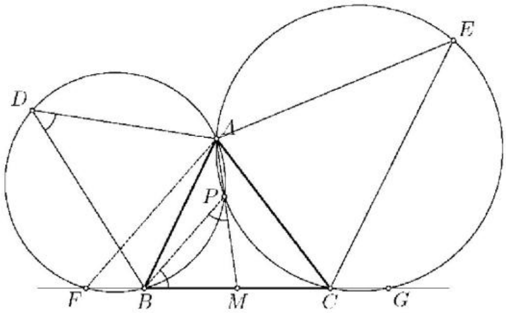
MMO 2019

Problem 1. Let ABC be an acute triangle, M be midpoint of the segment BC and the centers of the excircles with respect of M of the triangles AMB and AMC are D and E, respectively. The The circumcircle of the triangle ABD meets the line BC at the points B and F. The circumcircle of the triangle ACE meets the line BC at the points C and G. Prove that $BF = CG$.

Solution. (BMO shortlist) Obviously, we have $\measuredangle ADB = 90° - \frac{1}{2}\measuredangle AMB$ and $\measuredangle AEC = 90° - \frac{1}{2}\measuredangle AMC$.



Let the circumcircles of ΔADB and ΔAEC meet again the line AM at the points P and P', respectively. Let we notice that the point M is outside of the circumcircles of ΔADB and ΔAEC, since $\measuredangle ADB + \measuredangle AMB < 180°$ and $\measuredangle AEC + \measuredangle AMC < 180°$, so P and P' lie on the ray MA. Furthermore, $\measuredangle BPM = \measuredangle BDA = 90° - \frac{1}{2}\measuredangle PMB$, hence the triangle BPM is isosceles, so $MP = MB$. Analogously, $MP' = MC = MB$, so $P' \equiv P$.

Now, using the power of the point M, we obtain $MB \cdot MF = MP \cdot MA = MC \cdot MG$, i.e. $MF = MG = MA$, hence $BF = CG$.

Problem 2. Let n be a positive integer. If $r \equiv n(\bmod 2)$ and $r \in \{0,1\}$, then find the number of the integer solutions of the system of equations

$$\begin{cases} x+y+z=r \\ |x|+|y|+|z|=n \end{cases}.$$

Solution. Let n be a even positive integer, that is $r=0$. Then the problem can be reformulated as to find the number of integer solutions of the system of equations

$$\begin{cases} x+y+z=0 \\ |x|+|y|+|z|=n \end{cases} \quad .....(1)$$

Lemma. 1) At least one of the numbers x, y, z has absolute value $\frac{n}{2}$.

2) Each of x, y, z has absolute value $\leq \frac{n}{2}$.

Proof. It is clear that one of the numbers x, y, z must be positive; otherwise we obtain contradiction with the first equation of the system of equations (1). Without loss the generality, we may assume $x>0$.
Indeed, if $x>\frac{n}{2}$, from $x=-(y+z)$, and from $|y|+|z| \geq |y+z| > \frac{n}{2}$ we obtain contradiction with the second equation of the system of equations (1).

If $0<x<\frac{n}{2}$, then at leas one of the numbers y, z is smaller than 0. We consider two cases: Case 1. $y<0, z<0$, and Case 2. $yz<0$.
Case 1. $|y+z|=|y|+|z|$ and $y+z=-x$, so $|x|+|y|+|z|<\frac{n}{2}+\frac{n}{2}$, which is contradiction.

Case 2. Let $y<0<z$. In this case $x+z=-y$, that is $|y|=|x+z|=|x|+|z|$ from where we obtain

$$2|y|=|y|+|x+z|=|x|+|y|+|z|=n \text{ or } |y|=\frac{n}{2}.$$

The case when $x<0$ is analogues. This completes the lemma.

Continuation of the solution. Let only one of the numbers x, y, z be positive. Without loss of generality, let $x>0$, and then $x=\frac{n}{2}$ and $y+z=-\frac{n}{2}$. From the lemma, it follows that all the ordered triples

$$\left(\frac{n}{2},-\frac{n}{2},0\right),\left(\frac{n}{2},-\frac{n}{2}+1,-1\right),\left(\frac{n}{2},-\frac{n}{2}+2,-2\right),\ldots,\left(\frac{n}{2},0,-\frac{n}{2}\right)$$

are solution of the system of equations (1), and those are $\frac{n}{2}+1$ solutions. Changing the position of $\frac{n}{2}$ (at the second and at the third coordinate) and applying the same discussion, we obtain $3\left(\frac{n}{2}+1\right)$ ordered triples which are solution of the system of equations (1). Let any two of x, y, z are positive. Without loss of generality, let $x>0, y>0$. Then $z=-\frac{n}{2}$ and $x+y=\frac{n}{2}$. From the lemma, it follows that all the ordered triples

$$\left(1,\frac{n}{2}-1,-\frac{n}{2}\right),\left(2,\frac{n}{2}-2,-\frac{n}{2}\right),\left(3,\frac{n}{2}-3,-\frac{n}{2}\right),\ldots,\left(\frac{n}{2}-1,1,-\frac{n}{2}\right)$$

are solution of the system of equations (1), and those are $\frac{n}{2}-1$ solutions. Changing the position of $-\frac{n}{2}$ (at the first and at the second coordinate) and applying the same discussion, we obtain $3\left(\frac{n}{2}-1\right)$ ordered triples which are solution of the system of equations (1). Finally, we obtain that the total number of solutions of the system of equations (1) is

$$3\left(\frac{n}{2}+1\right)+3\left(\frac{n}{2}-1\right)=3n.$$

Now, let n be a odd positive integer, that is $r=1$. Then, the system (1) can be written as

$$\begin{cases} x+y+z=1 \\ |x|+|y|+|z|=n \end{cases}.$$

In analogues way as the case when n is even, (using the appropriate lemma obtained when replacing $\frac{n}{2}$ with $\frac{n+1}{2}$), we obtain that the total number of solutions of the system of equations (1) is

$$3\left(\frac{n-1}{2}+1\right)+3\left(\frac{n-1}{2}\right)=3n.$$

Problem 3. Let ABC be an isosceles triangle ($AB = AC$) and let M be a midpoint of the segment BC. The point P is chosen such that $PB < PC$ and PA is parallel to BC. Let X and Y are point from the lines PB and PC, respectively, such that the point B is on the segment PX, C is on the segment PY and $\measuredangle PXM = \measuredangle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

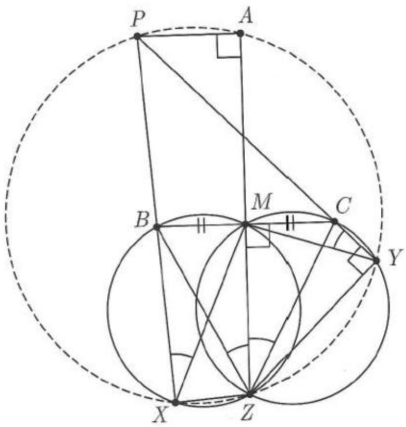
Solution (IMO Shortlist). Since $AB = AC$, AM is a axes of symmetry of the segment BC, we have $\measuredangle PAM = \measuredangle AMC = 90°$.

Let Z be intersection point of the line AM and the normal of PC, passing through Y (notice that Z is on the ray AM after the point M). We have, $\measuredangle PAZ = \measuredangle PYZ = 90°$. Hence, the points P, A, Y and Z are concyclic.

Since $\measuredangle CMZ = \measuredangle CYZ = 90°$, the quadrilateral $CYZM$ is cyclic, so $\measuredangle CZM = \measuredangle CYM$. By the condition of the problem, $\measuredangle CYM = \measuredangle BXM$, and since ZM is axes of symmetry of the angle $\measuredangle BZC$, we have $\measuredangle CZM = \measuredangle BZM$. So, $\measuredangle BXM = \measuredangle BZM$. Now, we have that the points B, X, Z and M are concyclic, so $\measuredangle BXZ = 180° - \measuredangle BMZ = 90°$.

Finally, e obtain that $\measuredangle PXZ = \measuredangle PYZ = \measuredangle PAZ = 90°$, hence the points P, A, X, Y, Z are concyclic, i.e. the quadrilateral $APXY$ is cyclic.

Remark. The construction of the point Z, can be made in a different ways. One way is the point Z to be second intersection of the circle CMY and the line AM. Another way to introduce the point Z is as a second intersection point of the circumcircles of the triangles CMY and BMX.



Problem 4. Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that

$$n!+f(m)!\mid f(n)!+f(m!)$$

for all $m,n\in\mathbb{N}$.

Solution. (BMO Shortlist) Taking $m=n=1$ in (*) yields $1+f(1)!\mid f(1)!+f(1)$ and hence $1+f(1)!\mid f(1)-1$. Since, $|f(1)-1|<f(1)!+1$, that is $1+f(1)!\mid f(1)-1$ it follows that $f(1)-1=0$, i.e. $f(1)=1$.

For $m=1$ in (*) we have $n!+1\mid f(n)!+1$, which implies $n!\le f(n)!$, i.e. $n\le f(n)$. On the other hand, taking $(m,n)=(1,p-1)$ for any prime number p and using Wilson's theorem we obtain $p\mid(p-1)!+1\mid f(p-1)!+1$, implying $f(p-1)<p$. But $f(p-1)\ge p-1$ and from $f(p-1)<p$ we conclude that

$$f(p-1)=p-1.$$

Next, fix a positive integer m. For any prime number p, setting $n=p-1$ in (*) yields $(p-1)!+f(m)!\mid(p-1)!+f(m!)$, hence

$$(p-1)!+f(m)!\mid f(m!)-f(m)!,$$

For all prime numbers p. This implies $f(m!)=f(m)!$, for all $m\in\mathbb{N}$, so (*) can be rewritten as $n!+f(m)!\mid f(n)!+f(m)!$. This implies

$$n!+f(m)!\mid f(n)!-n!,$$

for all $m,n\in\mathbb{N}$. Fixing $n\in\mathbb{N}$ and taking m large enough, we conclude that $f(n)!=n!$, i.e. $f(n)=n$, for all $n\in\mathbb{N}$.

Problem 5. Let n be a given positive integer. Sisyphus performs a sequence on a board consisting of $n+1$ squares in a row, numbered from 0 to n, starting from left to right. At the beginning, n stones are put into square numbered 0, and the other squares are empty. At any turn, Sisyphus chooses any nonempty square with k stones, takes on of these stones and moves it to the right but at most k squares (the chosen stone should stay within the board). The goal of Sisyphus is to place all n stones at the square n. Prove that Sisyphus can not achieve his goal in less that $\left\lceil\frac{n}{1}\right\rceil+\left\lceil\frac{n}{2}\right\rceil+\left\lceil\frac{n}{3}\right\rceil+\ldots+\left\lceil\frac{n}{n}\right\rceil$ moves. Notation $\lceil x\rceil$ stands for the least integer not smaller than x.

Solution. (IMO Shortlist) The stones are indistinguishable, and all have the same origin and the same final position. So, at any position we can prescribe which stone from the chosen square to move. We do it in the following manner. Number the stone from 1 to n. At any turn, after choosing a square, Sisyphus moves the stone with the largest number from the square.

This way, when stone k is moved from some square, that square contains not more than k stones (since all their numbers are at most k). Therefore, stone k is moved by at most k squares at each turn. Since the total shift of the stone is exactly n, at least $\left\lceil\frac{n}{k}\right\rceil$ moves of stone k should have been made, for every $k=1,2,\ldots,n$.

By summing up over all $k=1,2,\ldots,n$, we get the required estimate.