JBMO 2006

SOLUTIONS

ALGEBRA

Problem A1. Let a,b,c and m_a,m_b,m_c be respectively the lengths of the sides and the medians of an acute-angled triangle ABC. Prove that

$$\frac{m_a^2}{b^2+c^2-a^2}+\frac{m_b^2}{c^2+a^2-b^2}+\frac{m_c^2}{a^2+b^2-c^2}\geq\frac{9}{4}.$$

Solution: Taking into consideration that the triangle ABC is acute-angled, using the formulae for medians

$$4m_a^2=2b^2+2c^2-a^2,\quad 4m_b^2=2a^2+2c^2-b^2,\quad 4m_c^2=2a^2+2b^2-c^2$$

and using the notations $x=b^2+c^2-a^2>0$, $y=a^2+c^2-b^2>0$, $z=a^2+b^2-c^2>0$ we shall prove the equivalent inequality

$$\frac{4x+y+z}{x}+\frac{4y+z+x}{y}+\frac{4z+x+y}{z}\geq 18.$$

But

$$\frac{4x+y+z}{x}+\frac{4y+z+x}{y}+\frac{4z+x+y}{z}=12+\left(\frac{x}{y}+\frac{y}{x}\right)+\left(\frac{y}{z}+\frac{z}{y}\right)+\left(\frac{z}{x}+\frac{x}{z}\right)\geq 18.$$

Equality holds iff $x=y=z$, that is for the equilateral triangle.

Problem A2. Let x, y, z be real non-negative numbers such that $x+2y+3z=\frac{11}{12}$. Prove that

$$6(3xy+4xz+2yz)+6x+3y+4z+72xyz\leq\frac{107}{18}.$$

When does equality hold?

Solution: Let x, y, z be real non-negative numbers such that

$$x+2y+3z=\frac{11}{12}. \qquad (1)$$

Using the notations $a=6x$, $b=3y$ and $c=4z$, from (1) we obtain the equality

$$2a+8b+9c=11. \qquad (2)$$

So, for real non-negative numbers a, b, c, which satisfy the equality (2), we shall prove that

$$abc+ab+ac+bc+a+b+c+1=(a+1)(b+1)(c+1)\leq\frac{125}{18}.$$

By using the AM-GM inequality we have

$$144\cdot(a+1)(b+1)(c+1)=(2a+2)(8b+8)(9c+9)\leq$$

$$\left(\frac{2a+2+8b+8+9c+9}{3}\right)^3=\left(\frac{2a+8b+9c+19}{3}\right)^3=10^3.$$

From the last inequality it follows that $(a+1)(b+1)(c+1)\leq\frac{125}{18}$.

Equality holds iff $2a+2=8b+8=9c+9=10$ or $a=4, b=\frac{1}{4}, c=\frac{1}{9}$, that is

$x=\frac{2}{3}, y=\frac{1}{12}, z=\frac{1}{36}$.

Remark. The authors' original statement gave $s=x+2y+3z=\frac{5}{12}$; it is readily seen that we need $s\geq\frac{2}{3}$ in order for x, y, z of the equality case to be non-negative (as $\frac{5}{12}<\frac{2}{3}$, the Problem Selection Committee changed the value to be used for s).

Problem A3. Let $n\geq 3$ be an integer. A set of positive integers $x_1, x_2, \ldots, x_n$ is called *summable* if $\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}=1$.

(i) Prove that for every $n\geq 3$ there exists a summable set of n integers having the value of the largest element greater than $2^{2^{n-2}}$;

(ii) Prove that for every $n\geq 3$ there exists a summable set of n distinct integers having the value of the largest element less than n^2.

Solution: For a given n, denote by t_n the value of the largest element in a considered summable set $x_1, x_2, \ldots, x_n$.

(i) We shall use the identity

$$\frac{1}{m}=\frac{1}{m+1}+\frac{1}{m(m+1)}. \qquad (1)$$

We have the identity $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$ for $n=3$ and $t_3=6>2^{2^{n-2}}=4$.

By successively breaking, according to (1), $\frac{1}{6}=\frac{1}{7}+\frac{1}{42}$, then $\frac{1}{42}=\frac{1}{43}+\frac{1}{42\cdot 43}$, etc. we get summable sets with $n>3$ elements. But we have $t_{n+1}=t_n(t_n+1)>t_n^2$ hence for $t_n>2^{2^{n-2}}$ it follows that $t_{n+1}>2^{2^{(n+1)-2}}$.

The assertion (i) is thus proved by step-by-step constructing summable sets.

(ii) Using the identity (1), written as

$$\frac{1}{m(m+1)} = \frac{1}{m} - \frac{1}{m+1}$$

we get

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{(n-1)n} + \frac{1}{n} = 1.$$

If $n \neq m(m+1)$ for all m, then $t_n = (n-1)n < n^2$.
If $n = m(m+1)$ for some m, then $n-1 \neq k(k+1)$ for all k, as otherwise

$$1 = n-(n-1) = m(m+1) - k(k+1) = (m-k)(m+k+1)$$

implying $m + k = 0$, absurd. Let

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{(n-2)(n-1)} + \frac{1}{n-1} = 1.$$

Now write $\frac{1}{2\cdot 3} = \frac{1}{10} + \frac{1}{15}$ (as $n = m(m+1)$, $n > 3$, we will have $n-1 \neq 10, 15$).
We thus obtain a summable set

$$\frac{1}{1\cdot 2} + \frac{1}{10} + \frac{1}{3\cdot 4} + \frac{1}{15} + \frac{1}{4\cdot 5} + \dots + \frac{1}{(n-2)(n-1)} + \frac{1}{n-1} = 1$$

with n elements and $t_n = \max\{15, (n-2)(n-1)\} < n^2$.
The assertion (ii) is proved.

Remark. It can be shown that $t_n < 2^{n!}$ for any summable set of n elements (see A. Engel., Problem Solving Strategies). The authors do not have a better lower bound for t_n in the case of summable sets of n distinct elements.

GEOMETRY

Problem G1. Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $AB > CD$. It is given that the sum of its angles at the base AB is equal to 90^0. Prove that the distance between the midpoints of the parallel sides is equal to $\frac{1}{2}(AB - CD)$.

Solution: Let E be the intersection point of AD and BC. The point E is on the line through the midpoints K and F of the parallel sides. Let G and H be the points on AB, such that $CG \parallel EF$ and $CH \parallel AE$. Then the triangle HBC is right-angled with the right angle at C and the median CG. So,

$$KF = CG = \frac{1}{2}HB = \frac{1}{2}(AB - AH) = \frac{1}{2}(AB - CD).$$

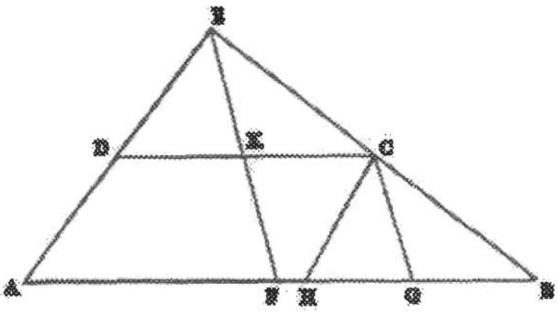


Figure G1

Problem G2.1 (first variant). Let the trapezoid $ABCD$ ($AB \parallel CD$) be inscribed in the circle γ, and $AB = 2CD$. The straight lines AD and BC meet at Q, and the tangents to γ at B and D meet at P. Let S denote the area of the triangle BDP. Find the area of the quadrilateral $ABPQ$.

Solution: Any inscribed trapezium $ABCD$ is isosceles. We have $DP = BP$ as common tangents from P, and $\angle DBP = \angle DBC + \angle CBP = \angle DBC + \angle ABD = \angle ABC$. Then $\angle BPD = 180^0 - 2\angle ABC = \angle AQB = \angle DQB$. It follows that the points D, B, P and Q are cocyclic. This implies $\angle BQP = \angle BDP = \angle ABC = \angle ABQ$. So, $QP \parallel AB$. From $AB = 2CD$ and $CD \parallel AB$ we obtain that CD is midsegment in the triangle AQB, i.e. D is the midpoint of AQ.

Let h be the height of the trapezoid $ABPQ$. We have

$$S_{DPQ} + S_{DAB} = \frac{1}{2}\cdot\frac{h}{2}\cdot QP + \frac{1}{2}\cdot\frac{h}{2}\cdot AB =$$

$$\frac{1}{2}\cdot\frac{AB + PQ}{2}\cdot h = \frac{1}{2}\cdot S_{ABPQ}.$$

Thus, we obtain $S_{ABPQ} = 2S_{BDP} = 2S$.

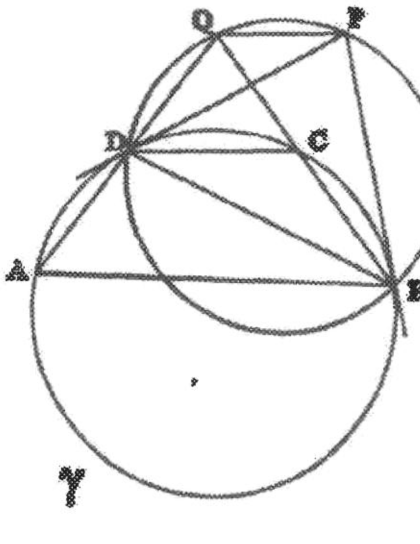


Figure G2.1

Problem G2.2 (second variant). Let the trapezoid $ABCD$ ($AB \parallel CD$) be inscribed in the circle γ of diameter AB, and $AB = 2CD$. The straight lines AD and BC meet at Q, and the tangents to γ at B and D meet at P. Let S denote the area of the triangle PQD. Find the area of the quadrilateral $ABPQ$.

Solution: Any inscribed trapezium $ABCD$ is isosceles. We have $DP = BP$ common tangents from P, and $\angle DBP = \angle DBC + \angle CBP = \angle DBC + \angle ABD = \angle ABC$. Then $\angle BPD = 180^0 - 2\angle ABC = \angle AQB = \angle DQB$. It follows that the points D, B, P and Q are cocyclic. This implies $\angle BQP = \angle BDP = \angle ABC = \angle ABQ$. So, $QP \parallel AB$. From $AB = 2CD$ and $CD \parallel AB$ we obtain that CD is midsegment in the triangle AQB, i.e. D is the midpoint of AQ. From $DO = OC = DC$ we have $\angle DOC = \angle DAB = 60^0$ and $\angle PDC = \angle CAB = \angle QDP = 30^0$. Let $DP \cap BQ = \{M\}$. From $DM = MP$ and $CM = \frac{1}{2}CQ = \frac{1}{2}BC$ it follows that the point C is the centroid of the triangle BDP. Then $S_{DBP} = 3S_{DCP} = 3S$.

Hence, we obtain $S_{ABPQ} = 2S_{BDP} = 6S$.

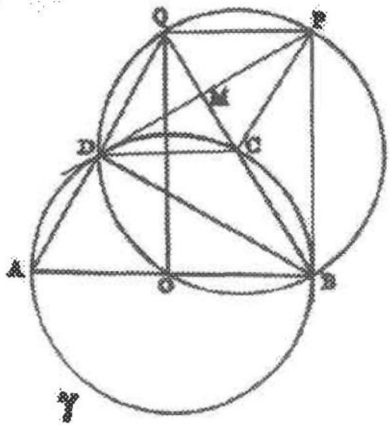


Figure G2.2

Problem G3. Let ABC be an isosceles triangle with $AB = AC$ and $\angle A < 60^0$. Let D and E be internal points lying on the side AC such that $EB = ED$ and $\angle ABD = \angle CBE$. The bisectors of the angles ACB and BDC meet at O. Compute $\angle COD$.

Solution: Let D and E be internal on the side AC such that $EB = ED$ and $\angle ABD = \angle CBE$. Suppose $E \in (AD)$. Then $\angle C + \angle CBD = \angle BDE = \angle EBD < \angle B$, contradiction. So, it follows that $D \in (AE)$. We have

$$\angle DBC = \angle DBE + \angle EBC = \angle EDB + \angle EBC =$$

$$\angle A + \angle ABD + \angle EBC = \angle A + 2\cdot\angle EBC =$$

$$180^0 - 2\cdot\angle B + 2\cdot\angle EBC = 180^0 - 2\cdot(\angle DBC + \angle ABD) +$$

$$+ 2\cdot\angle EBC = 180^0 - 2\cdot(\angle DBC + \angle EBC) +$$

$$+ 2\cdot\angle EBC = 180^0 - 2\cdot\angle DBC.$$

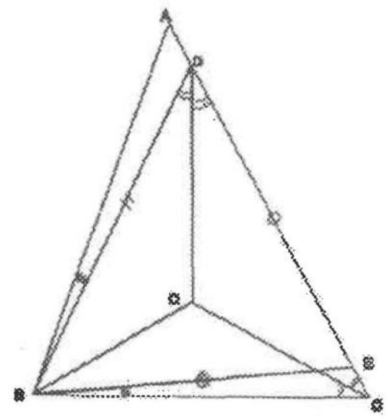


Figure G3

Hence $\angle DBC = 60^0$. Because the point O is the incenter of the triangle DBC, we obtain

$$\angle COD = 180^0 - (\angle OCD + \angle ODC) = 180^0 - \frac{1}{2}\cdot(\angle BDC + \angle DCB) =$$

$$180^0 - \frac{1}{2}\cdot(180^0 - \angle DBC) = 90^0 + \frac{1}{2}\cdot\angle DBC = 90^0 + 30^0 = 120^0.$$

Problem G4. Two circles γ_1 and γ_2 meet at A and B. Let M be an internal point lying on the segment AB. A straight line through M (other than AB) meets the circles γ_1 and γ_2 at Z, D and E, C respectively, such that E and D lie on the segment ZC. The perpendicular lines on EB and ZB at B, and the straight line AD meet the circle γ_2 a second time at F, K and N respectively. Prove that $KF = NC$.

Solution: We have

$$\angle NAC = \angle BAC - \angle BAN = \angle BAC - \angle BAD = \angle BEC - \angle BAD. \quad (1)$$

From $\angle BEC = \angle BZE + \angle ZBE$ and $\angle BAD = \angle BZD = \angle BZE$ we obtain

$$\angle ZBE = \angle BEC - \angle BZE =$$

$$\angle BEC - \angle BAD = \angle NAC.$$

The angles $\angle ZBE$ and $\angle KBF$ have the corresponding sides perpendicular, hence are equal.

From the equality $\angle ZBE = \angle KBF$ it follows that $\angle NAC = \angle KBF$. Thus the chords KF and NC are equal.

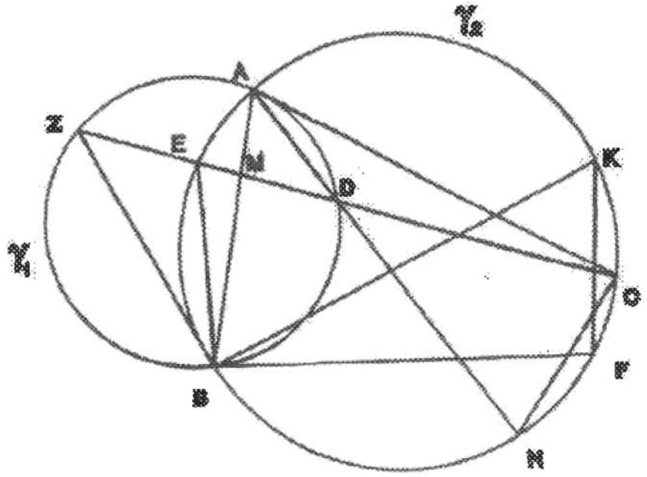


Figure G4

Problem G5. Let ABC be an equilateral triangle of center O. For an arbitrary internal point M lying on the side BC let K and L be its projections on the sides AB and AC respectively. Prove that the line OM passes through the midpoint of the segment KL.

Solution: Let N and Q be the midpoints of BC and AM respectively. From K, L and N the segment AM is seen under 90^0, so the points A, K, L, M and N are cocyclic and the triangles QKN and LQN are equilateral. Let $QN \cap KL = \{D\}$. Then $QN \perp KL$ and $\angle KQD = \angle LQD$, hence QN is the perpendicular bisector of the segment KL and D is the midpoint of the segment QN. The triangle AMN is right angled at N and NQ is the median corresponding to the hypotenuse.

By applying Menelaus' theorem for the triangle AQN and the points M, D, O, we obtain that they are collinear, because

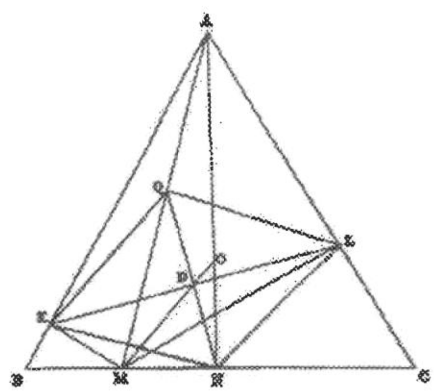


Figure G5

$$\frac{AM}{MQ} \cdot \frac{QD}{DN} \cdot \frac{NO}{OA} = 1, \text{ as } AM = 2MQ,\ QD = DN, \text{ and } AO = 2NO.$$

Problem G6. Let A_1 and B_1 be internal points lying on the sides BC and AC of the triangle ABC respectively, and segments AA_1 and BB_1 meet at O. The areas of the triangles AOB_1, AOB and BOA_1 are distinct prime numbers, and the area of the quadrilateral A_1OB_1C is an integer. Find the least possible value of the area of the triangle ABC, and argue the existence of such a triangle.

Solution: Let $S_{AOB_1}=p$, $S_{BOA_1}=q$, $S_{AOB}=r$, where p, q and r are distinct prime numbers. For two adjacent triangles that have a common height, the ratio of their areas is equal to the ratio of their base sides. This being the case for the pairs of triangles { AOB_1 and A_1OB_1 },{ BOA_1 and BOA },{ BCB_1 and ABB_1 }, { CA_1B_1 and AB_1A_1 }, we obtain

$$\frac{OA_1}{OA}=\frac{S_{OA_1B_1}}{S_{AOB_1}}=\frac{S_{BOA_1}}{S_{BOA}},\quad \frac{CB_1}{AB_1}=\frac{S_{CA_1B_1}}{S_{AB_1A_1}}=\frac{S_{BCB_1}}{S_{ABB_1}}.$$

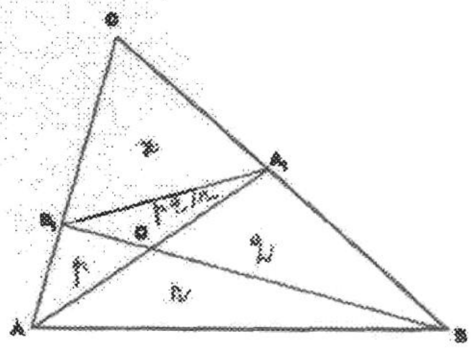


Figure G6

Let $S_{CA_1B_1}=x$. From the previous equalities we have

$$\frac{S_{A_1OB_1}}{p}=\frac{q}{r}\ \Rightarrow\ S_{A_1OB_1}=\frac{pq}{r},\quad \frac{x}{p+\frac{pq}{r}}=\frac{x+\frac{pq}{r}+q}{p+r}\ \Rightarrow\ x=\frac{pq(p+r)(q+r)}{r(r^2-pq)}.$$

From the last equality it follows that $r^2>pq$. We obtain

$$S_{A_1OB_1C}=\frac{pq}{r}+\frac{pq(p+r)(q+r)}{r(r^2-pq)}=\frac{pq(2r+p+q)}{r^2-pq},\quad S_{ABC}=\frac{r(r+p)(r+q)}{r^2-pq}.$$

Since the number S_{ABC} must be an integer, and as the numbers r and r^2-pq are mutually prime, we obtain that r^2-pq must be a divisor of $(r+p)(r+q)$.

If $\frac{(r+p)(r+q)}{r^2-pq}=k$ then $r^2+(p+q)r+(k+1)pq=kr^2$.

Since r and pq are coprime, from the last equality it follows that r must divide $k+1$. From here $k+1\ge r$ and $k\ge r-1$. So

$$S_{ABC}=kr\ge r(r-1)\qquad (1)$$

For $r=2$ or $r=3$ there are no distinct prime numbers p, q and r, such that $r^2>pq$.
For $r=5$ the pairs of distinct prime numbers (p,q), such that $pq<25$, are: (2;3), (2;7), (2;11) and (3;7). The first three do not make S_{ABC} an integer, while the fourth one yields $S_{ABC}=120$.
For $r=7$ we find that $p=2$, $q=11$ yield $S_{ABC}=42$, which is the best we can achieve, as seen in (1). Furthermore 42 is a lower value than 120 from the case above.
For $r>7$ we have the inequality $S_{ABC}>42$, as seen in (1), therefore the least possible value of the area of the triangle ABC is 42.

The existence of such a triangle is guaranteed by the very requirement $r^2>pq$. The value of r^2-pq is an indicator for AB_1A_1B being a trapezoid ($AB_1 \parallel A_1B$) when $r^2-pq=0$, while for $r^2-pq<0$ the triangle ABC is forming on the other side of AB, in disagreement with the conditions of the problem. Finally, for $r^2-pq>0$, one can always build a triangle ABC (the lines AB_1 and A_1B will meet on the "right" side of AB).

NUMBER THEORY

Problem NT1. Find the largest positive integer m such that the equation

$$2005x+2007y=m$$

has a unique solution with x and y positive integers.

Solution: Let the equation $2005x+2007y=m$ have a unique solution (x_0, y_0). If $x_0>2007$, i.e. $x_0=2007+t$, where t is a positive integer, then

$$2005\cdot t+2007\cdot(y_0+2005)=m.$$

So, the given equation has also the solution (t, y_0+2005). Hence $x_0\le 2007$.
Similarly $y_0\le 2005$.
Let $x_0=2007$, $y_0=2005$ and $m=2\cdot 2007\cdot 2005$. If (a,b) is a solution of the equation

$$2005\cdot x+2007\cdot y=2\cdot 2005\cdot 2007 \qquad (1)$$

then a is divisible by 2007 and b is divisible by 2005, i.e. $a=2007c$, $b=2005d$, where c and d are positive integers. So $c+d=2$, which implies $c=d=1$, i.e. the equation (1) has a unique solution (2007,2005). Thus $m=2\cdot 2005\cdot 2007$.

Problem NT2. A positive integer is called *perfect* if the sum of all its distinct divisors is equal to twice its value. Find all perfect numbers n such that $n-1$ and $n+1$ are both prime numbers.

Solution: The least perfect number, 6, obviously has this property. We shall prove that there are no more such numbers. Let $n-1$ and $n+1$ be prime numbers, for $n>6$. Then, for some integer $k>1$, we have that $n-1=6k-1$, $n+1=6k+1$ and $n=6k$. The numbers $\frac{n}{2},\frac{n}{3},\frac{n}{6}$ are distinct divisors of n, greater than 1, and $1+\frac{n}{2}+\frac{n}{3}+\frac{n}{6}=n+1>n$. Since the sum of four proper divisors of n is greater than n, the number n cannot be perfect.

Problem NT3. Find all pairs of positive integers (m, n) such that $m^2=nk+2$, where k is the number obtained from n by writing 1 on the left of its decimal representation.

Solution: Let $t\ge 1$ be the number of digits of n. Then $k=10^t+n$, and the given relation can be written in the following equivalent form

$$m^2=n\cdot(10^t+n)+2 \quad \text{that is} \quad m^2-n^2=n\cdot 10^t+2.$$

The last equality implies that both m, n are even or both m, n are odd.

If $t=1$, then $10n+2$ is divisible by 4 and n must be odd. From the equation $(n+5)^2-m^2=23$ we obtain the only solution (11,7). So $m=11$ and $n=7$.

If $t>1$, then 4 is a divisor of the number m^2-n^2, but $n\cdot 10^t+2$ is not divisible by 4. In this case there are no pairs (m,n) with the desired property.

Hence $(m,n)=(11,7)$.

Problem NT4. Prove that for any non-prime integer $n>4$ the number $2n$ divides $(n-1)!$.

Solution: As $n>4$ is not a prime, we can write $n=ab$ with $2\le a\le b,\ 2<b$.
From the relations

$$n-1-(a+b)=ab-1-(a+b)=(a-1)(b-1)-2\ge 0$$

we obtain that $n-1\ge a+b$.

We divide the solution into the cases:

(i) $a=b>2$; then $2b=b+a\le n-1$.
It follows that $2n=2\cdot b\cdot b$ divides $1\cdot ...\cdot b\cdot ...\cdot 2b$ which is a divisor of $(n-1)!$;

(ii) $a<b$

(1) $a>2$; then we have $2\cdot a\cdot b$ divides $1\cdot 2\cdot ...\cdot a\cdot ...\cdot b\cdot ...\cdot (a+b)$ which is a divisor of $(n-1)!$;

(2) $a=2$. If $b>4$ then $2n=2\cdot 2\cdot b$ divides $1\cdot 2\cdot 3\cdot 4\cdot ...\cdot b\cdot (b+1)\cdot (b+2)=(b+2)!$ which is a divisor of $(n-1)!$;

If $b=4$ then $2n=16$ divides $(n-1)!=7!$;

If $b=4$ then $2n=12$ divides $(n-1)!=5!$.

Comments to problem NT4. The following assertion is a stronger version of the problem:

Problem NT4.1. Prove that for any non-prime integer $n>4$ the number kn divides $(n-1)!$, for all integers k which satisfy $1\le k\le[\sqrt{n-1}]$.

Solution: As $n>4$ is not a prime, we can write $n=ab$ with $2\le a\le b,\ 2<b$.
From the relations

$$n-1-(a+b)=ab-1-(a+b)=(a-1)(b-1)-2\ge 0$$

we obtain that $n-1\ge a+b$. From $n=ab>n-1$ follows $b^2\ge ab>n-1$. Hence $b>\sqrt{n-1}$ and thus $b>[\sqrt{n-1}]$. So, the number k divides $(b-1)!$ for all $1\le k\le[\sqrt{n-1}]$, and kb divides the number $b!$.
It is well-known that the number $b!$ divides $(a+1)(a+2)\cdot ...\cdot (a+b)$.
We obtain that the number $kn=kab=a\cdot (kb)$ divides the number $a!b!$, which divides the number $a!\cdot (a+1)(a+2)\cdot ...\cdot (a+b)=(a+b)!$. Because $(a+b)!$ divides the number $(n-1)!$, we obtain that kn divides $(n-1)!$.

Problem NT5. Find all four-digit numbers $\overline{abcd}$ such that

$$\overline{abcd}=11\cdot (a+b+c+d)^2.$$

Solution: The given equality implies the estimations $10\le a+b+c+d\le 30$. Because the number $\overline{abcd}$ is divisible by 11, then $d-c+b-a$ is also divisible by 11. Therefore we have only the following three cases:

(i) $b+d=a+c$; (ii) $b+d=a+c+11$; (iii) $b+d=a+c-11$.

Let

$$1000\cdot a+100\cdot b+10\cdot c+d=11\cdot(a+b+c+d)^2. \qquad (1)$$

(i) If $b+d=a+c$ we get $5\le a+c\le 15$ and

$$b=\frac{(a+c)[4(a+c)-1]}{9}-10a.$$

As $(a+c)$ and $[4(a+c)-1]$ are coprime, the divisibility by 9 will be treated separately.
If $a+c$ is divisible by 9, then $a+c=9$ and $b=35-10a$. So, $a=3$, $b=5$, $c=6$, $d=4$ and $\overline{abcd}=3564$.
If $4(a+c)-1$ is divisible by 9, then $a+c=7$ and $b=21-10a$. Thus, $a=2, b=1, c=5$, $d=6$ and $\overline{abcd}=2156$.

(ii) If $b+d=a+c+11$, then we have $1\le a+c\le 7$ and

$$b=\frac{(a+c+1)[4(a+c+1)-1]}{9}+4(a+c+1)+9-10a.$$

As $(a+c+1)$ and $[4(a+c+1)-1]$ are coprime, the divisibility by 9 will be treated separately. Because $a+c+1$ cannot be divisible by 9, then $4(a+c+1)-1$ is divisible by 9. We obtain that $a+c=6$ and $b=58-10a$. So, $a=5$, $b=8$, $c=1$, $d=9$ and $\overline{abcd}=5819$.

(iii) If $b+d=a+c-11$, then we have $11\le a+c\le 18$ and

$$b=\frac{(a+c-1)[4(a+c-1)-1]}{9}-4(a+c-1)+9-10a.$$

As $(a+c-1)$ and $[4(a+c-1)-1]$ are coprime, the divisibility by 9 will be treated separately. Because $a+c-1$ cannot be divisible by 9, then $4(a+c-1)-1$ is divisible by 9. We obtain that $a+c=16$ and $b=54-10a$. So, $a\ge 7$ and $a\le 5$, contradiction.

Thus, all the numbers with the desired property are: 2156, 3564 and 5819.

Problem NT6. Prove that there is no integer $n>9$ whose decimal representation has no digit equal to zero, such that all numbers obtained by permuting the digits of n are perfect squares.

Solution: We will make repeated use of the fact that a number which is a multiple of 4 plus 2 or 3 cannot be a perfect square.
Let us make the following observations:

(*) {The digits in the decimal representation of n } $\subseteq \{1,4,5,6,9\}$ because the last digit of a perfect square belongs to it;

(**) 5 is not a digit of n because a number (obtained by a permutation) ending in 5 has to actually end in 25, being a perfect square, but 2 is not a digit of n;

(***) 1 and 9 cannot be at the same time digits of n, because a number (obtained by a permutation) ending in 19 is equal to $4m+3$.

Using these facts we will discuss the following cases:

(1) All digits of n are the same, so

$n=\overline{aa...a}=a\cdot\frac{10^k-1}{9}$, where k is the number of digits in the decimal represeantation of n

but then $a \notin \{1,4,9\}$, because $10^k - 1 = \overline{99...99} = \overline{99...96} + 3 = 4m+3$,
and $a \neq 6$, because 2 does not divide $10^k - 1$.

(2) n has (at least) two distinct digits in its decimal representation. Denote two of them by a and b ($a>b$). Let $n_1 = p^2$ and $n_2 = q^2$ be two numbers (obtained by permutations) ending in $\overline{ab}$ and $\overline{ba}$ respectively. Then $n_1 - n_2 = \overline{ab} - \overline{ba} = 9(a-b) \leq 45$, because of (***), thus $45 \geq p^2 - q^2 \geq p^2 - (p-1)^2 = 2p-1$ therefore $p \leq 23$ and $n_1 \leq 529$.

The case of n having three decimal digits, (as the only perfect squares, under the facts gathered above, and smaller than 529, are 144 and 441, when 414 is not a square); and the case of n having two decimal digits (n = 16, 25, 36, 49, 64, 81, when their "reverses" are not perfect squares), lead again to no solution.

Problem C1. Let A be a subset of $\{1,2,3,\ldots,2006\}$ containing 1004 elements. Show that

(i) there exist three distinct elements a, b, c of A such that the greatest common divisor of a, b divides c;

(ii) there exist three distinct elements a, b, c of A such that the greatest common divisor of a, b does not divide c.

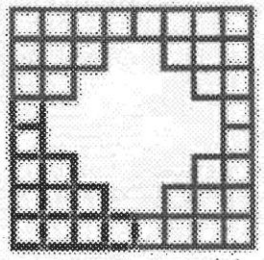
Solution: (i) Since the subset A has 1004 elements, two of them will be consecutive, hence coprime.

(ii) As the subset A has 1004 elements, it must contain at least one odd number: be it c. If it has at least two even numbers, a and b, then the greatest common divisor of a, b is even and cannot divide c. Otherwise the subset A will have to contain all odd numbers between 1 and 2006, and in this case we can take $a=3$, $b=9$ and $c=1$.

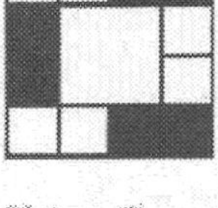
Remark.1. The Problem Selection Committee is fully aware that 1004 is far from being an optimal bound for sets A as asked for in the problem. In fact that bound is probably $2+\pi(2006)$.

2. The authors' original statement gave an equivalent characterization for the numbers a, b, c in terms of existence of solutions for the linear Diophantine equation $ax+by=c$, which may be beyond the competitors' level of knowledge. As well, the Problem Selection Committee lowered the original bound of 1005 in order to make the problem slightly more challenging.

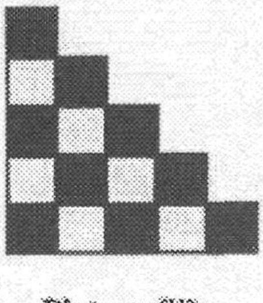
Problem C2. Let $n \geq 2$ be an integer. From a $2n \times 2n$ square table (made of 1×1 small squares) some of the small squares are removed: the middle two from row 2, the middle four from row 3, ..., the middle $2n-2$ from rows n and $n+1$, ..., the middle four from row $2n-2$ and the middle two from row $2n-1$, to obtain a new figure (as shown on the picture for $n=4$). What is the largest number of 2×1 dominoes that can be placed on such a figure, without overlapping, such that each domino covers exactly two small squares?



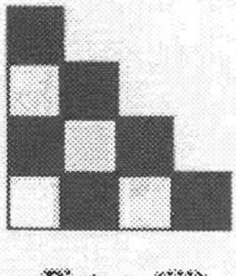
Solution: For $n=2$ the figure can be covered as shown on Picture (i), so the largest number in this case is 6.



Picture (i)



Picture (ii)



Picture (iii)

For $n > 2$, we cut the figure in four congruent parts along the lines between rows n and $n+1$ and columns n and $n+1$. Let us color those parts alternately white and black (as on a chessboard), as shown on Picture (ii) for $n = 5$ and Picture (iii) for $n = 4$. Each domino covers one black and one white small square.

Then the largest number of dominoes that can be placed on one of those parts is at most equal to the number of white small squares. On each of these four parts (without the top and the far right small square), dominoes may be placed as follows:
vertical dominoes on the bottom two rows, starting from the left edge, as far as it goes, and horizontal dominoes on each other row, starting from the left edge, as far as it goes.

For $n = 2k-1$, where $k \geq 2$ is an integer, on each of those parts there are

$$2\cdot[(k-1)+(k-2)+\ldots+2+1] = k(k-1)$$

white small squares. So the largest number of dominoes is at most $4k(k-1)$, plus four dominos that connect every two adjacent parts, i.e. at most $4k(k-1)+4$ dominoes. Using the method explained above we can place $k(k-1)$ dominoes, which account for all the white squares. Hence in this case the largest number of dominoes that can be placed on the figure is equal to

$$4k(k-1)+4 = 4k^2-4k+4 = (2k-1)^2+3 = n^2+3.$$

For $n = 2k$, where $k \geq 2$ is an integer, we have

$$(2k-1)+(2k-3)+\ldots+3+1 = k^2$$

white small squares. The same argument as above shows that in this case the largest number of dominoes that can be placed on the figure is equal to $4k^2+4 = (2k)^2+4 = n^2+4$.

Problem C3. Let $n \geq 5$ be an integer. Show that the set of the first n positive integers $\{1,2,3,\ldots,n\}$ can be partitioned into two non-empty subsets, S_n and P_n, such that the sum of all elements in S_n be equal to the product of all elements in P_n.

Solution: Denote by $|P_n|$ the number of elements (the cardinal) of the set P_n. To start with, notice that we need $|P_n| \geq 2$, otherwise, if $|P_n| = 1$, then $n < (1+2+\ldots+n)-n = \frac{n(n-1)}{2}$.

If $|P_n| = 2$, i.e. $P_n = \{p,q\}$, we obtain the equation

$$pq = (1+2+\ldots+n)-p-q \quad \Leftrightarrow \quad pq = \frac{n(n+1)}{2}-p-q$$

or

$$(p+1)(q+1) = \frac{n(n+1)}{2}+1, \qquad p \neq q, \qquad p,q \in \{1,2,\ldots,n\}. \qquad (1)$$

Unfortunately the equation (1) has no solutions for most values of n, which makes this attempt futile (for example, for $n = 8$, we obtain the equation $(p+1)(q+1) = 37$, which has no solutions for $p,q \in \{1,2,\ldots,8\}$).
Effectively calculating the first few cases yields the following unique solutions: $P_5 = \{1,2,4\}$, $P_6 = \{1,2,6\}$, $P_7 = \{1,3,6\}$, suggesting to try $|P_n| = 3$, with 1 being one of its elements.

Thus, for $|P_n| = 3$ we take $P_n = \{1,p,q\}$ with $p \neq q$ and $p,q \in \{2,3\ldots,n\}$. We obtain the equation

$$pq = (1+2+\ldots+n)-1-p-q \quad \Leftrightarrow \quad (p+1)(q+1) = \frac{n(n+1)}{2}. \qquad (2)$$

Let $n = 2k + 1$, where $k \geq 2$ is an integer. The equation $(p + 1)(q+1) = (k + 1)(2k + 1)$ has the solution $(p, q) = (k, 2k)$. So, we obtain the subsets

$$P_{2k+1} = \{1, k, 2k\}, \qquad S_{2k+1} = \{1,2,3,\ldots,n\} - \{1, k, 2k\}.$$

Let $n = 2k$, where $k \geq 3$ is an integer. The equation $(p + 1)(q+1) = k\,(2k + 1)$ has the solution $(p, q) = (k-1, 2k)$. So, we obtain the subsets

$$P_{2k} = \{1, k-1, 2k\}, \qquad S_{2k} = \{1,2,3,\ldots,n\} - \{1, k-1, 2k\}.$$

The problem is solved.

Comments to problem C3. The following extra assertion may be added to the problem, in order to increase its difficulty (the meaning of it is to show that the family of solutions found before does not provide all the solutions in all cases) :

Find (at least) an example each of odd and even n, such that this can be done in more than one way.

Solution: This may be done either by inspecting the case $|P_n| = 2$ (where no general solution exists), looking for some particular favorable factorization; this yields $P_{10}' = \{6,7\}$, where $P_{10} = \{1,4,10\}$, and $P_{17}' = \{10,13\}$, where $P_{17} = \{1,8,16\}$; or by looking for an alternative favorable factorization in the case $|P_n| = 3$; this yields $P_{15}' = \{1,9,11\}$, where $P_{15} = \{1,7,14\}$, and $P_{20}' = \{1,13,14\}$, where $P_{20} = \{1,9,20\}$.