SEEMOUS
BULGARIA • DEVIN • MARCH 12-17 • 2019

**Problem** (1). *We shall call the numerical sequence* $\{x_n\}$ *a "Devin" sequence if* $0 \leq x_n \leq 1$ *and for each function* $f \in \mathrm{C}[0,1]$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) = \int_0^1 f(x)\mathrm{d}x.$$

*Prove that the numerical sequence* $\{x_n\}$ *is a "Devin" sequence if and only if* $\forall k \geq 0$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i^k = \frac{1}{k+1}\cdot$$

**Problem** (2). *Let* $m$ *and* $n$ *be positive integers. Prove that for any matrices* $A_1, A_2, \ldots, A_m \in \mathcal{M}_n(\mathbb{R})$ *there exist* $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m \in \{-1, 1\}$ *such that*

$$\mathrm{Tr}\left((\varepsilon_1 A_1 + \varepsilon_2 A_2 + \ldots + \varepsilon_m A_m)^2\right) \geq \mathrm{Tr}\left(A_1^2\right) + \mathrm{Tr}\left(A_2^2\right) + \ldots + \mathrm{Tr}\left(A_m^2\right).$$

**Problem** (3). *Let* $n \geq 2$ *and* $A, B \in \mathcal{M}_n(\mathbb{C})$ *such that* $B^2 = B$. *Prove that*

$$\mathrm{rank}(AB - BA) \leq \mathrm{rank}(AB + BA).$$

**Problem** (4). *(a) Let* $n \geq 1$ *be an integer. Calculate* $\int_0^1 x^{n-1} \ln x \, dx$.

*(b) Calculate*

$$\sum_{n=0}^{\infty} (-1)^n \left( \frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} - \cdots \right).$$