APMO SOLUTIONS

<u>PROBLEM 1.</u> Let F be the set of all n-tuples $(A_1, A_2, \ldots, A_n)$ where each A_i, $i = 1, 2, \ldots, n$ is a subset of $\{1, 2, \ldots, 1998\}$. Let $|A|$ denote the number of elements of the set A. Find the number

$$\sum_{(A_1, A_2, \ldots, A_n)} |A_1 \cup A_2 \cup \ldots \cup A_n|.$$

MARKING SCHEME:

Let M be a subset of the set $\{1, 2, \ldots, 1998\}$ and let $|M| = k$. Then the set M can be obtained as the union of t sets $A_1, A_2, \ldots, A_t$ in $(2^t - 1)^k$ different ways since each element $x \in M$ can belong to $2^t - 1$ nonempty families of subsets $A_1, A_2, \ldots, A_t$.

3 points for describing the correct counting method

Thus we have

$$\sum_{(A_1, A_2, \ldots, A_t) \in F} |A_1 \cup A_2 \cup \ldots \cup A_t| = \sum_{k=1}^{1998} k \binom{1998}{k} (2^t - 1)^k$$

2 points for setting up the above formula

$$= 1998(2^t - 1) \sum_{k=0}^{1997} \binom{1997}{k} (2^t - 1)^k = 1998(2^t - 1) 2^{1997t}.$$

2 points for correctly carrying out the computation

<u>PROBLEM 2.</u> Show that for any positive integers a and b, $(36a+b)(a+36b)$ cannot be a power of 2.

MARKING SCHEME:

Suppose that $(36a+b)(a+36b)$ is a power of 2 for some positive integers a and b. Write $36a+b=2^m=r$ and $a+36b=2^n=s$. Then

$$36r - s = 35 \times 37a \text{ and } 36s - r = 35 \times 37b.$$

Hence

$$1/36 < r/s = 2^{m-n} < 36, \text{or} -6 < m-n < 6.$$

2 points for the basic setting and observation

Furthermore,

$$4^n(4^{m-n}-1) = r^2 - s^2 = 35 \times 37(a^2-b^2).$$

Thus

$$4^{m-n} \equiv 1 (mod 37).$$

2 points for this congruence

Observe that the 9th power of 4 is the smallest power of 4 that is congruent to $1(mod 37)$. Thus $9|(m-n)$. Also note that $m \neq n$. Hence $|m-n| \geq 9$, which is not possible because $|m-n| < 6$. *3 points for the final conclusion*

Problem 3. Let a, b, c be positive real numbers. Prove that

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \geq 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right).$$

MARKING SCHEME:

Let

$$x = \frac{a}{\sqrt[3]{abc}}, \quad y = \frac{b}{\sqrt[3]{abc}}, \quad z = \frac{c}{\sqrt[3]{abc}}.$$

We need to show that

$$\left(1+\frac{x}{y}\right)\left(1+\frac{y}{z}\right)\left(1+\frac{z}{x}\right) \geq 2(1+x+y+z)$$

or, since $xyz = 1$,

$$(x+y)(y+z)(z+x) \geq 2 + 2(x+y+z) \quad (1)$$

2 points for (1) by making change of variables or by assuming $abc = 1$ without loss of generality

which is equivalent to

$$(x+y+z)(xy+yz+zx-2) - xyz \geq 2 \quad (2)$$

2 points for reducing to (2)

Since $xyz = 1$, the AM-GM inequality implies

$$x+y+z \geq 3 and xy+yz+zx \geq 3$$

So, (2) follows. *3 points for successful application of AM-GM inequality*

ALTERNATE SOLUTION

2 points for (1) by making change of variables or by assuming $abc = 1$ without loss of generality

¿From (1), one can proceed as follows:

$$(x+y)(y+z)(z+x) = 2xyz+x^2(y+z)+y^2(z+x)+z^2(x+y) = 2+x\left(\frac{1}{y}+\frac{1}{z}\right)+y\left(\frac{1}{z}+\frac{1}{x}\right)+z\left(\frac{1}{x}+\frac{1}{y}\right).$$

Thus, (1) is equivalent to

$$x\left(\frac{1}{y}+\frac{1}{z}\right)+y\left(\frac{1}{z}+\frac{1}{x}\right)+z\left(\frac{1}{x}+\frac{1}{y}\right) \geq 2(x+y+z).3$$

2 points for reducing to (3)

Assume that $x \geq y \geq z$. Then

$$\frac{1}{y}+\frac{1}{z} \geq \frac{1}{z}+\frac{1}{x} \geq \frac{1}{x}+\frac{1}{y}$$

hence we can apply Chebyshev's inequality

1 points for checking the hypothesis of Chevyshev's inequality

to get

$$x\left(\frac{1}{y}+\frac{1}{z}\right)+y\left(\frac{1}{z}+\frac{1}{x}\right)+z\left(\frac{1}{x}+\frac{1}{y}\right) \geq \frac{1}{3}(x+y+z)\left(\frac{2}{x}+\frac{2}{y}+\frac{2}{z}\right).$$

¿From the AM-GM inequality and the fact that $xyz = 1$ we obtain

$$\left(\frac{2}{x}+\frac{2}{y}+\frac{2}{z}\right) \geq 6$$

and hence (3) follows. *2 points for successful applications of Chevyshev's and AM-GM inequalities*

Problem 4. Let ABC be a triangle and D the foot of the altitude from A. Let E and F be on a line passing through D such that AE is perpendicular to BE, AF is perpendicular to CF, and E and F are different from D. Let M and N be the midpoints of the line segments BC and EF, respectively. Prove that AN is perpendicular to NM.

MARKING SCHEME:

Let P be such that $ADMP$ is a rectangle. Choose points Q and R on the line AP such that $QBDA$ and $ADCR$ are rectangles. Points Q, B and D lie on the circle of diameter AB, hence $ADEQ$ is a cyclic quadrilateral. Similarly, R, C and D lie on the circle of diameter AC, hence $ADFR$ is a cyclic quadrilateral.

1 point for proving $ADEQ$ and/or $ADFR$ are cyclic

The two quadrilaterals share a side, and have the same supporting lines for other two sides. Since they are cyclic, the remaining two sides EQ and RF must be parallel. Thus E, Q, R and F are vertices of a trapezoid.

1 point for proving $EQ \parallel RF$

On the other hand, in rectangle $QBCR$ M is the midpoint of BC, and MP is parallel to QB, so P is the midpoint of QR. Since N is the midpoint of EF, we obtain that, in trapezoid $QEFR$, NP is parallel to QE.

2 points for proving $NP \parallel QE$

This implies that quadrilateral $ADNP$ is cyclic, having the sides parallel to the sides of $ADFR$. Moreover, A lies on the circle circumscribed to this quadrilateral, because the other three vertices of the rectangle $ADMP$ lie on it. Hence the quadrilateral $ADMN$ is cyclic.

2 points for proving $ADNP$ and $ADMN$ are cyclic

Consequently, $\angle ANM = 180° - \angle ADM = 90°$.

1 point for proving $\angle ANM = 180° - \angle ADM = 90°$

Problem 5. Determine the largest of all integers n with the property that n is divisible by all positive integers that are less than $\sqrt[3]{n}$.

MARKING SCHEME:

Observation from that $lcm(2,3,4,5,6,7) = 420$ is divisible by every integer less than or equal to $7 = [\sqrt[3]{420}]$ and that $lcm(2,3,4,5,6,7,8) = 840$ is not divisible by $9 = [\sqrt[3]{840}]$, one may guess 420 is the required integer. *2 points for the correct guess*

Let N be the required integer and suppose $N > 420$. Put $t = [\sqrt[3]{N}]$. Then

$$N \le t(t^3 + 3t + 3) \quad (1)$$

Since $t \ge 7$, $lcm(2,3,4,5,6,7) = 420$ should divide N and hence $N \ge 840$, which implies $t \ge 9$. But then $lcm(2,3,4,5,6,7,8,9) = 2520$ should divide N, which implies $t \ge 13 = [\sqrt[3]{2520}]$.

1 points for $t \ge 13$

Observe that any four consecutive integers are divisible by 8 and that any two out of four consecutive integers have *gcd* either 1, 2, or 3. So, we have $t(t-1)(t-2)(t-3)$ divides $6N$ and in particular,

$$t(t-1)(t-2)(t-3) \le 6N \quad (2)$$

2 points for $t(t-1)(t-2)(t-3) \le 6N$

¿From (1) and (2) follows

$$t(t-1)(t-2)(t-3) \le 6t(t^3 + 3t + 3) \frac{12}{t} + \frac{7}{t^2} + \frac{24}{t^3} \ge 1.$$

Since $t \ge 13$,

$$\frac{12}{t} + \frac{7}{t^2} + \frac{24}{t^3} < 1,$$

which is a contradiction.

2 points for the contradiction