam is radiated from A and intersects sides BC, CD, DA respectively and reaches the midpoint of side AB. What is the length of the path that the beam has taken?



Proposed by Mahdi Etesamifard - Iran

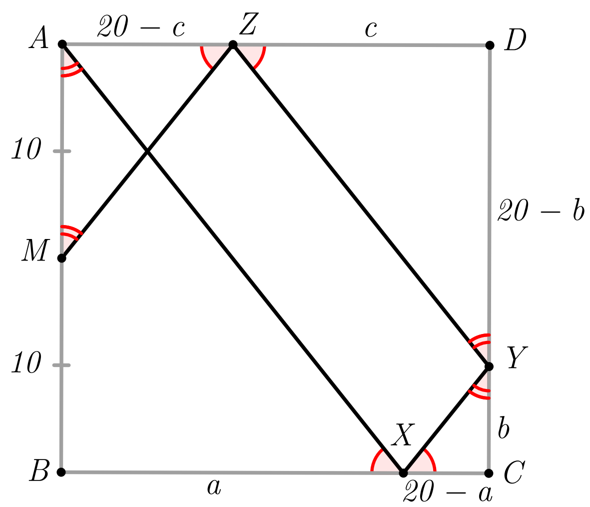
Solution. Assume that the intersection of the light beam with the sides BC, CD, DA is referred to as X, Y, Z. Also, we will call the lengths of the line segments BX, CY, DZ as a, b, c. Since the length of the square's side is 20, we will have:

$$CX = 20 - a, DY = 20 - b, AZ = 20 - c$$

Additionally, in triangles ABX, XCY, YDZ, ZAM, based on the existing angles, we will have:

$$\angle BAX = 90 - \angle AXB = 90 - \angle YXC$$

$$= \angle XYC = \angle ZYD = 90 - \angle DZY = 90 - \angle AZM = \angle AMZ$$



As a result, triangles ABX, XCY, YDZ, ZAM are similar. Now, based on the similarity ratios, we will have:

$$\triangle ABX \sim \triangle YCX \sim \triangle YDZ \sim \triangle MAZ \rightarrow \frac{AB}{BX} = \frac{YC}{CX} = \frac{YD}{DZ} = \frac{MA}{AZ}$$

$$\rightarrow \frac{20}{a} = \frac{b}{20 - a} = \frac{20 - b}{c} = \frac{10}{20 - c}$$

According to algebraic relations, for $2n$ real numbers $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$, if:

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n}$$

then:

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n} = \frac{a_1 + a_2 + ... + a_n}{b_1 + b_2 + ... + b_n}$$

so we have:

$$\frac{20}{a} = \frac{b}{20-a} = \frac{20-b}{c} = \frac{10}{20-c} = \frac{20 + b + (20-b) + 10}{a + (20-a) + c + (20-c)} = \frac{50}{40} = \frac{5}{4}$$

As a result, by calculating the fractions and setting them equal to the number $\frac{5}{4}$, we will have:

$$\frac{20}{a} = \frac{5}{4} \to a = 16$$
$$\frac{b}{20-a} = \frac{b}{4} = \frac{5}{4} \to b = 5$$
$$\frac{20-b}{c} = \frac{15}{c} = \frac{5}{4} \to c = 12$$

so, by Pythagoras theorem, we can calculate lengths AX, XY, YZ, ZM as follow:

$$AX = \sqrt{656}, XY = \sqrt{41}, YZ = \sqrt{369}, ZM = \sqrt{164}$$

$$\to AX + XY + YZ + ZM = \sqrt{41}(4 + 1 + 3 + 2) = \sqrt{41}.10$$

Problem 3. Inside a convex quadrilateral $ABCD$ with $BC > AD$, a point T is chosen. S lies on the segment AT such that $DT = BC, \angle TSD = 90°$. Prove that if $\angle DTA + \angle TAB + \angle ABC = 180°$, then $AB + ST \geqslant CD + AS$.

Proposed by Aleksander Tereshin - Russia

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Solution. Let K be a point such that $ABCK$ is a parallelogram. Now we have:

$$\angle KAT = \angle KAB - \angle TAB = 180° - \angle ABC - \angle TAB = \angle DTA$$

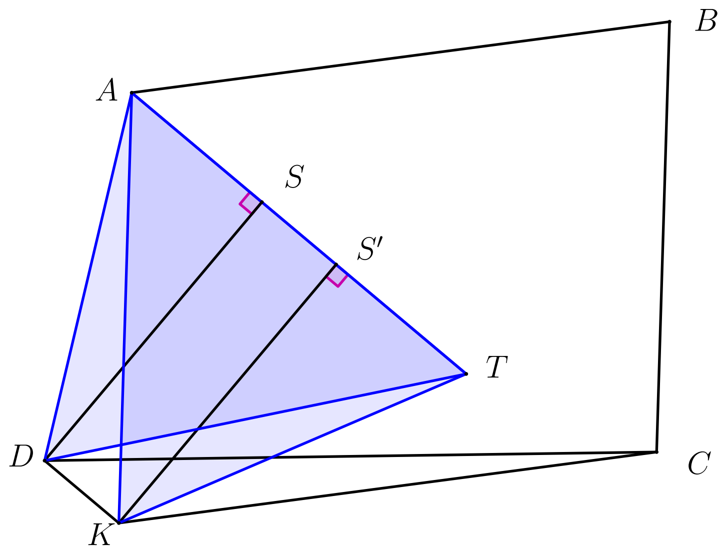
Therefore $\overset{\triangle}{ADT}$ and $\overset{\triangle}{TKA}$ are congruent, so $ADKT$ is an isosceles trapezoid.
Note that $ST > AS$, since $DT = BC > AD$. Let S' be the projection of K on AT, since $ADKT$ is an isosceles trapezoid, we have:

$$DK = SS' = ST - TS' = ST - AS$$

By triangle inequality, we have:

$$DK \geq DC - CK \Rightarrow ST - AS \geq DC - AB \Rightarrow AB + ST \geq CD + AS$$

and so we are done.



Problem 4. An inscribed n-gon ($n > 3$), is divided into $n-2$ triangles by diagonals which meet only in vertices. What is the maximum possible number of congruent triangles obtained? (An inscribed n-gon is an n-gon where all its vertices lie on a circle)

Proposed by Buris Frenkin - Russia

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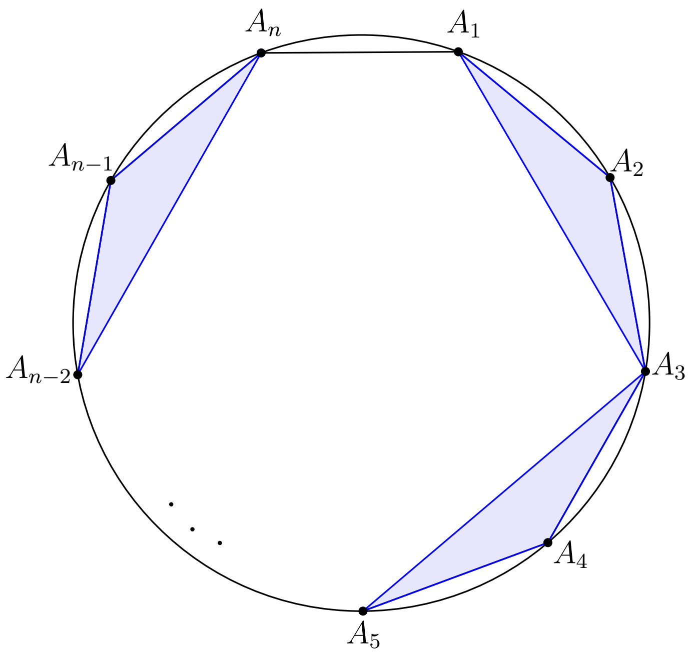
Solution.
Answer. $\lfloor \frac{n}{2} \rfloor$
Example. Consider a regular n-gon with consecutive vertices $A_1, A_2, ..., A_n$. Draw the diagonals between consecutive vertices with odd numbers. We get $\lfloor \frac{n}{2} \rfloor$ triangles congruent to $\triangle A_1A_2A_3$. The rest of the n-gon triangulate arbitrarily under the condition of the problem.



Example for odd n

Estimate. If one of triangles from the condition is acute (resp. right) then it contains the center of the circle inside it (resp. on its boundary). Thus the number of acute (resp. right) triangles in the triangulation is 1 (resp. 2) at most.
Now consider the obtuse triangles. If some of them are congruent then their sides opposite to obtuse angles are equal. The midpoints of these sides are equidistant from the center O of the circumscribed circle Ω, so they lie on a circle ω also centered at O. Consider two consecutive midpoints on ω, and let the respective sides of triangles have endpoints A_1, A_2 and B_1, B_2 on Ω (listing clockwise). Then B_1 cannot lie between A_1 and A_2, otherwise the sides intersect inside the n-gon.
Thus the pairs of vertices of the triangles in question are located on Ω consecutively (of course a vertex can belong to two such triangles). The vertex of the obtuse angle of such a triangle is located between the two others. Hence the vertices of the obtuse angles are separated by the other vertices of the triangles. So if the number k of congruent triangles in the triangulation is k then the total number of vertices of the n-gon is $2k$ at least, thus $k \leq \lfloor \frac{n}{2} \rfloor$ as required.

Problem 5. Points Y, Z lie on the smaller arc BC of the circumcircle of an acute triangle $\triangle ABC$ (Y lies on the smaller arc BZ). Let X be a point such that the triangles $\triangle ABC, \triangle XYZ$ are similar (in this exact order) with A, X lying on the same side of YZ. Lines XY, XZ intersect sides AB, AC at points E, F respectively. Let K be the intersection of lines BY, CZ. Prove that one of the intersections of the circumcircles of triangles $\triangle AEF, \triangle KBC$ lie on the line KX.

Proposed by Amirparsa Hosseini Nayeri - Iran

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Solution. First note that A, E, F, X lie on the same circle named as ω(since $\angle EXF = \angle EAF = 180° - \angle BAC$). Let Q be the second intersection of the circumcirle of triangles $B\overset{\Delta}{K}C, Y\overset{\Delta}{K}Z$. we will show that Q lies on ω. By angle chasing we have:

$$\angle BEY = \angle ABY - \angle BYE = \angle ABC + \angle CBY - \angle XYK$$

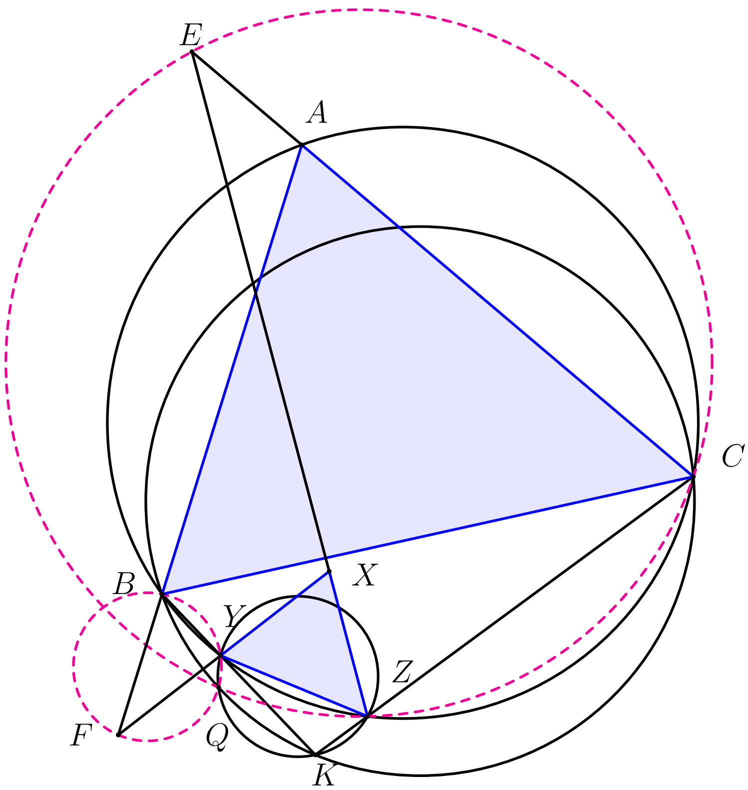
$$= \angle ABC + \angle CBY - \angle XYZ - \angle KYZ = \angle KZY - \angle KCB$$

$$= (180° - \angle KCB) - (180° - \angle KZY) = \angle KQB - \angle KQY = \angle BQY$$

(Note that $\angle ABC = \angle XYZ$, since $A\overset{\Delta}{B}C \sim X\overset{\Delta}{Y}Z$)
so $BQEY$ is cyclic and similarly, $CZQF$ is cyclic. so we have:

$$\angle QEX = \angle QBY = \angle QCZ = \angle QFX$$

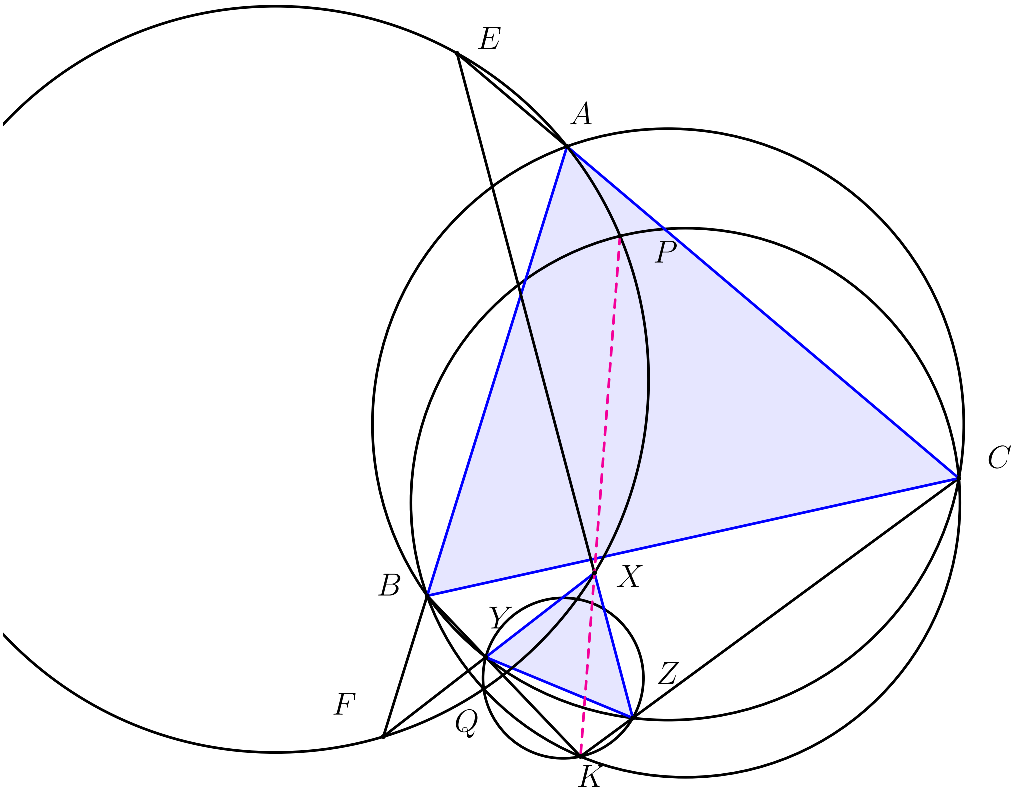
Therefore Q lies on ω.



Now let P be the second intersection of circumcircle of triangles $\overset{\Delta}{AEF}, \overset{\Delta}{BKC}$. we claim that P, X, K are collinear.

$$\angle QPX = \angle QEY = \angle QBK = \angle QPK$$

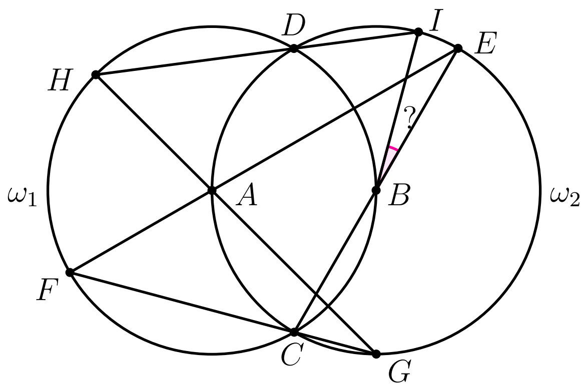
so P, X, K are collinear and we are done.



Intermediate Level

Problems

Problem 1. In the figure below points A, B are the centers of the circles ω_1, ω_2. Starting from the line BC points E, F, G, H, I are obtained respectively. Find the angle $\angle IBE$.



(→ p.17)

Problem 2. Points X, Y lie on the side CD of a convex pentagon $ABCDE$ with X between Y and C. Suppose that the triangles $\triangle XCB, \triangle ABX, \triangle AXY, \triangle AYE, \triangle YED$ are all similar (in this exact order). Prove that circumcircles of the triangles $\triangle ACD, \triangle AXY$ are tangent.

(→ p.18)

Problem 3. Let $\triangle ABC$ be an acute triangle with a point D on side BC. Let J be a point on side AC such that $\angle BAD = 2\angle ADJ$, and ω be the circumcircle of triangle $\triangle CDJ$. The line AD intersects ω again at a point P, and Q is the feet of the altitude from J to AB.
Prove that if $JP = JQ$, then the line perpendicular to DJ through A is tangent to ω.

(→ p.19)

Problem 4. Eric has assembled a convex polygon P from finitely many centrally symmetric (not necessarily congruent or convex) polygonal tiles. Prove that P is centrally symmetric.

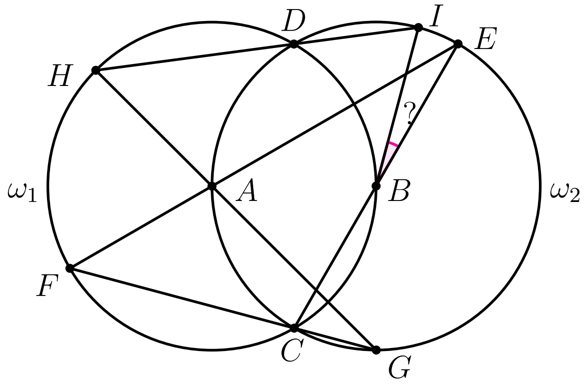
(→ p.21)

Problem 5. Point P is the intersection of diagonals AC, BD of the trapezoid $ABCD$ with $AB \parallel CD$. Reflections of the lines AD and BC into the internal angle bisectors of $\angle PDC$ and $\angle PCD$ intersects the circumcircles of $\triangle APD$ and $\triangle BPC$ at D' and C'. Line $C'A$ intersects the circumcircle of $\triangle BPC$ again at Y and $D'C$ intersects the circumcricle of $\triangle APD$ again at X. Prove that P, X, Y are collinear.

(→ p.22)

Solutions

Problem 1. In the figure below points A, B are the centers of the circles ω_1, ω_2. Starting from the line BC points E, F, G, H, I are obtained respectively. Find the angle $\angle IBE$.

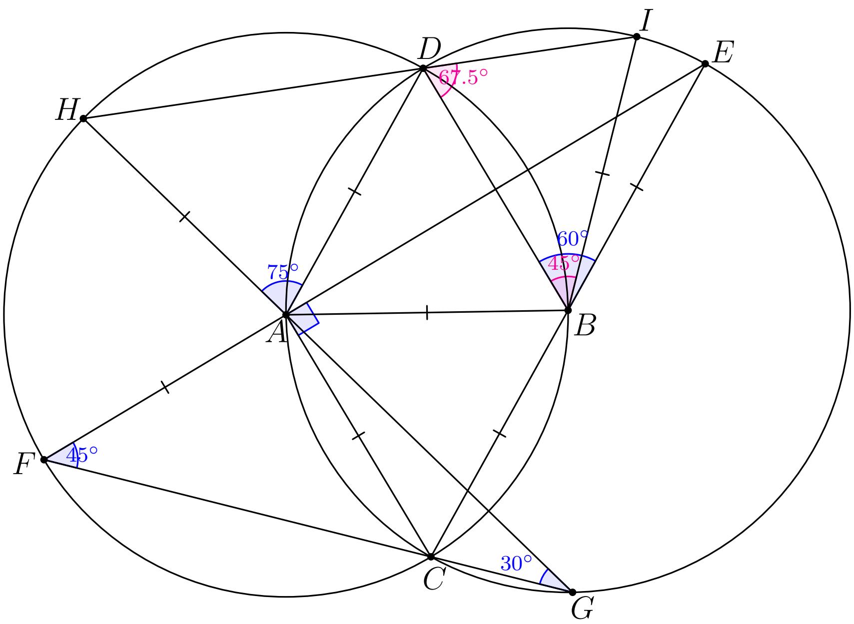


Proposed by - Syria

Solution. Since the circles are congruent, the triangles $\triangle ABC$ and $\triangle ABD$ are equilateral. Also $\triangle ACE$ is a right triangle, since CE passes through B. Therefore:

$$\angle CEA = \angle BAE = \angle EAD = \angle CGA = 30^\circ$$

Hence B being the center yields $\angle DBE = 60^\circ$.
Since $\angle CAE = 90^\circ$, then $\angle AFC = 45^\circ$, so $\angle GAE = \angle AGF + \angle AFG = 75^\circ$, which yields $\angle HAD = 180^\circ - \angle GAD = 75^\circ$ and $\angle HDA = 52.5^\circ$. Finally we have $\angle BDI = 180^\circ - \angle HDB = 67.5^\circ$ or $\angle DBI = 180^\circ - 2\angle BDI = 45^\circ$. Thus $\angle IBE = \angle DBE - \angle DBI = 15^\circ$



Problem 2. Points X, Y lie on the side CD of a convex pentagon $ABCDE$ with X between Y and C. Suppose that the triangles $\triangle XCB, \triangle ABX, \triangle AXY, \triangle AYE, \triangle YED$ are all similar (in this exact order). Prove that circumcircles of the triangles $\triangle ACD, \triangle AXY$ are tangent.

Proposed by Pouria MahmoodKhan Shirazi - Iran

Solution. Let's begin the solution with a claim

Claim 1. *Triangles $\triangle AED, \triangle ABC$ are similar.*

Proof. Note that $\triangle AEY \sim \triangle XBC$ and $\triangle YED \sim \triangle ABX$, hence $\angle AEY = \angle XBC$ and $\angle YED = \angle XBA$:

$$\angle AED = \angle AEY + \angle YED = \angle XBC + \angle ABX = \angle ABC \tag{1}$$

$\triangle ABX \sim \triangle YED$:

$$\frac{AB}{BX} = \frac{EY}{ED} \tag{2}$$

$\triangle AEY \sim \triangle XBC$:

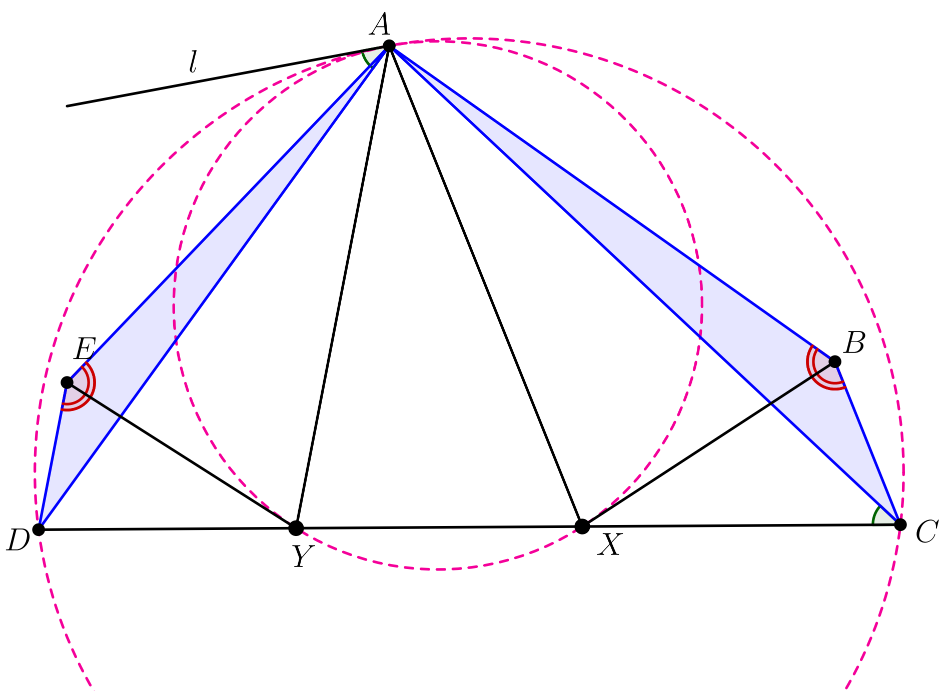
$$\frac{AE}{EY} = \frac{BX}{BC} \tag{3}$$

By (2) and (3), $AB.ED = EY.BX = AE.BC$ hence:

$$\frac{AB}{AE} = \frac{BC}{ED} \tag{4}$$

By (1) and (4) the triangles $\triangle AED, \triangle ABC$ are similar. ∎

Note that $\angle CAX = \angle BAX - \angle BAC = \angle YAE - \angle EAD = \angle YAD$. Let l be the line tangent at point A to the circumcircle $\triangle ADC$, then $\angle lAD = \angle ACD$, thus $\angle lAY = \angle lAD + \angle DAY = \angle ACD + \angle XAC = \angle AXY$ and l is tangent to the circumcircle of triangle $\triangle AXY$.



Problem 3. Let $\triangle ABC$ be an acute triangle with a point D on side BC. Let J be a point on side AC such that $\angle BAD = 2\angle ADJ$, and ω be the circumcircle of triangle $\triangle CDJ$. The line AD intersects ω again at a point P, and Q is the feet of the altitude from J to AB.
Prove that if $JP = JQ$, then the line perpendicular to DJ through A is tangent to ω.

Proposed by Ivan Chan - Malaysia

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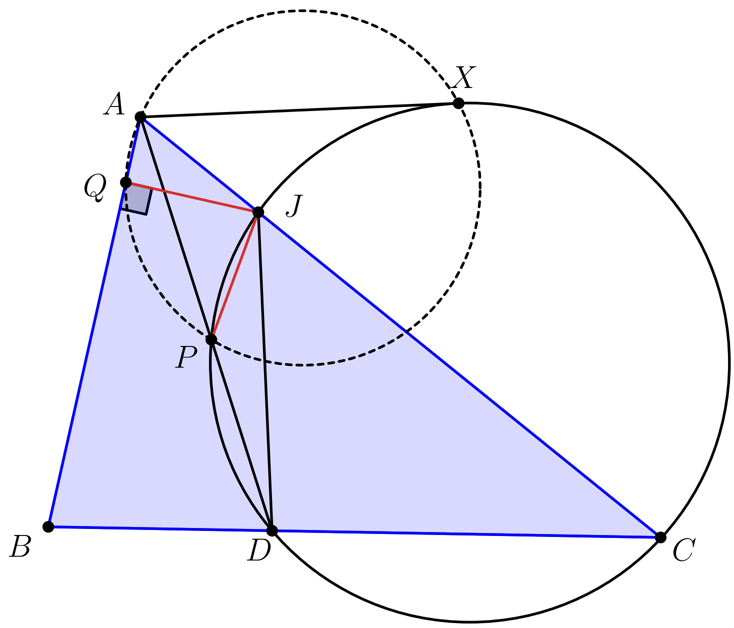
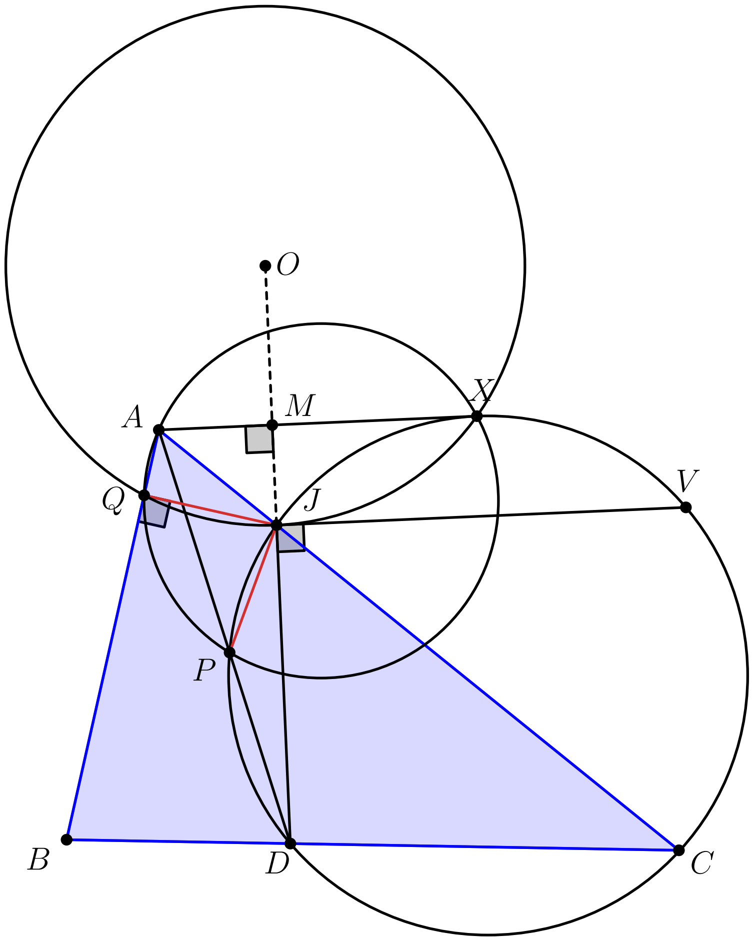
Solution 1. Let circle (AQP) intersect ω again at $X = P$. We claim that $AX \perp DJ$ and AX is tangent to ω.
First let's note that $\angle QXP = \angle QAP = 2\angle ADJ = 2\angle PXJ$, so JX is the angle bisector of $\angle QXP$. As $JP = JQ$, then JX is the perpendicular bisector of PQ. As A and X lie on the same side of line PQ, then $XP = XQ$ implies AX is the external angle bisector of $\angle QAP$. But DJ is parallel to the internal angle bisector of $\angle QAP$, so $AX \perp DJ$.
It suffices to prove that $\angle AXP = \angle XDP$. But this is because:

$$\angle AXP = \angle BQP = 90^\circ - \angle JQP = 90^\circ - \angle JPQ = 180^\circ - \angle XJP = \angle XDP$$

This completes the solution.



Solution 2. This is another solution which proves that AX is tangent to ω by proving $\angle AXJ = \angle XCJ$.
Let DJ intersect AX at M, then we get $\angle AXJ = 90° - \angle MJX = 90° - \angle XCD$. So we want to prove that $\angle XCJ = 90° - \angle XCD$. Let V be a point on ω such that DV is a diameter, then it suffice to prove that $\angle VCX = 90° \angle XCD = \angle XCJ$, namely the angle subtended by the arcs XJ and XV are equal. So it suffices to prove that $XJ = XV$. Let the center of the circle (JQX) be O, then we claim that O lies on DJ. Note that $\angle XJO = 90° - \angle XQJ = \angle AQX = \angle APX = \angle XCD = \angle XJM$, so the points J, O, M are colinear, and hence O lies on DJ.



By the claim above, then since DJ passes through the center of circle (JQX), while we have $DJ \perp JV$, then JV is tangent to circle (JQX). Finally, this implies that $\angle VJX = \angle XQJ = \angle XPJ = \angle XVJ$. So $XV = VJ$, as desired.

Problem 4. Eric has assembled a convex polygon P from finitely many centrally symmetric (not necessarily congruent or convex) polygonal tiles. Prove that P is centrally symmetric.

Proposed by Josef Tkadlec - Czech Republic

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Solution. Denote the tiles by $T_1, \cdots, T_n$.

Fix a side of P. Without loss of generality, suppose it is vertical.

For each tile Ti imagine an ant Ai walking around the perimeter of the tile in the clockwise direction, and record all distances of its journey when it walked exactly up or exactly down. Since the tile Ti is centrally symmetric, if the ant Ai moved up for a total distance d (possibly split over several sections), it has also moved down for a total distance d (split over the same sections). Summing over all ants, the total distance traveled up by all the ants combined is equal to the total distance traveled down. Any time two tiles share (a section of) a side, the contributions of the ants traveling along those sections cancel. The only contributions that do not cancel are those that lie on the boundary of the polygon P. Thus, along the boundary of P, the ants in total traveled the same distance up as down. Since P is convex, there must be only one side where the ant (or ants) traveled up, and similarly only one side where the ant (or ants) traveled down, and those two traveled distances are equal. The same argument applies to any side of P.

Now imagine a beetle walking around the perimeter of P in the clockwise direction. If we erase the two sides of P where the beetle walked exactly up and exactly down, the rest of the perimeter of P is split into two arcs. Since for every side that the beetle traverses, there is an equal side that the beetle traverses in the exact opposite absolute direction, in each such pair of sides one side belongs to the "top" arc and one side belongs to the "bottom" arc. Thus, the vertical sides are the opposite sides of P.

Polygon P has congruent and parallel opposite sides, thus it is centrally symmetric.

Remark. Note that the claim is not true if we do not require P to be convex. As a counterexample, a 1×1 square and a 2×1 rectangle can be assembled into a (non-symmetric) L-shape.

Problem 5. Point P is the intersection of diagonals AC, BD of the trapezoid $ABCD$ with $AB \parallel CD$. Reflections of the lines AD and BC into the internal angle bisectors of $\angle PDC$ and $\angle PCD$ intersects the circumcircles of $\triangle APD$ and $\triangle BPC$ at D' and C'. Line $C'A$ intersects the circumcircle of $\triangle BPC$ again at Y and $D'C$ intersects the circumcricle of $\triangle APD$ again at X. Prove that P, X, Y are collinear.

Proposed by Iman Maghsoodi - Iran

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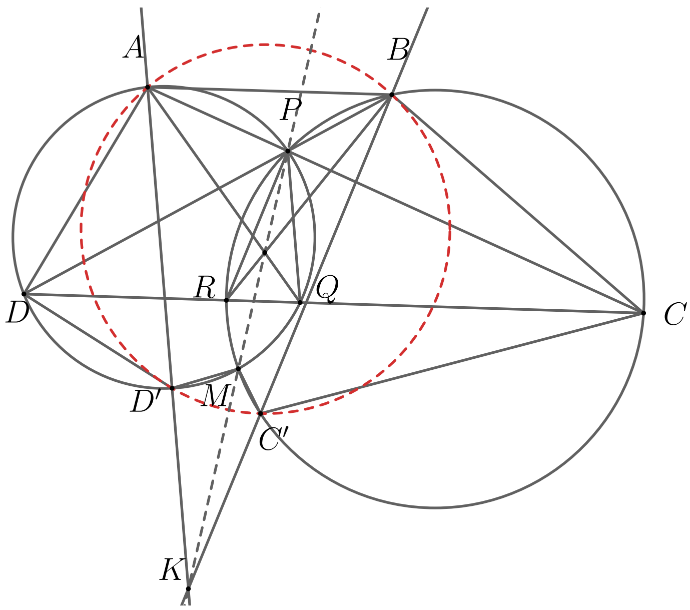
Solution. Let ω_1, ω_2 be the circumcircle of the triangles $\overset{\triangle}{APD}, \overset{\triangle}{BPC}$, respectively and let M be the second intersetion of ω_1 and ω_2. We shall prove the following lemma:
Lemma . *$ABC'D'$ is cyclic.*

Proof. By Radical axis theorem, it suffices to show that the lines AD', BC', PM are concurrent. Let K be the intersection point of AD' and BC' and let Q, R be the second intersection of CD with ω_1, ω_2, respectively. Since $\angle D'DC = \angle ADC$, so we have $\overset{\frown}{D'Q} = \overset{\frown}{AP}$, therefore $AD' || PQ$, and similarly, $BC' || PR$. Note that $\angle AQD = \angle APD = \angle BPC = \angle BRC = \angle ABR$, so $ABQR$ is cyclic and by Radical axis theorem, BR, AQ, PM are concurrent.

Now since $AB || QR$, so two triangles $\overset{\triangle}{PQR}$ and $\overset{\triangle}{ABK}$ are homothetic and therefore BR, AQ, PK are concurrent, which means that K lies on PM and we are done.



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Now let E be the intersection of AD' and CD. By angle chasing we have:

$$\angle MC'D' = \angle BC'D' - \angle BC'M = (180^\circ - \angle BAD') - \angle MPD = \angle AEQ - \angle MPD$$

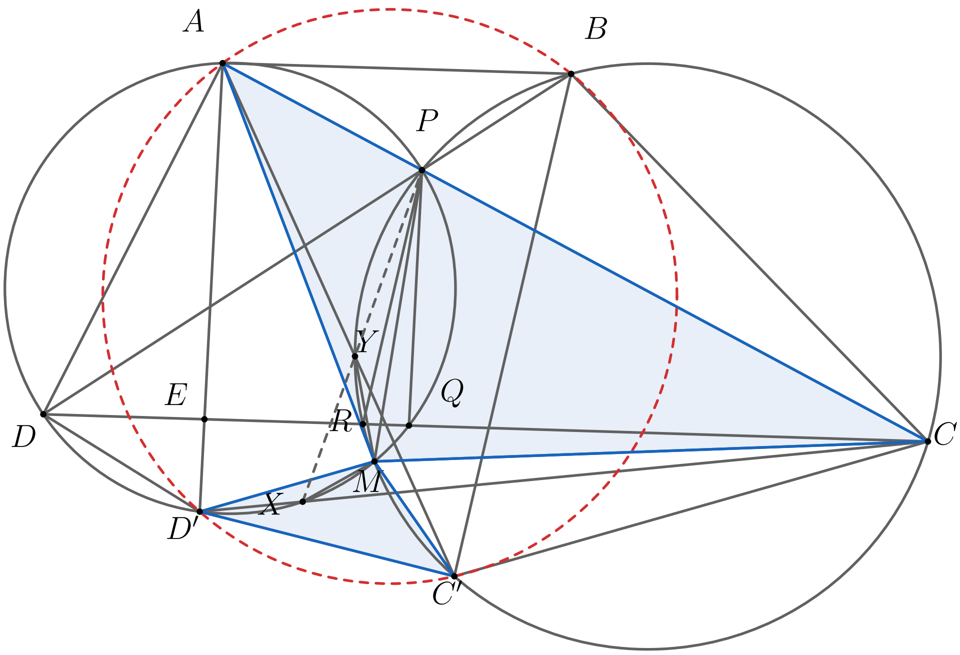
$$= \frac{\overset{\frown}{AQ} + \overset{\frown}{DD'}}{2} - \frac{\overset{\frown}{DD'} + \overset{\frown}{D'M}}{2} = \frac{\overset{\frown}{PD'} - \overset{\frown}{D'M}}{2} = \frac{\overset{\frown}{PM}}{2} = \angle MAC$$

Similarly we can show that $\angle MD'C' = \angle MCA$, so $\overset{\triangle}{MAC}$ and $\overset{\triangle}{MC'D'}$ are similar, and therefore $\overset{\triangle}{MC'A}$ and $\overset{\triangle}{MD'C}$ are similar

We have:

$$\angle MPX = \angle MD'X = \angle MD'C = \angle MC'A = \angle MC'Y = \angle MPY$$

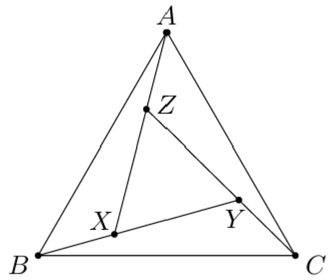
So P, X, Y are collinear.



Advanced Level

Problems

Problem 1. An equilateral triangle is split into 4 triangles with equal area; three congruent triangles $\triangle ABX, \triangle BCY, \triangle CAZ$, and a smaller equilateral triangle $\triangle XYZ$, as shown. Prove that the points X, Y, Z lie on the incircle of triangle $\triangle ABC$.



(→ p.29)

Problem 2. Point P lies on the side CD of the cyclic quadrilateral $ABCD$ such that $\angle CBP = 90°$. Let K be the intersection of AC, BP such that $AK = AP = AD$. H is the projection of B onto the line AC. Prove that $\angle APH = 90°$.

(→ p.30)

Problem 3. In the triangle $\triangle ABC$ let D be the foot of the altitude from A to the side BC and I, I_A, I_C be the incenter, A-excenter, and C-excenter, respectively. Denote by $P \neq B$ and $Q \neq D$ the other intersection points of the circle $\triangle BDI_C$ with the lines BI and DI_A, respectively. Prove that $AP = AQ$.

(→ p.31)

Problem 4. Point P is inside the acute triangle $\triangle ABC$ such that $\angle BPC = 90°$ and $\angle BAP = \angle PAC$. Let D be the projection of P onto the side BC. Let M and N be the incenters of the triangles $\triangle ABD$ and $\triangle ADC$ respectively. Prove that the quadrilateral $BMNC$ is cyclic.

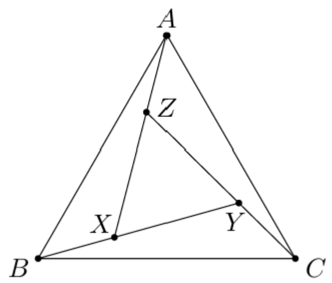
(→ p.34)

Problem 5. Cyclic quadrilateral $ABCD$ with circumcircle ω is given. Let E be a fixed point on segment AC. M is an arbitrary point on ω, lines AM and BD meet at a point P. EP meets AB and AD at points R and Q, respectively, S is the intersection of BQ, DR and lines MS and AC meet at a point T. Prove that as M varies the circumcircle of triangle $\triangle CMT$ passes through a fixed point other than C.

(→ p.37)

Solutions

Problem 1. An equilateral triangle is split into 4 triangles with equal area; three congruent triangles $\triangle ABX, \triangle BCY, \triangle CAZ$, and a smaller equilateral triangle $\triangle XYZ$, as shown. Prove that the points X, Y, Z lie on the incircle of triangle $\triangle ABC$.



Proposed by Josef Tkadlec - Czech Republic

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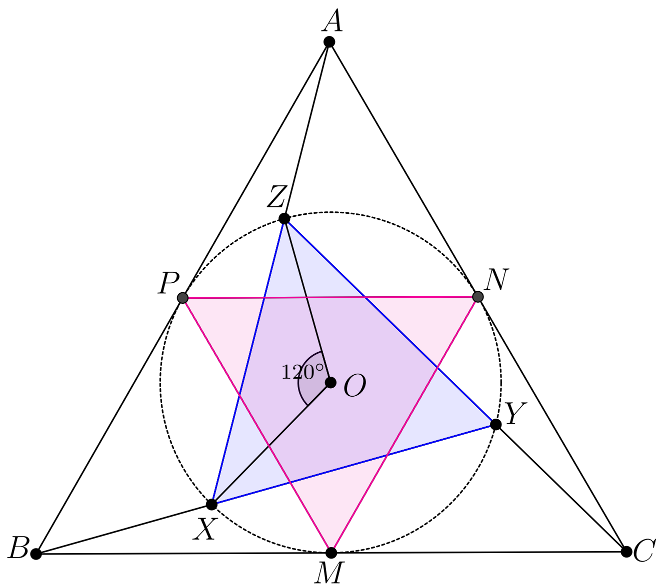
Solution. Let O be the center of the triangle $\triangle ABC$. Counter Clockwise rotation from O by $120°$, takes A to B, B to C and C to A. But since the three triangles on each side are congruent, the triangle $\triangle ABX$, becomes the triangle $\triangle BYC$, hence the rotation takes X to Y, and similarly, Y to Z and Z to X. Therefore O is the center of the triangle $\triangle XYZ$. Now notice that the triangles $\triangle XYZ$, and the medial triangle of $\triangle ABC$, are both equilateral, have the same area and the same center, so the radius of their circumcircles is equal too, which means that X, Y, Z are on the incircle as needed.



Problem 2. Point P lies on the side CD of the cyclic quadrilateral $ABCD$ such that $\angle CBP = 90°$. Let K be the intersection of AC, BP such that $AK = AP = AD$. H is the projection of B onto the line AC. Prove that $\angle APH = 90°$.

Proposed by Iman Maghsoodi - Iran

Solution. Let K' be the reflection of K about A. Notice that $AK = AP = AD = AK'$, so $KPDK'$ is a cyclic quadrilateral with circumcenter A. BA intersects $K'D$ at E. We have:

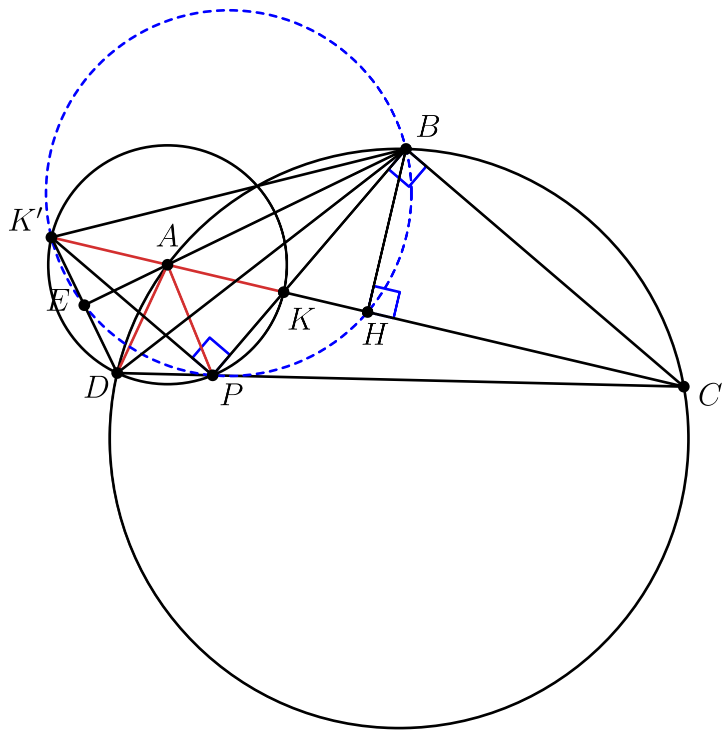
$$\angle ADK' = \angle KK'D = \angle KPC = 90° - \angle BCD = 90° - \angle EAD$$

So $BE \perp K'D$ and this result with $AK' = AD$, shows that BE is perpendicular bisector of $K'D$. Clearly $\angle K'PK = 90°$, hence $K'BHP$ is cyclic and we have:

$$\angle APK' = \angle AK'P = \angle HK'P = \angle HBP = \angle BCA = \angle BDA = \angle BK'A = \angle BPH$$

$$\Leftrightarrow \angle APH = \angle BPH + \angle APK = \angle APK' + \angle APK = 90°$$

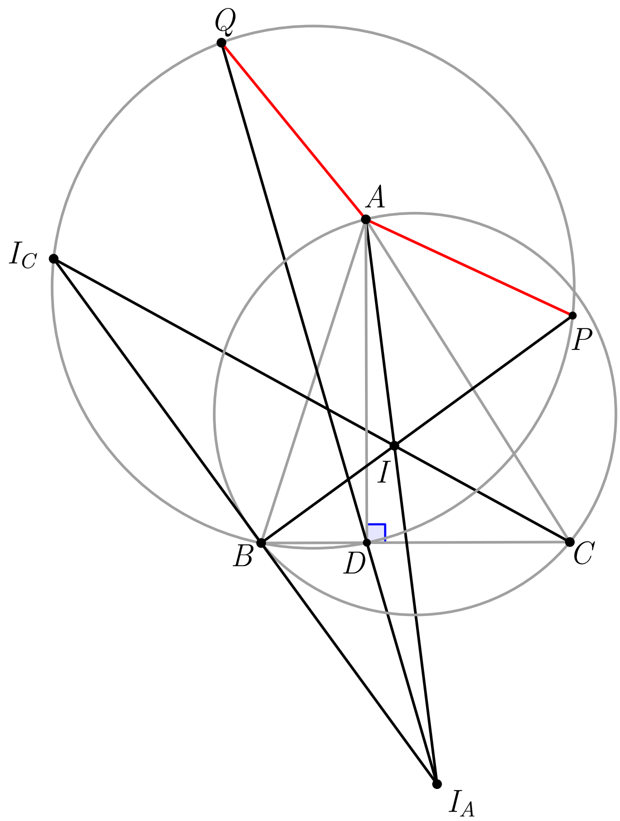
Which completes the proof.



Problem 3. In the triangle $\triangle ABC$ let D be the foot of the altitude from A to the side BC and I, I_A, I_C be the incenter, A-excenter, and C-excenter, respectively. Denote by $P \neq B$ and $Q \neq D$ the other intersection points of the circle $\triangle BDI_C$ with the lines BI and DI_A, respectively. Prove that $AP = AQ$.

Proposed by Michal Jan'ik - Czech

Solution.



We shall use direct angles throughout the whole solution. We should rather prove that $\angle APQ = \angle PQA$. It is known that A, I, B, I_C are concyclic. So as Q, D, B, I_C are concyclic, by power of the point I_A we get that Q, A, I, D are concyclic. Now we can write

$$\angle APQ = \angle BPQ - \angle APB = \angle BDQ - \angle APB$$

and

$$\angle PQA = \angle PQD - \angle DQA = \angle PBD - \angle DII_A = \angle PBD - \angle IDA - \angle DAI$$

First, we will prove that $\angle BDQ = 90 - \angle IDA$.
AS $\angle BDQ = \angle CDI_A$ and $\angle IDC = 90 - \angle IDA$, we just need to prove that CD is the angle bisector of $\angle IDI_A$.

Lemma 1. *In triangle ABC with the foot of A-Altitude D, incentre I and A-excentre I_A, the side BC is the angle bisector of $\angle IDI_A$.*

Proof. Let E, F be the touch points of incircle and A-excircle with the side BC, respectively. Then it suffices to prove that $\frac{FD}{ED} = \frac{FI_A}{EI}$ as then triangles DEI and DFI_A would be similar, because of the right angles $\angle DEI$ and $\angle DFI_A$. Let a, b, c and s be the lengths of BC, AC, AB and the semiperimeter, respectively. Then it is known that $BE = s - b$ and $BF = s - c$, so $ED = s - b - BD$, $FD = s - c - BD$ and then

$$\frac{FD}{ED} = \frac{s - c - BD}{s - b - BD} = 1 + \frac{b - c}{s - b - BD}$$

It is also known that the lengths of tangents from A to incircle and A-excircle are equal to $s-a$ and s, respectively. Thus $\frac{FI_A}{EI} = \frac{s}{s-a} = 1 + \frac{a}{s-a}$ and it suffices to prove that

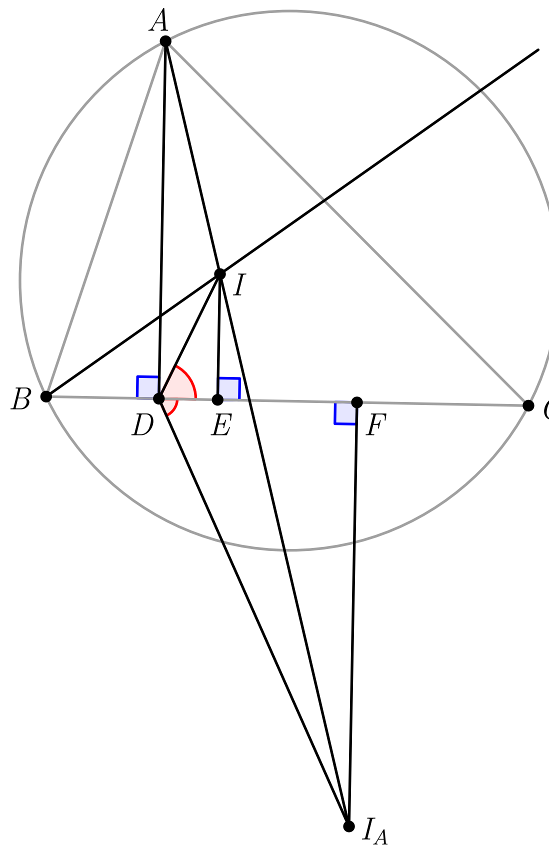
$$\frac{b-c}{s-b-BD} = \frac{a}{s-a}$$

or equivalently

$$2(b-c)(s-a) = 2a(s-b-BD) \Leftrightarrow (b-c)(b+c-a) = a(a+c-b-2BD)$$

$$b^2 - c^2 - ab + ac = a^2 + ac - ab - 2a.BD \Leftrightarrow b^2 = c^2 - BD^2 + (a-BD)^2 = AD^2 + (a-BD)^2$$

where in the last equality, we used Pythagorean theorem for triangle ADB and the last equation is true as it is the Pythagorean theorem for triangle ADC. ■



Now we return to the original problem.We need to prove that

$$\angle BDQ - \angle APB = \angle PBD - \angle IDA - \angle DAI$$

We know that

$$\angle BDQ = 90 - \angle IDA$$

Hence we need to prove the following

$$90 - \angle APB = \angle PBD - \angle DAI$$

we may further simplify the right hand side, as

$$\angle PBD - \angle DAI = \angle IBD - \frac{1}{2}(\angle B - \angle C)$$

so in the end,we may prove that

$$\angle APB = 90 - \frac{1}{2}\angle C = \angle BI_cA$$

We shall now prove this. Let B' be the intersection point of line AD with the circumcircle of $\triangle BDI_C$ different from D. Then BB' is a diameter of this circle since $\angle BDB' = 90°$. I_CP is a diameter of this circle as well, because of the right angle $\angle I_CBP$, we know that $BPB'I_C$ is a rectangle. Consider the point A' such that BI_CAA' is a parallelogram. Then $B'AA'P$ is also a parallelogram. We know that

$$\angle PA'A = \angle AB'P = \angle DB'P = \angle DBP = \angle PBA$$

So, P, A', A, B' are concyclic. But then $\angle BI_CA = \angle AA'B = \angle APB$
as desired.

Problem 4. Point P is inside the acute triangle $\triangle ABC$ such that $\angle BPC = 90°$ and $\angle BAP = \angle PAC$. Let D be the projection of P onto the side BC. Let M and N be the incenters of the triangles $\triangle ABD$ and $\triangle ADC$ respectively. Prove that the quadrilateral $BMNC$ is cyclic.

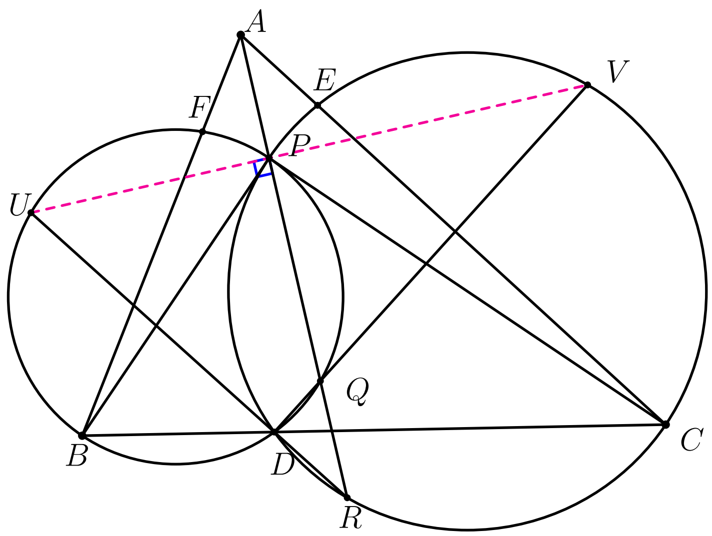
Proposed by Hussein Khayou - Syria

Solution. Let the circumcircles of triangles BPD, CPD intersect the line AP at points Q, R and the sides AB, AC at points F, E, respectively. Lines RD, QD intersect the circumcircles BPD, CPD at points U, V, respectively.

Claim 1. *Points P, U, V are collinear and $UV \perp AP$*

Proof. Note that

$$\angle PUD + \angle PRD = \angle PBC + \angle PCD = 90°$$
$$\angle PVD + \angle PQV = \angle PCB + \angle PBC = 90°$$



Therefore points P, U, V are collinear and $UV \perp AP$. ■

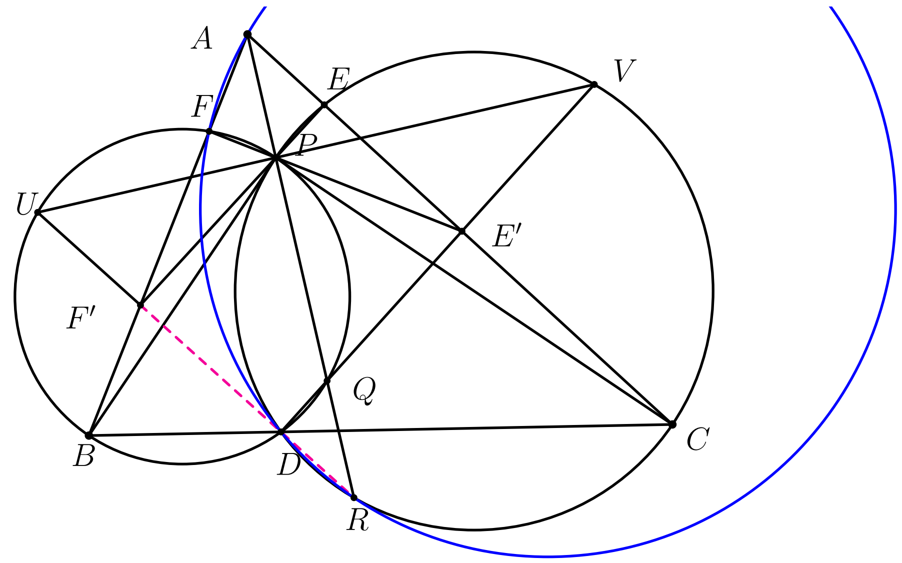
Triangles BPC, PDB, PDC are all similar. Let F', E' be the intersection points of the lines PE, PF with the sides AB, AC.

Claim 2. *D, R, F' and D, Q, E' are collinear.*

Proof. Since the quadrilaterals $FPDB$ and $PDRC$ are cyclic we have:

$$\angle PQD = \angle PCB = \angle DPB = \angle DFB$$

Hence the quadrilateral $AFDR$ is cyclic.



It is easy to see that points E, F lie on the circle with diameter AP, hence F' is the radical center of the circumcircles $AEPF, AFRD, EPDC$. Therefore F' lies on the line DR. similarly E' lies on the line DQ. ■

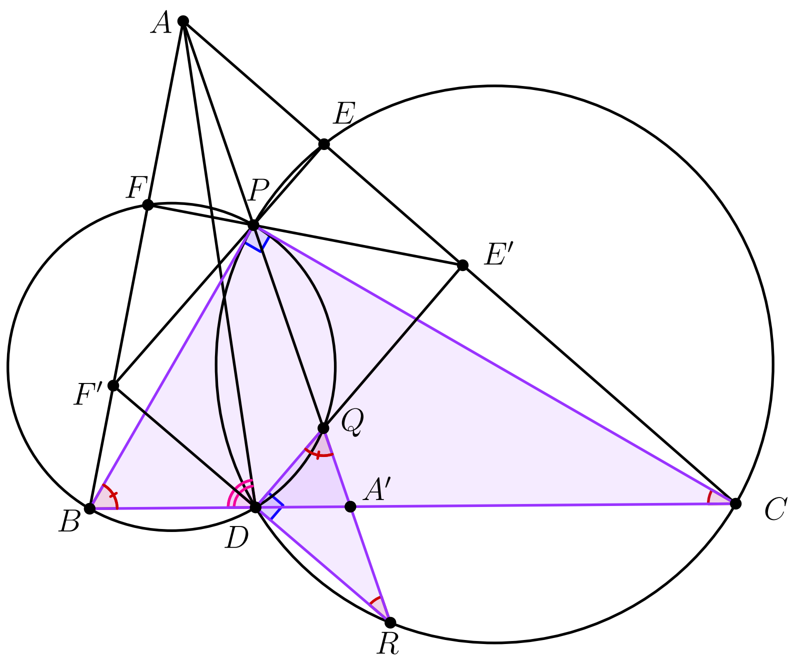
Let AP intersect the side BC at A'.

Claim 2. $\angle ADF' = \angle F'DB$

Proof. Note that $\angle QRD = \angle PCD$ and $\angle RQD = \angle PBC$ hence $\triangle DQR \sim \triangle PBC$. $\angle A'DR = \angle A'PC$ therefore $\frac{QA'}{A'R} = \frac{BA'}{A'C} = \frac{AB}{AC}$. Observing the power of the point A with respect to circumcircles FPD, EPD :

$$AF.AB = AP.AQ$$
$$AE.AC = AP.AR$$
$$\frac{AQ}{AR} = \frac{AB}{AC}$$

hence $(AA'; QR) = -1$ and since $\angle QDR = 90^\circ$ then the line DQ or DE' bisects the angle $\angle ADA'$. Similarly DF' bisects the angle ADA' and $\angle ADF' = \angle F'DB$ ■



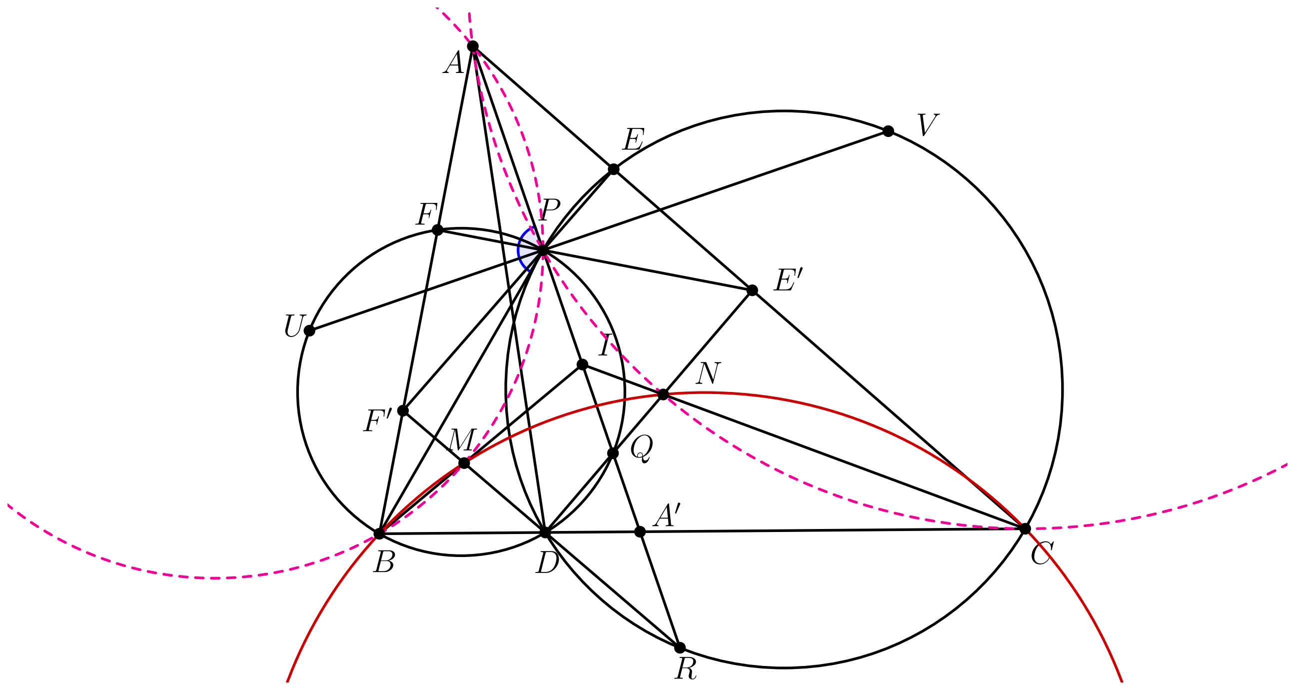
Now note that M, N lie on the lines DF', DE', respectively.
Claim 3. *Quadrilateral $APMB$ is cyclic.*

Proof.

$$\frac{1}{2}\angle ADB = \angle F'DB = \angle UDB = \angle UPB$$
$$\angle APB = \angle UPB + \angle APU = \frac{1}{2}\angle ADB + 90^\circ$$

But M is the incenter of triangle ADB and $\angle AMB = 90^\circ + \frac{1}{2}\angle ADB$, thus $\angle APB = \angle AMB$ and $APMB$ is cyclic. ∎

Similarly $APNC$ is also cyclic. Let I be the incenter of triangle $\triangle ABC$. Since I lies on the radical axis of circumcircles $APMB, APNC$ then $IM.IB = IP.IA = IN.IC$ hence quadrilateral $BMNC$ is cyclic.



Problem 5. Cyclic quadrilateral $ABCD$ with circumcircle ω is given. Let E be a fixed point on segment AC. M is an arbitrary point on ω, lines AM and BD meet at a point P. EP meets AB and AD at points R and Q, respectively, S is the intersection of BQ, DR and lines MS and AC meet at a point T. Prove that as M varies the circumcircle of triangle $\triangle CMT$ passes through a fixed point other than C.

Proposed by Chunlai Jin - China

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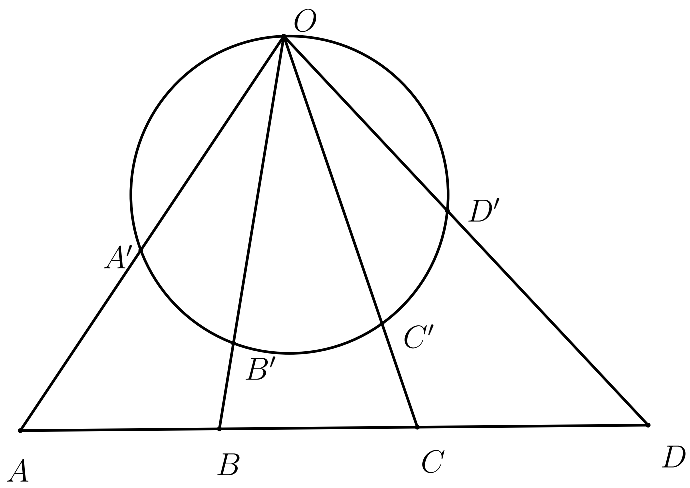
Solution.
Lemma 1. *Let A, B, C, D be four points on a line l in this order. Then*

$$AB.CD + AD.BC = AC.BC$$

Proof. Consider an inversion with respect to a point O not lying on l and radius 1, let A' be the inverse of A. Define B', C', D' similarly.Since A, B, C, D are collinear then $OA'B'C'D'$ is cyclic. Note that

$$AB = \frac{A'B'}{PA'.PB'}$$



Since $A'B'C'D'$ is cyclic by *Ptolemy*'s theorem:

$$A'B'.C'D' + A'D'.B'C' = A'C'.B'D'$$

Multiplying the latter by $\frac{1}{OA'}.\frac{1}{OB'}.\frac{1}{OC'}.\frac{1}{OD'}$ results in

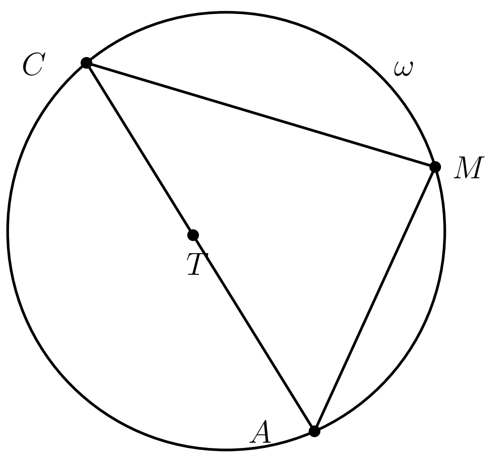
$$AB.CD + AD.BC = AC.BD$$

■

Lemma 2. *Points A, C lie on a fixed circle ω. Points M, T are moving along ω, AC such that*

$$\frac{CT}{AT} + \lambda\frac{CM}{AM} = k$$

for constants λ, k. Then the circumcircle of triangle $\triangle CMT$ passes through a fixed point.



Proof. Consider an inversion with center C and radius 1. Let A', M', T' be the inverse points of A, M, T. Note that

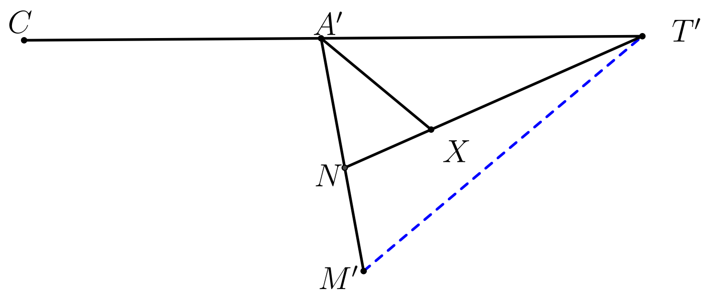
$$CT' = \frac{1}{CT}, A'T' = \frac{AT}{CA.CT}$$
$$M'C = \frac{1}{MC}, M'A' = \frac{MA}{CM.CA}$$

T' lies on the line CA' and M lies on a fixed line passing through A such that

$$\frac{CA'}{A'T'} + \lambda\frac{CA'}{A'M'} = k$$

Therefore $\frac{1}{A'T'} + \lambda\frac{1}{A'M'}$ is fixed. Let N be a point on line $A'M'$ such that $\lambda A'N = A'M'$. then

$$\frac{1}{A'N} + \frac{1}{A'T'} \Rightarrow fixed$$



Letting $A'X$ be the angle bisector of $\angle NA'T'$, by the *Trigonometric Collinearity* Lemma :

$$\frac{\sin A}{\sin\frac{A}{2}}.\frac{1}{AX} = \frac{1}{AN} + \frac{1}{AM'}$$

Hence AX is fixed and $M'N$ passes through the fixed point X. Note that the homothety centered at A maps $M' \to N$. Intersecting the line NX with CA' maps $N \to T'$, hence $M' \to N \to T'$ is a projective map. Note that A' maps to itself hence $M'T'$ passes through a fixed point. ■

Let AC, BD intersect at a point F. By **Lemma 1** :

$$AT.CF + AC.TF = AF.CT \tag{1}$$

U, V are the intersections of the line AS with BD, QR respectively. Denote by h_F, h_U, h_P the distances from F, U, P to the line SM, respectively. Note that

$$h_U = \frac{FU}{FP}.h_P + \frac{PU}{FP}.h_F$$

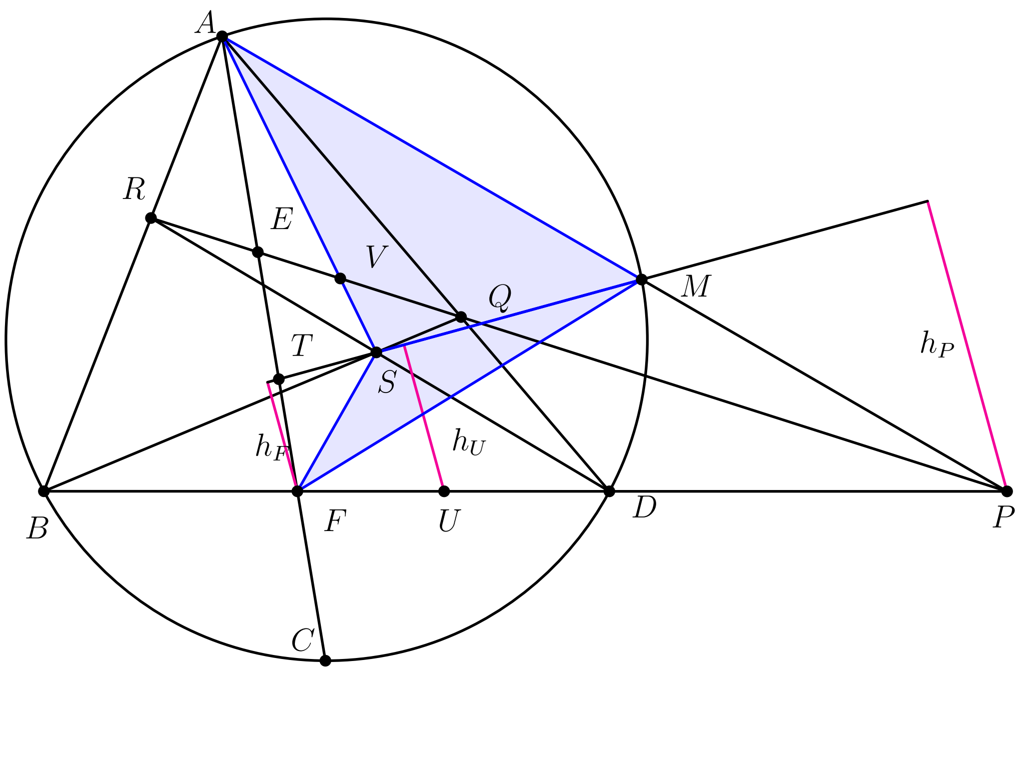
Denote by S_{XYZ} the area of triangle $\triangle XYZ$:

$$S_{FSM} = \frac{FP}{PU}S_{USM} - \frac{FU}{PU}S_{PSM}$$

hence

$$\frac{FT}{AT} = \frac{S_{FSM}}{S_{ASM}} = \frac{FP}{PU}.\frac{S_{USM}}{S_{ASM}} - \frac{FU}{PU}.\frac{S_{PSM}}{S_{ASM}}$$

$$= \frac{FP}{PU}.\frac{SU}{AS} - \frac{FU}{PU}.\frac{MP}{AM} \tag{2}$$



Note that $(AS;UV)=-1$, by properties of harmonic divisions and **lemma 1**:

$$SU.AV+VS.AU=AS.VU$$
$$SU.AV=VS.AU$$
$$\frac{UV}{SU.AV}=\frac{UV}{VS.AU}=\frac{2UV}{AS.UV}=\frac{2}{AS}$$
$$\Rightarrow\frac{SU}{AS}=\frac{1}{2}.\frac{UV}{AV}$$

By $Menelaus$'s theorem in triangle $\triangle AFU$ and points E,V,P :

$$\frac{FP}{UP}.\frac{UV}{AV}.\frac{AE}{EF}=1$$

hence

$$\frac{FP}{PU}.\frac{SU}{AS}=\frac{FP}{PU}.\frac{UV}{2AV}=\frac{1}{2EF} \tag{3}$$

Note that $(BD;UP)=-1$, by properties of harmonic divisions :

$$\frac{1}{PU}=\frac{1}{2}(\frac{1}{PB}+\frac{1}{PD})$$
$$\frac{FU}{PU}=\frac{1}{2}(\frac{FU}{PB}+\frac{FU}{PD})=\frac{1}{2}(\frac{BU-BF}{PB}+\frac{DF-DU}{PD})=\frac{1}{2}(\frac{DF}{PD}-\frac{BF}{PB})$$

Since $AMDB$ is cyclic $\angle PAD=\angle PBM,\angle PDM=\angle PAB$

$$\triangle PBM\sim\triangle PAD\Rightarrow\frac{PM}{PD}=\frac{BM}{AD}$$
$$\triangle PDM\sim\triangle PAB\Rightarrow\frac{PM}{PB}=\frac{DM}{AB}$$

hence

$$\frac{FU}{PU}.\frac{MP}{AM}=\frac{1}{2}(\frac{DF}{PD}.\frac{MP}{AM}-\frac{BF}{BP}.\frac{PM}{AM}=\frac{1}{2}(\frac{DF}{AD}.\frac{BM}{AM}-\frac{BF}{AB}.\frac{DM}{AM})$$

Since $ADCB$ is cyclic $\angle ADF=\angle BCF,\angle FAB=\angle FDC$

$$\triangle FAD\sim\triangle FBC\Rightarrow\frac{DF}{AD}=\frac{CF}{BC}$$
$$\triangle FAB\sim\triangle FCD\Rightarrow\frac{BF}{AB}=\frac{CF}{CD}$$

hence

$$\frac{FU}{PU}.\frac{MP}{AM}=\frac{1}{2}(\frac{CF}{BC}.\frac{BM}{AM}-\frac{CF}{CD}.\frac{DM}{AM})$$

By $Ptolemy$'s theorem on cyclic quadrilaterals $AMCB$ and $AMDC$:

$$AB.CM+AM.BC=AC.BM$$
$$AM.CD+AC.DM=AD.CM$$
$$\frac{BM}{AM}=\frac{BC}{AC}+\frac{AB.CM}{AC.AM},\frac{DM}{AM}=\frac{AD.CM}{AC.AM}-\frac{CD}{AC}$$

therefore

$$\frac{FU}{PU}.\frac{MP}{AM} = \frac{1}{2}.\frac{CF}{BC}(\frac{BC}{AC} + \frac{AB.CM}{AC.AM}) - \frac{1}{2}.\frac{CF}{CD}(\frac{AD.CM}{AC.AM} - \frac{CD}{AC})$$

$$= \frac{CF}{AC} + \frac{1}{2}.\frac{CF}{AC}(\frac{AB}{BC} - \frac{AD}{CD})\frac{CM}{AM} \tag{4}$$

By (2), (3), (4) :

$$\frac{FT}{AT} = \frac{1}{2}.\frac{EF}{AE} - \frac{CF}{AC} - \frac{1}{2}.\frac{CF}{AC}(\frac{AB}{BC} - \frac{AD}{CD})\frac{CM}{AM}$$

By (1) :

$$\frac{CT}{AT} = \frac{CF}{AF} + \frac{FT}{AT}.\frac{AC}{AF}$$

$$= \frac{CF}{AF} + \left[\frac{1}{2}.\frac{EF}{AE} - \frac{CF}{AC} - \frac{1}{2}.\frac{CF}{AC}(\frac{AB}{BC} - \frac{AD}{CD})\frac{CM}{AM}\right].\frac{AC}{AF}$$

$$= \frac{1}{2}.\frac{EF}{AE}.\frac{AC}{CF} - \frac{1}{2}.\frac{CF}{AF}(\frac{AB}{BC} - \frac{AD}{CD})\frac{CM}{AM}$$

Let $\lambda = \frac{1}{2}.\frac{CF}{AF}(\frac{AB}{BC} - \frac{AD}{CD})$ and $k = \frac{1}{2}.\frac{EF}{AE}.\frac{AC}{CF}$ then

$$\frac{CT}{AT} + \lambda\frac{CM}{AM} = k$$

thus by **Lemma 2** the circumcircle of triangle $\triangle CMT$ passes through a fixed point.