The 9^{th} Romanian Master of Mathematics Competition

Day 1 — Solutions

Problem 1. **(a)** Prove that every positive integer n can be written uniquely in the form

$$n = \sum_{j=1}^{2k+1} (-1)^{j-1} 2^{m_j},$$

where $k \geq 0$ and $0 \leq m_1 < m_2 < \cdots < m_{2k+1}$ are integers.
This number k is called the *weight* of n.

(b) Find (in closed form) the difference between the number of positive integers at most 2^{2017} with even weight and the number of positive integers at most 2^{2017} with odd weight.

Vjekoslav Kovač, Croatia

Solution. **(a)** We show by induction on the integer $M \geq 0$ that every integer n in the range -2^M+1 through 2^M can uniquely be written in the form $n = \sum_{j=1}^{\ell}(-1)^{j-1}2^{m_j}$ for some integers $\ell \geq 0$ and $0 \leq m_1 < m_2 < \cdots < m_\ell \leq M$ (empty sums are 0); moreover, in this unique representation ℓ is odd if $n > 0$, and even if $n \leq 0$. The integer $w(n) = \lfloor \ell/2 \rfloor$ is called the *weight* of n.

Existence once proved, uniqueness follows from the fact that there are as many such representations as integers in the range -2^M+1 through 2^M, namely, 2^{M+1}.

To prove existence, notice that the base case $M = 0$ is clear, so let $M \geq 1$ and let n be an integer in the range $-2^M + 1$ through 2^M.

If $-2^M + 1 \leq n \leq -2^{M-1}$, then $1 \leq n + 2^M \leq 2^{M-1}$, so $n + 2^M = \sum_{j=1}^{2k+1}(-1)^{j-1}2^{m_j}$ for some integers $k \geq 0$ and $0 \leq m_1 < \cdots < m_{2k+1} \leq M-1$ by the induction hypothesis, and $n = \sum_{j=1}^{2k+2}(-1)^{j-1}2^{m_j}$, where $m_{2k+2} = M$.

The case $-2^{M-1}+1 \leq n \leq 2^{M-1}$ is covered by the induction hypothesis.

Finally, if $2^{M-1}+1 \leq n \leq 2^M$, then $-2^{M-1}+1 \leq n - 2^M \leq 0$, so $n - 2^M = \sum_{j=1}^{2k}(-1)^{j-1}2^{m_j}$ for some integers $k \geq 0$ and $0 \leq m_1 < \cdots < m_{2k} \leq M-1$ by the induction hypothesis, and $n = \sum_{j=1}^{2k+1}(-1)^{j-1}2^{m_j}$, where $m_{2k+1} = M$.

(b) First Approach. Let $M \geq 0$ be an integer. The solution for part **(a)** shows that the number of even (respectively, odd) weight integers in the range 1 through 2^M coincides with the number of subsets in $\{0, 1, 2, \ldots, M\}$ whose cardinality has remainder 1 (respectively, 3) modulo 4. Therefore, the difference of these numbers is

$$\sum_{k=0}^{\lfloor M/2 \rfloor} (-1)^k \binom{M+1}{2k+1} = \frac{(1+\mathrm{i})^{M+1} - (1-\mathrm{i})^{M+1}}{2\mathrm{i}} = 2^{(M+1)/2} \sin \frac{(M+1)\pi}{4},$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. Thus, the required difference is 2^{1009}.

Second Approach. For every integer $M \geq 0$, let $A_M = \sum_{n=-2^M+1}^{0}(-1)^{w(n)}$ and let $B_M = \sum_{n=1}^{2^M}(-1)^{w(n)}$; thus, B_M evaluates the difference of the number of even weight integers in the range 1 through 2^M and the number of odd weight integers in that range.

Notice that

$$w(n) = \begin{cases} w(n+2^M)+1 & \text{if } -2^M+1 \leq n \leq -2^{M-1}, \\ w(n-2^M) & \text{if } 2^{M-1}+1 \leq n \leq 2^M, \end{cases}$$

to get

$$
\begin{aligned}
A_M &= -\sum_{n=-2^M+1}^{-2^{M-1}} (-1)^{w(n+2^M)} + \sum_{n=-2^{M-1}+1}^{0} (-1)^{w(n)} = -B_{M-1} + A_{M-1}, \\
B_M &= \sum_{n=1}^{2^{M-1}} (-1)^{w(n)} + \sum_{n=2^{M-1}+1}^{2^M} (-1)^{w(n-2^M)} = B_{M-1} + A_{M-1}.
\end{aligned}
$$

Iteration yields

$$
\begin{aligned}
B_M &= A_{M-1} + B_{M-1} = (A_{M-2} - B_{M-2}) + (A_{M-2} + B_{M-2}) = 2A_{M-2} \\
&= 2A_{M-3} - 2B_{M-3} = 2(A_{M-4} - B_{M-4}) - 2(A_{M-4} + B_{M-4}) = -4B_{M-4}.
\end{aligned}
$$

Thus, $B_{2017} = (-4)^{504} B_1 = 2^{1008} B_1$; since $B_1 = (-1)^{w(1)} + (-1)^{w(2)} = 2$, it follows that $B_{2017} = 2^{1009}$.

Problem 2. Determine all positive integers n satisfying the following condition: for every monic polynomial P of degree at most n with integer coefficients, there exists a positive integer $k \leq n$, and $k+1$ distinct integers $x_1, x_2, \ldots, x_{k+1}$ such that

$$P(x_1) + P(x_2) + \cdots + P(x_k) = P(x_{k+1}).$$

Semen Petrov, Russia

Note. A polynomial is *monic* if the coefficient of the highest power is one.

Solution. There is only one such integer, namely, $n = 2$. In this case, if P is a constant polynomial, the required condition is clearly satisfied; if $P = X + c$, then $P(c-1) + P(c+1) = P(3c)$; and if $P = X^2 + qX + r$, then $P(X) = P(-X-q)$.

To rule out all other values of n, it is sufficient to exhibit a monic polynomial P of degree at most n with integer coefficients, whose restriction to the integers is injective, and $P(x) \equiv 1 \pmod{n}$ for all integers x. This is easily seen by reading the relation in the statement modulo n, to deduce that $k \equiv 1 \pmod{n}$, so $k = 1$, since $1 \leq k \leq n$; hence $P(x_1) = P(x_2)$ for some distinct integers x_1 and x_2, which contradicts injectivity.

If $n = 1$, let $P = X$, and if $n = 4$, let $P = X^4 + 7X^2 + 4X + 1$. In the latter case, clearly, $P(x) \equiv 1 \pmod{4}$ for all integers x; and P is injective on the integers, since $P(x) - P(y) = (x-y)\big((x+y)(x^2+y^2+7)+4\big)$, and the absolute value of $(x+y)(x^2+y^2+7)$ is either 0 or at least 7 for integral x and y.

Assume henceforth $n \geq 3$, $n \neq 4$, and let $f_n = (X-1)(X-2)\cdots(X-n)$. Clearly, $f_n(x) \equiv 0 \pmod{n}$ for all integers x. If n is odd, then f_n is non-decreasing on the integers; and if, in addition, $n > 3$, then $f_n(x) \equiv 0 \pmod{n+1}$ for all integers x, since $f_n(0) = -n! = -1 \cdot 2 \cdot \cdots \cdot \frac{n+1}{2} \cdot \cdots \cdot n \equiv 0 \pmod{n+1}$.

Finally, let $P = f_n + nX + 1$ if n is odd, and let $P = f_{n-1} + nX + 1$ if n is even. In either case, P is strictly increasing, hence injective, on the integers, and $P(x) \equiv 1 \pmod{n}$ for all integers x.

Remark. The polynomial $P = f_n + nX + 1$ works equally well for even $n > 2$. To prove injectivity, notice that P is strictly monotone, hence injective, on non-positive (respectively, positive) integers. Suppose, if possible, that $P(a) = P(b)$ for some integers $a \leq 0$ and $b > 0$. Notice that $P(a) \geq P(0) = n! + 1 > n^2 + 1 = P(n)$, since $n \geq 4$, to infer that $b \geq n+1$. It is therefore sufficient to show that $P(x) > P(n+1-x) > P(x-1)$ for all integers $x \geq n+1$. The former inequality is trivial, since $f_n(x) = f_n(n+1-x)$ for even n. For the latter, write

$$\begin{aligned} P(n+1-x) - P(x-1) &= (x-1)\cdots(x-n) - (x-2)\cdots(x-n-1) + n(n+2-2x) \\ &= n\big((x-2)\cdots(x-n) + (n-2) - 2(x-2)\big) \geq n(n-2) > 0, \end{aligned}$$

since $(x-3)\cdots(x-n) \geq 2$.

Problem 3. Let n be an integer greater than 1 and let X be an n-element set. A non-empty collection of subsets $A_1, \ldots, A_k$ of X is *tight* if the union $A_1 \cup \cdots \cup A_k$ is a proper subset of X and no element of X lies in exactly one of the A_is. Find the largest cardinality of a collection of proper non-empty subsets of X, no non-empty subcollection of which is tight.

Note. A subset A of X is *proper* if $A \neq X$. The sets in a collection are assumed to be distinct. The whole collection is assumed to be a subcollection.

Alexander Polyansky, Russia

Solution 1. (*Ilya Bogdanov*) The required maximum is $2n-2$. To describe a $(2n-2)$-element collection satisfying the required conditions, write $X = \{1, 2, \ldots, n\}$ and set $B_k = \{1, 2, \ldots, k\}$, $k = 1, 2, \ldots, n-1$, and $B_k = \{k-n+2, k-n+3, \ldots, n\}$, $k = n, n+1, \ldots, 2n-2$. To show that no subcollection of the B_k is tight, consider a subcollection $\mathcal{C}$ whose union U is a proper subset of X, let m be an element in $X \smallsetminus U$, and notice that $\mathcal{C}$ is a subcollection of $\{B_1, \ldots, B_{m-1}, B_{m+n-1}, \ldots, B_{2n-2}\}$, since the other B's are precisely those containing m. If U contains elements less than m, let k be the greatest such and notice that B_k is the only member of $\mathcal{C}$ containing k; and if U contains elements greater than m, let k be the least such and notice that B_{k+n-2} is the only member of $\mathcal{C}$ containing k. Consequently, $\mathcal{C}$ is not tight.

We now proceed to show by induction on $n \geq 2$ that the cardinality of a collection of proper non-empty subsets of X, no subcollection of which is tight, does not exceed $2n-2$. The base case $n=2$ is clear, so let $n > 2$ and suppose, if possible, that $\mathcal{B}$ is a collection of $2n-1$ proper non-empty subsets of X containing no tight subcollection.

To begin, notice that $\mathcal{B}$ has an empty intersection: if the members of $\mathcal{B}$ shared an element x, then $\mathcal{B}' = \{B \smallsetminus \{x\} \colon B \in \mathcal{B},\ B \neq \{x\}\}$ would be a collection of at least $2n-2$ proper non-empty subsets of $X \smallsetminus \{x\}$ containing no tight subcollection, and the induction hypothesis would be contradicted.

Now, for every x in X, let $\mathcal{B}_x$ be the (non-empty) collection of all members of $\mathcal{B}$ not containing x. Since no subcollection of $\mathcal{B}$ is tight, $\mathcal{B}_x$ is not tight, and since the union of $\mathcal{B}_x$ does not contain x, some x' in X is covered by a single member of $\mathcal{B}_x$. In other words, *there is a single set in $\mathcal{B}$ covering x' but not x.* In this case, draw an arrow from x to x'. Since there is at least one arrow from each x in X, some of these arrows form a (minimal) cycle $x_1 \to x_2 \to \cdots \to x_k \to x_{k+1} = x_1$ for some suitable integer $k \geq 2$. Let A_i be the unique member of $\mathcal{B}$ containing x_{i+1} but not x_i, and let $X' = \{x_1, x_2, \ldots, x_k\}$.

Remove $A_1, A_2, \ldots, A_k$ from $\mathcal{B}$ to obtain a collection $\mathcal{B}'$ each member of which either contains or is disjoint from X': for if a member B of $\mathcal{B}'$ contained some but not all elements of X', then B should contain x_{i+1} but not x_i for some i, and $B = A_i$, a contradiction. This rules out the case $k = n$, for otherwise $\mathcal{B} = \{A_1, A_2, \ldots, A_n\}$, so $|\mathcal{B}| < 2n-1$.

To rule out the case $k < n$, consider an extra element x^* outside X and let

$$\mathcal{B}^* = \{B \colon B \in \mathcal{B}',\ B \cap X' = \varnothing\} \cup \{(B \smallsetminus X') \cup \{x^*\} \colon B \in \mathcal{B}',\ X' \subseteq B\};$$

thus, in each member of $\mathcal{B}'$ containing X', the latter is collapsed to singleton x^*. Notice that $\mathcal{B}^*$ is a collection of proper non-empty subsets of $X^* = (X \smallsetminus X') \cup \{x^*\}$, no subcollection of which is tight. By the induction hypothesis, $|\mathcal{B}'| = |\mathcal{B}^*| \leq 2|X^*| - 2 = 2(n-k)$, so $|\mathcal{B}| \leq 2(n-k) + k = 2n - k < 2n-1$, a final contradiction.

Solution 2. Proceed again by induction on n to show that the cardinality of a collection of proper non-empty subsets of X, no subcollection of which is tight, does not exceed $2n-2$.

Consider any collection $\mathcal{B}$ of proper non-empty subsets of X with no tight subcollection (we call such collection *good*). Assume that there exist $M, N \in \mathcal{B}$ such that $M \cup N$ is distinct from M, N, and X. In this case, we will show how to modify $\mathcal{B}$ so that it remains good, contains the same number of sets, but the total number of elements in the sets of $\mathcal{B}$ increases.

Consider a maximal (relative to set-theoretic inclusion) subcollection $\mathcal{C} \subseteq \mathcal{B}$ such that the set $C = \bigcup_{A \in \mathcal{C}} A$ is distinct from X and from all members of $\mathcal{C}$. Notice here that the union of *any* subcollection $\mathcal{D} \subset \mathcal{B}$ cannot coincide with any $K \in \mathcal{B} \setminus \mathcal{D}$, otherwise $\{K\} \cup \mathcal{D}$ would be tight. Surely, $\mathcal{C}$ exists (since $\{M, N\}$ is an example of a collection satisfying the requirements on $\mathcal{C}$, except for maximality); moreover, $C \notin \mathcal{B}$ by the above remark.

Since $C \neq X$, there exists an $L \in \mathcal{C}$ and $x \in L$ such that L is the unique set in $\mathcal{C}$ containing x. Now replace in $\mathcal{B}$ the set L by C in order to obtain a new collection $\mathcal{B}'$ (then $|\mathcal{B}'| = |\mathcal{B}|$). We claim that $\mathcal{B}'$ is good.

Assume, to the contrary, that $\mathcal{B}'$ contained a tight subcollection $\mathcal{T}$; clearly, $C \in \mathcal{T}$, otherwise $\mathcal{B}$ is not good. If $\mathcal{T} \subseteq \mathcal{C} \cup \{C\}$, then C is the unique set in $\mathcal{T}$ containing x which is impossible. Therefore, there exists $P \in \mathcal{T} \setminus (\mathcal{C} \cup \{C\})$. By maximality of $\mathcal{C}$, the collection $\mathcal{C} \cup \{P\}$ does not satisfy the requirements imposed on $\mathcal{C}$; since $P \cup C \neq X$, this may happen only if $C \cup P = P$, i.e., if $C \subset P$. But then $\mathcal{G} = (\mathcal{T} \setminus \{C\}) \cup \mathcal{C}$ is a tight subcollection in $\mathcal{B}$: all elements of C are covered by $\mathcal{G}$ at least twice (by P and an element of $\mathcal{C}$), and all the rest elements are covered by $\mathcal{G}$ the same number of times as by $\mathcal{T}$. A contradiction. Thus $\mathcal{B}'$ is good.

Such modifications may be performed finitely many times, since the total number of elements of sets in $\mathcal{B}$ increases. Thus, at some moment we arrive at a good collection $\mathcal{B}$ for which the procedure no longer applies. This means that for every $M, N \in \mathcal{B}$, either $M \cup N = X$ or one of them is contained in the other.

Now let M be a minimal (with respect to inclusion) set in $\mathcal{B}$. Then each set in $\mathcal{B}$ either contains M or forms X in union with M (i.e., contains $X \setminus M$). Now one may easily see that the two collections

$$\mathcal{B}_+ = \{A \setminus M \colon A \in \mathcal{B},\ M \subset A,\ A \neq M\}, \quad \mathcal{B}_- = \{A \cap M \colon A \in \mathcal{B},\ X \setminus M \subset A,\ A \neq X \setminus M\}$$

are good as collections of subsets of $X \setminus M$ and M, respectively; thus, by the induction hypothesis, we have $|\mathcal{B}_+| + |\mathcal{B}_-| \leq 2n - 4$.

Finally, each set $A \in \mathcal{B}$ either produces a set in one of the two new collections, or coincides with M or $X \setminus M$. Thus $|\mathcal{B}| \leq |\mathcal{B}_+| + |\mathcal{B}_-| + 2 \leq 2n - 2$, as required.

Solution 3. We provide yet another proof of the estimate $|\mathcal{B}| \leq 2n - 2$, using the notion of a *good* collection from Solution 2. Arguing indirectly, we assume that there exists a good collection $\mathcal{B}$ with $|\mathcal{B}| \geq 2n - 1$, and choose one such for the minimal possible value of n. Clearly, $n > 2$.

Firstly, we perform a different modification of $\mathcal{B}$. Choose any $x \in X$, and consider the subcollection $\mathcal{B}_x = \{B \colon B \in \mathcal{B},\ x \notin B\}$. By our assumption, $\mathcal{B}_x$ is not tight. As the union of sets in $\mathcal{B}_x$ is distinct from X, either this collection is empty, or there exists an element $y \in X$ contained in a unique member A_x of $\mathcal{B}_x$. In the former case, we add the set $B_x = X \setminus \{x\}$ to $\mathcal{B}$, and in the latter we replace A_x by B_x, to form a new collection $\mathcal{B}'$. (Notice that if $B_x \in \mathcal{B}$, then $B_x \in \mathcal{B}_x$ and $y \in B_x$, so $B_x = A_x$.)

We claim that the collection $\mathcal{B}'$ is also good. Indeed, if $\mathcal{B}'$ has a tight subcollection $\mathcal{T}$, then B_x should lie in $\mathcal{T}$. Then, as the union of the sets in $\mathcal{T}$ is distinct from X, we should have $\mathcal{T} \subseteq \mathcal{B}_x \cup \{B_x\}$. But in this case an element y is contained in a unique member of $\mathcal{T}$, namely B_x, so $\mathcal{T}$ is not tight — a contradiction.

Perform this procedure for every $x \in X$, to get a good collection $\mathcal{B}$ containing the sets $B_x = X \setminus \{x\}$ for all $x \in X$. Consider now an element $x \in X$ such that $|\mathcal{B}_x|$ is maximal. As we have mentioned before, there exists an element $y \in X$ belonging to a unique member (namely, B_x) of $\mathcal{B}_x$. Thus, $\mathcal{B}_x \setminus \{B_x\} \subset \mathcal{B}_y$; also, $B_y \in \mathcal{B}_y \setminus \mathcal{B}_x$. Thus we get $|\mathcal{B}_y| \geq |\mathcal{B}_x|$, which by the maximality assumption yields the equality, which in turn means that $\mathcal{B}_y = (\mathcal{B}_x \setminus \{B_x\}) \cup \{B_y\}$.

Therefore, each set in $\mathcal{B} \setminus \{B_x, B_y\}$ contains either both x and y, or none of them. Collapsing $\{x, y\}$ to singleton x^*, we get a new collection of $|\mathcal{B}| - 2$ subsets of $(X \setminus \{x, y\}) \cup \{x^*\}$ containing no tight subcollection. This contradicts minimality of n.

Remarks. 1. Removal of the condition that subsets be proper would only increase the maximum by 1. The 'non-emptiness' condition could also be omitted, since the empty set forms a tight collection by itself, but the argument is a bit too formal to be considered.

2. There are many different examples of good collections of $2n-2$ sets. E.g., applying the algorithm from the first part of Solution 2 to the example shown in Solution 1, one may get the following example: $B_k = \{1, 2, \ldots, k\}$, $k = 1, 2, \ldots, n-1$, and $B_k = X \setminus \{k-n+1\}$, $k = n, n+1, \ldots, 2n-2$.

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Day 2 — Solutions

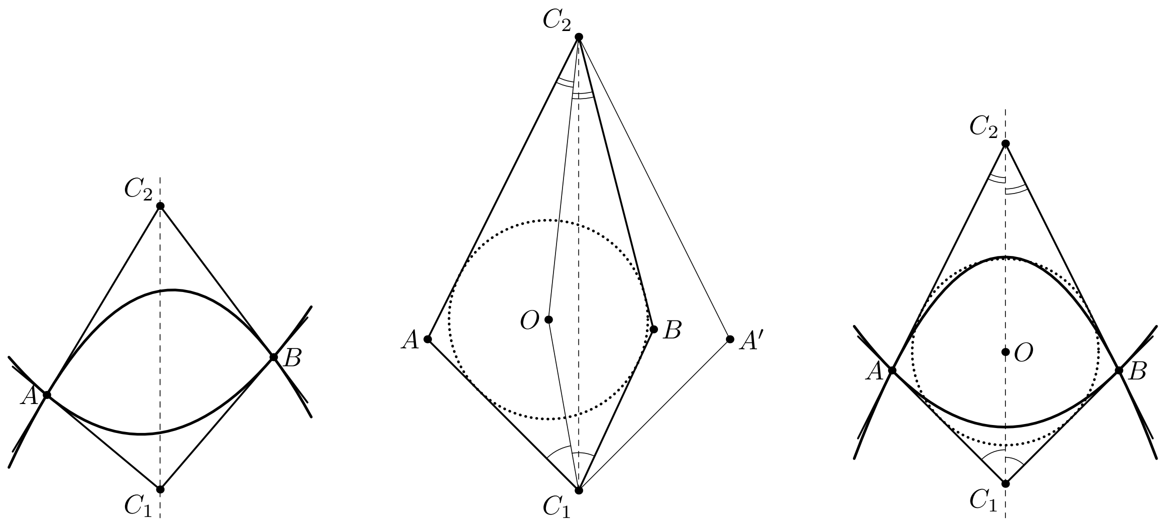
Problem 4. In the Cartesian plane, let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the graphs of the quadratic functions $f_1(x) = p_1x^2 + q_1x + r_1$ and $f_2(x) = p_2x^2 + q_2x + r_2$, where $p_1 > 0 > p_2$. The graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ cross at distinct points A and B. The four tangents to $\mathcal{G}_1$ and $\mathcal{G}_2$ at A and B form a convex quadrilateral which has an inscribed circle. Prove that the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same axis of symmetry.

Alexey Zaslavsky, Russia

Solution 1. Let $\mathcal{A}_i$ and $\mathcal{B}_i$ be the tangents to $\mathcal{G}_i$ at A and B, respectively, and let $C_i = \mathcal{A}_i \cap \mathcal{B}_i$. Since $f_1(x)$ is convex and $f_2(x)$ is concave, the convex quadrangle formed by the four tangents is exactly AC_1BC_2.

Lemma. *If CA and CB are the tangents drawn from a point C to the graph $\mathcal{G}$ of a quadratic trinomial $f(x) = px^2 + qx + r$, $A, B \in \mathcal{G}$, $A \neq B$, then the abscissa of C is the arithmetic mean of the abscissae of A and B.*

Proof. Assume, without loss of generality, that C is at the origin, so the equations of the two tangents have the form $y = k_ax$ and $y = k_bx$. Next, the abscissae x_A and x_B of the tangency points A and B, respectively, are multiple roots of the polynomials $f(x) - k_ax$ and $f(x) - k_bx$, respectively. By the Vieta theorem, $x_A^2 = r/p = x_B^2$, so $x_A = -x_B$, since the case $x_A = x_B$ is ruled out by $A \neq B$.



The Lemma shows that the line C_1C_2 is parallel to the y-axis and the points A and B are equidistant from this line.

Suppose, if possible, that the incentre O of the quadrangle AC_1BC_2 does not lie on the line C_1C_2. Assume, without loss of generality, that O lies inside the triangle AC_1C_2 and let A' be the reflection of A in the line C_1C_2. Then the ray C_iB emanating from C_i lies inside the angle AC_iA', so B lies inside the quadrangle $AC_1A'C_2$, whence A and B are not equidistant from C_1C_2 — a contradiction.

Thus O lies on C_1C_2, so the lines AC_i and BC_i are reflections of one another in the line C_1C_2, and $B = A'$. Hence $y_A = y_B$, and since $f_i(x) = y_A + p_i(x - x_A)(x - x_B)$, the line C_1C_2 is the axis of symmetry of both parabolas, as required.

Solution 2. Use the standard equation of a tangent to a smooth curve in the plane, to deduce that the tangents at two distinct points A and B on the parabola of equation $y = px^2 + qx + r$,

$p \neq 0$, meet at some point C whose coordinates are

$$x_C = \tfrac{1}{2}(x_A + x_B) \quad \text{and} \quad y_C = px_Ax_B + q \cdot \tfrac{1}{2}(x_A + x_B) + r.$$

Usage of the standard formula for Euclidean distance yields

$$CA = \tfrac{1}{2}|x_B - x_A|\sqrt{1 + (2px_A + q)^2} \quad \text{and} \quad CB = \tfrac{1}{2}|x_B - x_A|\sqrt{1 + (2px_B + q)^2},$$

so, after obvious manipulations,

$$CB - CA = \frac{2p(x_B - x_A)|x_B - x_A|(p(x_A + x_B) + q)}{\sqrt{1 + (2px_A + q)^2} + \sqrt{1 + (2px_B + q)^2}}.$$

Now, write the condition in the statement in the form $C_1B - C_1A = C_2B - C_2A$, apply the above formula and clear common factors to get

$$\frac{p_1(p_1(x_A + x_B) + q_1)}{\sqrt{1 + (2p_1x_A + q_1)^2} + \sqrt{1 + (2p_1x_B + q_1)^2}} = \frac{p_2(p_2(x_A + x_B) + q_2)}{\sqrt{1 + (2p_2x_A + q_2)^2} + \sqrt{1 + (2p_2x_B + q_2)^2}}.$$

Next, use the fact that x_A and x_B are the solutions of the quadratic equation $(p_1 - p_2)x^2 + (q_1 - q_2)x + r_1 - r_2 = 0$, so $x_A + x_B = -(q_1 - q_2)/(p_1 - p_2)$, to obtain

$$\frac{p_1(p_1q_2 - p_2q_1)}{\sqrt{1 + (2p_1x_A + q_1)^2} + \sqrt{1 + (2p_1x_B + q_1)^2}} = \frac{p_2(p_1q_2 - p_2q_1)}{\sqrt{1 + (2p_2x_A + q_2)^2} + \sqrt{1 + (2p_2x_B + q_2)^2}}.$$

Finally, since $p_1p_2 < 0$ and the denominators above are both positive, the last equality forces $p_1q_2 - p_2q_1 = 0$; that is, $q_1/p_1 = q_2/p_2$, so the two parabolas have the same axis.

Remarks. The are, of course, several different proofs of the Lemma in Solution 1 — in particular, computational. Another argument relies on the following consequence of focal properties: The tangents to a parabola at two points meet at the circumcentre of the triangle formed by the focus and the orthogonal projections of those points on the directrix. Since the directrix of the parabola in the lemma is parallel to the axis of abscissae, the conclusion follows.

Problem 5. Fix an integer $n \geq 2$. An $n \times n$ *sieve* is an $n \times n$ array with n cells removed so that exactly one cell is removed from every row and every column. A *stick* is a $1 \times k$ or $k \times 1$ array for any positive integer k. For any sieve A, let $m(A)$ be the minimal number of sticks required to partition A. Find all possible values of $m(A)$, as A varies over all possible $n \times n$ sieves.

Palmer Mebane and Nikolai Beluhov

Solution 1. Given A, $m(A) = 2n - 2$, and it is achieved, for instance, by dissecting A along all horizontal (or vertical) grid lines. It remains to prove that $m(A) \geq 2n - 2$ for every A.

By *holes* we mean the cells which are cut out from the board. The *cross* of a hole in A is the union of the row and the column through that hole.

Arguing indirectly, consider a dissection of A into $2n - 3$ or fewer sticks. Horizontal sticks are all labelled h, and vertical sticks are labelled v; 1×1 sticks are both horizontal and vertical, and labelled arbitrarily. Each cell of A inherits the label of the unique containing stick.

Assign each stick in the dissection to the cross of the unique hole on its row, if the stick is horizontal; on its column, if the stick is vertical.

Since there are at most $2n - 3$ sticks and exactly n crosses, there are two crosses each of which is assigned to at most one stick in the dissection. Let the crosses be c and d, centred at $a = (x_a, y_a)$ and $b = (x_b, y_b)$, respectively, and assume, without loss of generality, $x_a < x_b$ and $y_a < y_b$. The sticks covering the cells (x_a, y_b) and (x_b, y_a) have like labels, for otherwise one of the two crosses would be assigned to at least two sticks. Say the common label is v, so each of c and d contains a stick covering one of those two cells. It follows that the lower (respectively, upper) arm of c (respectively, d) is all-h, and the horizontal arms of both crosses are all-v, as illustrated below.

							h			
							h			
							h			
v	v	v	v	v	v	v	b	v	v	v
v	v	v	a	v	v	v	v	v	v	v
			h							
			h							
			h							

Each of the rows between the rows of a and b, that is, rows $y_a + 1$, $y_a + 2$, $\ldots$, $y_b - 1$, contains a hole. The column of each such hole contains at least two v-sticks. All other columns contain at least one v-stick each. In addition, all rows below a and all rows above b contain at least one h-stick each. This amounts to a total of at least $2(y_b - y_a - 1) + (n - y_b + y_a + 1) + (n - y_b) + (y_a - 1) = 2n - 2$ sticks. A contradiction.

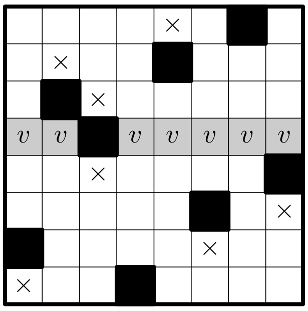
Remark. One may find a different argument finishing the solution. Since c and d are proven to contain one stick each, there is a third cross e centred at (x_*, y_*) also containing at most one stick. It meets the horizontal arms of c and d at two v-cells, so all the cells where two of the three crosses meet are labelled with v. Now, assuming (without loss of generality) that $y_a < y_* < y_b$, we obtain that both vertical arms of e contain v-cells, so e is assigned to two different v-sticks. A contradiction.

Solution 2. (*Ilya Bogdanov*) We provide a different proof that $m(A) \geq 2n - 2$.

Call a stick *vertical* if it is contained in some column, and *horizontal* if it is contained in some row; 1×1 sticks may be called arbitrarily, but any of them is supposed to have only one direction. Assign to each vertical/horizontal stick the column/row it is contained in. If each row and each column is assigned to some stick, then there are at least $2n$ sticks, which is even more than we want. Thus we assume, without loss of generality, that some *exceptional* row R is not assigned to any stick. This means that all $n-1$ existing cells in R belong to $n-1$ distinct vertical sticks; call these sticks *central*.

Now we mark $n-1$ cells on the board in the following manner. ($\downarrow$) For each hole c below R, we mark the cell just under c; ($\uparrow$) for each hole c above R, we mark the cell just above c; and ($\bullet$) for the hole r in R, we mark both the cell just above it and just below it. We have described $n+1$ cells, but exactly two of them are out of the board; so $n-1$ cells are marked within the board. A sample marking is shown in the figure below, where the marked cells are crossed.

R



Notice that all the marked cells lie in different rows, and all of them are marked in different columns, except for those two marked for ($\bullet$); but the latter two have a hole r between them. So no two marked cells may belong to the same stick. Moreover, none of them lies in a central stick, since the marked cells are separated from R by the holes. Thus the marked cells should be covered by $n-1$ different sticks (call them *border*) which are distinct from the central sticks. This shows that there are at least $(n-1)+(n-1) = 2n-2$ distinct sticks, as desired.

Solution 3. In order to prove $m(A) \geq 2n-2$, it suffices to show that there are $2n-2$ cells in A, no two of which may be contained in the same stick.

To this end, consider the bipartite graph G with parts G_h and G_v, where the vertices in G_h (respectively, G_v) are the $2n-2$ maximal sticks A is dissected into by all horizontal (respectively, vertical) grid lines, two sticks being joined by an edge in G if and only if they share a cell.

We show that G admits a perfect matching by proving that it fulfils the condition in Hall's theorem; the $2n-2$ cells corresponding to the edges of this matching form the desired set. It is sufficient to show that every subset S of G_h has at least $|S|$ neighbours (in G_v, of course).

Let L be the set of all sticks in S that contain a cell in the leftmost column of A, and let R be the set of all sticks in S that contain a cell in the rightmost column of A; let ℓ be the length of the longest stick in L (zero if L is empty), and let r be the length of the longest stick in R (zero if R is empty).

Since every row of A contains exactly one hole, L and R partition S; and since every column of A contains exactly one hole, neither L nor R contains two sticks of the same size, so $\ell \geq |L|$ and $r \geq |R|$, whence $\ell + r \geq |L| + |R| = |S|$.

If $\ell + r \leq n$, we are done, since there are at least $\ell + r \geq |S|$ vertical sticks covering the cells of the longest sticks in L and R. So let $\ell + r > n$, in which case the sticks in S span all n columns, and notice that we are again done if $|S| \leq n$, to assume further $|S| > n$.

Let $S' = G_h \setminus S$, let T be set of all neighbours of S, and let $T' = G_v \setminus T$. Since the sticks in S span all n columns, $|T| \geq n$, so $|T'| \leq n-2$. Transposition of the above argument (replace S by T'), shows that $|T'| \leq |S'|$, so $|S| \leq |T|$.

Remark. Here is an alternative argument for $s = |S| > n$. Add to S two *empty sticks* formally present to the left of the leftmost hole and to the right of the rightmost one. Then there are at least $s-n+2$ rows containing two sticks from S, so two of them are separated by at least $s-n$ other rows. Each hole in those $s-n$ rows separates two vertical sticks from G_v both of which are neighbours of S. Thus the vertices of S have at least $n+(s-n)$ neighbours.

Solution 4. Yet another proof of the estimate $m(A) \geq 2n-2$. We use the induction on n. Now we need the base cases $n=2,3$ which can be completed by hands.

Assume now that $n>3$ and consider any dissection of A into sticks. Define the *cross* of a hole as in Solution 1, and notice that each stick is contained in some cross. Thus, if the dissection contains more than n sticks, then there exists a cross containing at least two sticks. In this case, remove this cross from the sieve to obtain an $(n-1)\times(n-1)$ sieve. The dissection of the original sieve induces a dissection of the new array: even if a stick is partitioned into two by the removed cross, then the remaining two parts form a stick in the new array. After this operation has been performed, the number of sticks decreases by at least 2, and by the induction hypothesis the number of sticks in the new dissection is at least $2n-4$. Hence, the initial dissection contains at least $(2n-4)+2=2n-2$ sticks, as required.

It remains to rule out the case when the dissection contains at most n sticks. This can be done in many ways, one of which is removal a cross containing some stick. The resulting dissection of an $(n-1)\times(n-1)$ array contains at most $n-1$ sticks, which is impossible by the induction hypothesis since $n-1<2(n-1)-2$.

Remark. The idea of removing a cross containing at least two sticks arises naturally when one follows an inductive approach. But it is much trickier to finish the solution using this approach, **unless** one starts to consider removing **each** cross instead of removing a specific one.

Problem 6. Let $ABCD$ be any convex quadrilateral and let P, Q, R, S be points on the segments AB, BC, CD, and DA, respectively. It is given that the segments PR and QS dissect $ABCD$ into four quadrilaterals, each of which has perpendicular diagonals. Show that the points P, Q, R, S are concyclic.

Nikolai Beluhov

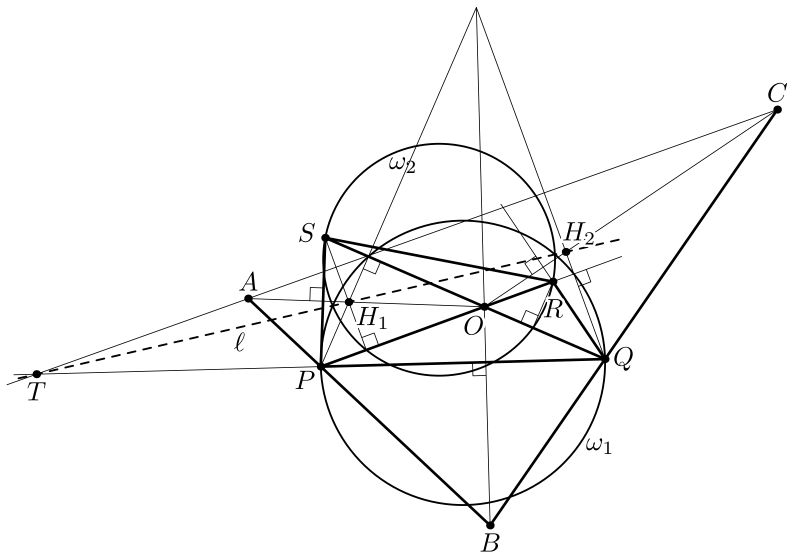
Solution 1. We start with a lemma which holds even in a more general setup.

Lemma 1. *Let $PQRS$ be a convex quadrangle whose diagonals meet at O. Let ω_1 and ω_2 be the circles on diameters PQ and RS, respectively, and let ℓ be their radical axis. Finally, choose the points A, B, and C outside this quadrangle so that: the point P (respectively, Q) lies on the segment AB (respectively, BC); and $AO \perp PS$, $BO \perp PQ$, and $CO \perp QR$. Then the three lines AC, PQ, and ℓ are concurrent or parallel.*

Proof. Assume first that the lines PR and QS are not perpendicular. Let H_1 and H_2 be the orthocentres of the triangles OSP and OQR, respectively; notice that H_1 and H_2 do not coincide.

Since H_1 is the radical centre of the circles on diameters RS, SP, and PQ, it lies on ℓ. Similarly, H_2 lies on ℓ, so the lines H_1H_2 and ℓ coincide.

The corresponding sides of the triangles APH_1 and CQH_2 meet at O, B, and the orthocentre of the triangle OPQ (which lies on OB). By Desargues' theorem, the lines AC, PQ and ℓ are concurrent or parallel.

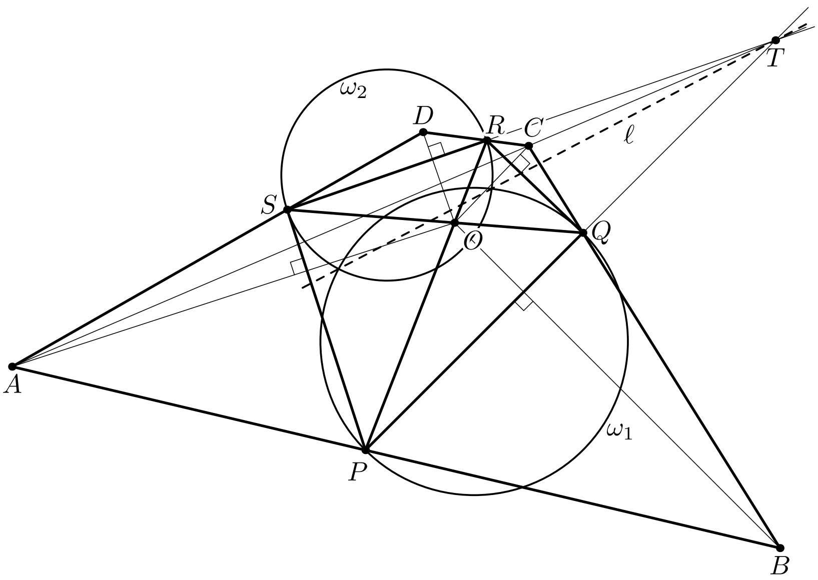


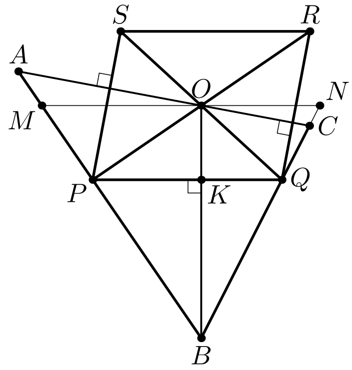
The case when $PR \perp QS$ may be considered as a limit case, since the configuration in the statement of the lemma allows arbitrarily small perturbations. The lemma is proved.

Back to the problem, let the segments PR and QS cross at O, let ω_1 and ω_2 be the circles on diameters PQ and RS, respectively, and let ℓ be their radical axis. By the Lemma, the three lines AC, ℓ, and PQ are concurrent or parallel, and similarly so are the three lines AC, ℓ, and RS. Thus, if the lines AC and ℓ are distinct, all four lines are concurrent or pairwise parallel.

This is clearly the case when the lines PS and QR are not parallel (since ℓ crosses OA and OC at the orthocentres of OSP and OQR, these orthocentres being distinct from A and C). In this case, denote the concurrency point by T. If T is not ideal, then we have $TP \cdot TQ = TR \cdot TS$ (as $T \in \ell$), so $PQRS$ is cyclic. If T is ideal (i.e., all four lines are parallel), then the segments PQ and RS have the same perpendicular bisector (namely, the line of centers of ω_1 and ω_2), and $PQRS$ is cyclic again.

Assume now PS and QR parallel. By symmetry, PQ and RS may also be assumed parallel: otherwise, the preceding argument goes through after relabelling. In this case, we need to prove that the parallelogram $PQRS$ is a rectangle.





Suppose, by way of contradiction, that $OP > OQ$. Let the line through O and parallel to PQ meet AB at M, and CB at N. Since $OP > OQ$, the angle SPQ is acute and the angle PQR is obtuse, so the angle AOB is obtuse, the angle BOC is acute, M lies on the segment AB, and N lies on the extension of the segment BC beyond C. Therefore: $OA > OM$, since the angle OMA is obtuse; $OM > ON$, since $OM : ON = KP : KQ$, where K is the projection of O onto PQ; and $ON > OC$, since the angle OCN is obtuse. Consequently, $OA > OC$.

Similarly, $OR > OS$ yields $OC > OA$: a contradiction. Consequently, $OP = OQ$ and $PQRS$ is a rectangle. This ends the proof.

Solution 2. (*Ilya Bogdanov*) To begin, we establish a useful lemma.

Lemma 2. *If P is a point on the side AB of a triangle OAB, then*

$$\frac{\sin AOP}{OB} + \frac{\sin POB}{OA} = \frac{\sin AOB}{OP}.$$

Proof. Let $[XYZ]$ denote the area of a triangle XYZ, to write

$$0 = 2([AOB] - [POB] - [POC]) = OA \cdot OB \cdot \sin AOB - OB \cdot OP \cdot \sin POB - OP \cdot OA \cdot \sin AOP,$$

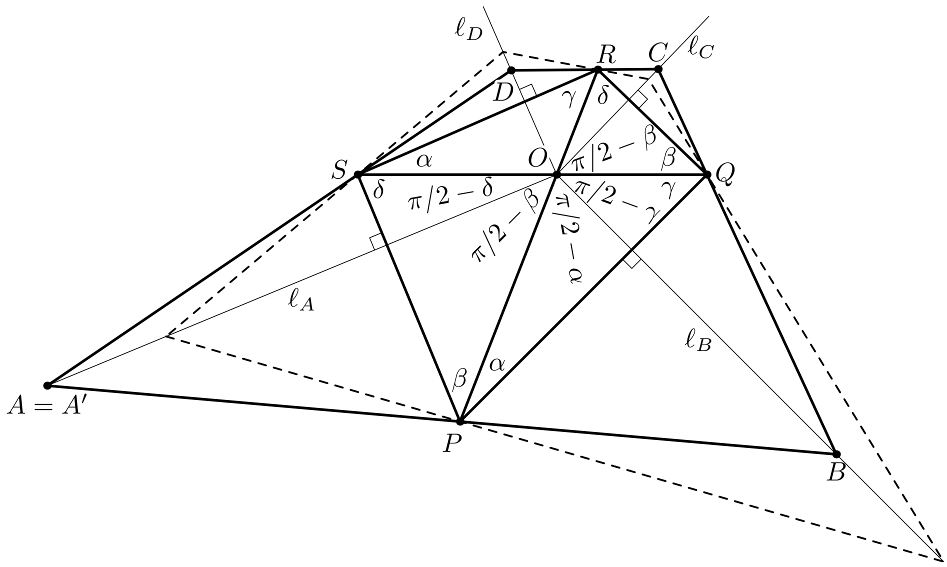
and divide by $OA \cdot OB \cdot OP$ to get the required identity.

A similar statement remains valid if the point C lies on the line AB; the proof is obtained by using signed areas and directed lengths.

We now turn to the solution. We first prove some sort of a converse statement, namely:

Claim. *Let $PQRS$ be a cyclic quadrangle with $O = PR \cap QS$; assume that no its diagonal is perpendicular to a side. Let ℓ_A, ℓ_B, ℓ_C, and ℓ_D be the lines through O perpendicular to SP, PQ, QR, and RS, respectively. Choose any point $A \in \ell_A$ and successively define $B = AP \cap \ell_B$, $C = BQ \cap \ell_C$, $D = CR \cap \ell_D$, and $A' = DS \cap \ell_A$. Then $A' = A$.*

Proof. We restrict ourselves to the case when the points A, B, C, D, and A' lie on ℓ_A, ℓ_B, ℓ_C, ℓ_D, and ℓ_A on the same side of O as their points of intersection with the respective sides of the quadrilateral $PQRS$. Again, a general case is obtained by suitable consideration of directed lengths.



Denote

$$\begin{aligned}
\alpha &= \angle QPR = \angle QSR = \pi/2 - \angle POB = \pi/2 - \angle DOS, \\
\beta &= \angle RPS = \angle RQS = \pi/2 - \angle AOP = \pi/2 - \angle QOC, \\
\gamma &= \angle SQP = \angle SRP = \pi/2 - \angle BOQ = \pi/2 - \angle ROD, \\
\delta &= \angle PRQ = \angle PSQ = \pi/2 - \angle COR = \pi/2 - \angle SOA.
\end{aligned}$$

By Lemma 2 applied to the lines APB, PQC, CRD, and DSA', we get

$$\frac{\sin(\alpha+\beta)}{OP} = \frac{\cos\alpha}{OA} + \frac{\cos\beta}{OB}, \qquad \frac{\sin(\beta+\gamma)}{OQ} = \frac{\cos\beta}{OB} + \frac{\cos\gamma}{OC},$$
$$\frac{\sin(\gamma+\delta)}{OR} = \frac{\cos\gamma}{OC} + \frac{\cos\delta}{OD}, \qquad \frac{\sin(\delta+\alpha)}{OS} = \frac{\cos\delta}{OD} + \frac{\cos\alpha}{OA'}.$$

Adding the two equalities on the left and subtracting the two on the right, we see that the required equality $A = A'$ (i.e., $\cos\alpha/OA = \cos\alpha/OA'$, in view of $\cos\alpha \neq 0$) is equivalent to the relation

$$\frac{\sin QPS}{OP} + \frac{\sin SRQ}{OR} = \frac{\sin PQR}{OQ} + \frac{\sin RSP}{OS}.$$

Let d denote the circumdiameter of $PQRS$, so $\sin QPS = \sin SRQ = QS/d$ and $\sin RSP = \sin PQR = PR/d$. Thus the required relation reads

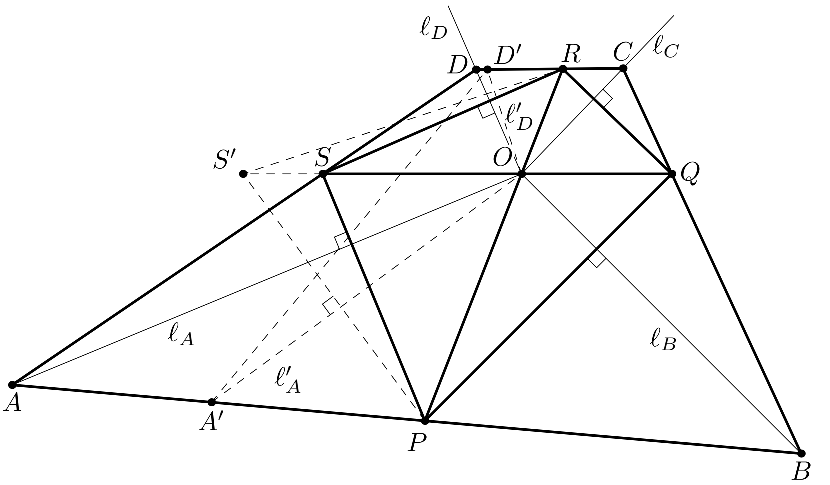
$$\frac{QS}{OP} + \frac{QS}{OR} = \frac{PR}{OS} + \frac{PR}{OQ}, \quad \text{or} \quad \frac{QS \cdot PR}{OP \cdot OR} = \frac{PR \cdot QS}{OS \cdot OQ}.$$

The last relation is trivial, due again to cyclicity.

Finally, it remains to derive the problem statement from our Claim. Assume that $PQRS$ is not cyclic, e.g., that $OP \cdot OR > OQ \cdot OS$, where $O = PR \cap QS$. Mark the point S' on the ray OS so that $OP \cdot OR = OQ \cdot OS'$. Notice that no diagonal of $PQRS$ is perpendicular to a side, so the quadrangle $PQRS'$ satisfies the conditions of the claim.

Let ℓ'_A and ℓ'_D be the lines through O perpendicular to PS' and RS', respectively. Then ℓ'_A and ℓ'_D cross the segments AP and RD, respectively, at some points A' and D'. By the Claim, the line $A'D'$ passes through S'. This is impossible, because the segment $A'D'$ crosses the segment OS at some interior point, while S' lies on the extension of this segment. This contradiction completes the proof.

Remark. According to the author, there is a remarkable corollary that is worth mentioning: Four lines dissect a convex quadrangle into nine smaller quadrangles to make it into a 3×3 array



of quadrangular cells. Label these cells 1 through 9 from left to right and from top to bottom. If the first eight cells are orthodiagonal, then so is the ninth.