ALGEBRA

A1. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2 + xy + f(y^2)) = xf(y) + y^2 + f(x^2),$$

for all real numbers x and y.

United Kingdom, Joe Benton

Solution 1. We claim that the only solutions are $f(x) = x$ and $f(x) = -x$; it is easy to verify that these are indeed solutions.

Now suppose f satisfies the given equation. Studying $(x, 0)$ and $(-x, 0)$ leads to

$$xf(0) + f(x^2) = f(x^2 + f(0)) = -xf(0) + f(x^2),$$

and so $f(0) = 0$. Then, studying $(0, y)$ and $(-y, y)$ leads to

$$y^2 = f\left(f(y^2)\right) = -yf(y) + y^2 + f(y^2),$$

and therefore

$$f(y^2) = yf(y). \tag{1}$$

Given (x, y), take $z = -(x + y)$, so that $x^2 + xy = z^2 + zy$. Then $f(x^2 + xy + f(y^2)) = f(z^2 + zy + f(y^2))$, hence from the original equation,

$$xf(y) + y^2 + f(x^2) = zf(y) + y^2 + f(z^2),$$

$$xf(y) - zf(y) = f(z^2) - f(x^2).$$

Eliminating z and using (1),

$$2xf(y) = f(z^2) - f(x^2) - yf(y) = f((x + y)^2) - f(x^2) - f(y^2). \tag{2}$$

But the RHS of (2) is symmetric in x, y and so $xf(y) = yf(x)$. In particular $f(x) = xf(1)$. Writing $c = f(1)$, and substituting back into the original equation, all the terms cancel except $c^2y^2 \equiv y^2$, hence $c = \pm 1$, so $f(x) = x$ and $f(x) = -x$ are the only solutions.

Solution 2. Letting $z = -(x + y)$, we have $z^2 + zy = x^2 + xy$. Substituting the pairs (x, y) and (z, y) into the given equation yields

$$xf(y) + y^2 + f(x^2) = f(x^2 + xy + f(y^2)) = f(z^2 + zy + f(y^2)) = zf(y) + y^2 + f(z^2),$$

or

$$f((x + y)^2) - f(x^2) = (x - z)f(y) = (2x + y)f(y).$$

Now, for any a, b, c we have

$$\begin{aligned} f((a + b + c)^2) - f(a^2) &= \left(f((a + b + c)^2) - f((a + b)^2)\right) + \left(f((a + b)^2) - f(a^2)\right) \\ &= (2a + 2b + c)f(c) + (2a + b)f(b). \end{aligned}$$

Swapping b and c, we similarly obtain

$$f((a + b + c)^2) - f(a^2) = (2a + 2c + b)f(b) + (2a + c)f(c).$$

Subtracting the two obtained equations, we get $bf(c) = cf(b)$, which yields $f(x) = xf(1)$ for all x.

It remains to check that the only functions of the form $f(x) = Cx$ are $f(x) = x$ and $f(x) = -x$.

Solution 3. Plug $y = -x$ into the given relation to write $f(f(x^2)) = xf(-x) + x^2 + f(x^2)$. Changing x to $-x$ in the latter and comparing the two yields $xf(-x) = -xf(x)$, so $f(-x) = -f(x)$ for all non-zero real numbers x.

Let $y = 0$ in the given relation, to get $f(x^2 + f(0)) = xf(0) + f(x^2)$ and infer thereby that $-xf(0) = xf(0)$ for all real numbers x, so $f(0) = 0$.

Plug $x = 0$ into the given relation, to get $f(f(y^2)) = y^2$, and refer to the first paragraph to write $f(x^2) = -xf(-x) = xf(x)$ for all real numbers x.

Let $x = -f(y)$ in the given relation and refer to the previous paragraph to write $f(f(y)^2) = -f(y)^2 + y^2 + f(f(y)^2)$, so $f(y)^2 = y^2$, i.e., $f(y) = \pm y$ for all real numbers y.

Suppose now, if possible, that $f(x) = -x$ and $f(y) = y$ for some non-zero real numbers x and y. Then $|x| \neq |y|$ and $f(x^2 + xy + y^2) = xy + y^2 - x^2$, so $x^2 + xy + y^2 = \pm(xy + y^2 - x^2)$ which is impossible.

Consequently, either $f(x) = x$ for all real numbers x or $f(x) = -x$ for all real numbers x.

A2. Given a positive integer n, determine the maximal constant C_n satisfying the following condition: For any partition of the set $\{1, 2, \ldots, 2n\}$ into two n-element subsets A and B, there exist labellings $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ of A and B, respectively, such that $(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2 \geq C_n$.

Belarus, Barys Serankou and Mikhail Karpuk

Solution 1. The required maximal constant is $C_n = n(13n^2 - 4)/12$ if n is even, and $C_n = n(13n^2 - 1)/12$ if n is odd.

For any partition $A \sqcup B$ and any labellings $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ of A and B, respectively, $\sum_{i=1}^n (a_i - b_i)^2 = \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 - 2\sum_{i=1}^n a_i b_i = \sum_{i=1}^n i^2 - 2\sum_{i=1}^n a_i b_i$. Thus, maximising $\sum_{i=1}^n (a_i - b_i)^2$ is equivalent to minimising $\sum_{i=1}^n a_i b_i$. By the rearrangement inequality, the latter achieves its minimum — and so, the former its maximum — over all possible labellings when $a_1 < a_2 < \cdots < a_n$ and $b_1 > b_2 > \cdots > b_n$.

We first deal with the case where n is even, say, $n = 2k$. Consider the partition $A_0 = \{1, 2, \ldots, k, 3k+1, 3k+2, \ldots, 4k\}$ and $B_0 = \{3k, 3k-1, \ldots, k+1\}$, the elements of both halves being labelled as displayed. It is readily checked that the corresponding sum is the C_n mentioned above. Since A_0 is labelled increasingly, and B_0 decreasingly, the previous paragraph shows that C_n cannot be increased for this partition.

We now show that, if $A \sqcup B$ is any partition, and A is labelled increasingly, $a_1 < a_2 < \cdots < a_n$, and B decreasingly, $b_1 > b_2 > \cdots > b_n$, then $\sum_{i=1}^n (a_i - b_i)^2 \geq C_n$. Let m be the number of elements in A that do not exceed n, and notice that the first m elements of B all exceed n. Let further $r_i = b_i - a_i > 0$, $i = 1, \ldots, m$, and $q_j = a_{n+1-j} - b_{n+1-j} > 0$, $j = 1, \ldots, n-m$, and notice that

$$r_i - r_{i+1} \geq 2, \quad i = 1, \ldots, m-1, \quad \text{and} \quad q_j - q_{j+1} \geq 2, \quad j = 1, \ldots, n-m-1, \tag{1}$$

and

$$r_1 + \cdots + r_m + q_1 + \cdots + q_{n-m} = (n+1+\cdots+2n) - (1+\cdots+n) = n^2. \tag{2}$$

Collect the r_i and the q_j together and arrange them in descending order to form an n-tuple $P = (P_1, P_2, \ldots, P_n)$, where $P_i \geq P_{i+1}$, $i = 1, \ldots, n-1$. Relation (1) shows that $P_i - P_{i+2} \geq 2$, $i = 1, \ldots, n-2$, and relation (2) reads $P_1 + \cdots + P_n = n^2$.

Since the function $x \mapsto x^2$ is convex, it suffices, by Karamata's inequality, to show that P majorises the n-tuple $I = (I_1, I_2, \ldots, I_n) = (3k-1, 3k-1, 3k-3, 3k-3, \ldots, k+1, k+1)$ corresponding to the partition $A_0 \sqcup B_0$; that is, $P_1 + \cdots + P_i \geq I_1 + \cdots + I_i$ for all indices i.

Suppose, if possible, this is not the case, and let s be the smallest index for which $P_1 + \cdots + P_s < I_1 + \cdots + I_s$; then $P_s < I_s$. Relation (2) shows that $s < n$. Without loss of generality, we may and will assume that P_s is some r_i, so the list $P_1, \ldots, P_s$ consists of $r_1, \ldots, r_i$ and $q_1, \ldots, q_{s-i}$. Consider the parity of s.

If s is odd, then $P_{s+1} \leq P_s < I_s = I_{s+1}$, so $P_{s+1+2j} \leq P_{s+2j} \leq P_{2s} - 2j < I_s - 2j = I_{s+2j} = I_{s+1+2j}$ for all non-negative j. Consequently,

$$\begin{aligned} n^2 = P_1 + \cdots + P_n &= (P_1 + \cdots + P_s) + P_{s+1} + \cdots + P_n \\ &< (I_1 + \cdots + I_s) + I_{s+1} + \cdots + I_n = I_1 + \cdots + I_n = n^2, \end{aligned}$$

which is a contradiction. This settles the case where s is odd.

If s is even (so $s \leq 2k-2$, since $s < n = 2k$), then $P_{s+1} \leq P_s \leq I_s - 1 = I_{s+1} + 1$, and $P_{s+2} \leq P_s - 2$, and $I_{s+2} = I_{s+1}$. Hence $P_{s+1} + P_{s+2} \leq 2P_s - 2 \leq 2I_s - 4 = I_{s+1} + I_{s+2}$. It then follows that $P_{s+1+2j}+P_{s+2+2j} \le P_{s+1}-2j+P_{s+2}-2j \le I_{s+1}-2j+I_{s+2}-2j = I_{s+1+2j}+I_{s+2+2j}$ for all non-negative j. Consequently,

$$\begin{aligned} n^2 = P_1+\cdots+P_n &= (P_1+\cdots+P_s)+(P_{s+1}+P_{s+2})+\cdots+(P_{n-1}+P_n) \\ &< (I_1+\cdots+I_s)+(I_{s+1}+I_{s+2})+\cdots+(I_{n-1}+I_n) = I_1+\cdots+I_n = n^2, \end{aligned}$$

which is again a contradiction. This settles the case where s is even, and concludes the proof for an even n.

To deal with the case where n is odd, say, $n = 2k-1$, consider the partition $A_0 = \{1, 2, \ldots, k-1, 3k-1, 3k, \ldots, 4k-2\}$ and $B_0 = \{3k-2, 3k-3, \ldots, k\}$, the elements of both halves being again labelled as displayed. As before, the corresponding sum is the C_n in the first paragraph, and this is maximal over all possible labellings for this partition.

To complete the proof, go through the previous argument all the way down to the case analysis involving the parity of s. Notice that the current I is $I = (3k-2, 3k-3, \ldots, k)$, and argue as follows: Clearly, $P_s < I_s$, so $P_{s+1} \le P_s \le I_s - 1 = I_{s+1}$. Recall that $P_{i+2} \le P_i - 2$ for all i, to infer that $P_{s+2j} < I_{s+2j}$ and $P_{s+1+2j} \le I_{s+1+2j}$ for all non-negative j, and reach thereby the same contradiction as before:

$$\begin{aligned} n^2 = P_1+\cdots+P_n &= (P_1+\cdots+P_s)+P_{s+1}+\cdots+P_n \\ &< (I_1+\cdots+I_s)+I_{s+1}+\cdots+I_n = I_1+\cdots+I_n = n^2. \end{aligned}$$

Solution 2. Notice that

$$S(a_1,\ldots,b_n) = \sum_{i=1}^{n}(a_i-b_i)^2 = \sum_{i=1}^{n}(a_i^2+b_i^2) - 2\sum_{i=1}^{n} a_ib_i = 2\left(\sum_{j=1}^{2n} j^2 - \sum_{i=1}^{n} a_ib_i\right).$$

Hence, for a given partition $A \sqcup B$, in order to maximize the sum $S(a_1,\ldots,b_n)$, one needs to minimize $P(a_1,\ldots,b_n) = \sum_i a_ib_i$. Due to the rearrangement inequality, this happens if the a_i and the b_i are ordered in the opposite way, i. e., the optimal value is achieved when $a_1 < \cdots < a_n$ and $b_1 > \cdots > b_n$. In the sequel, we always assume that the elements of A and B are arranged in such way, and denote by $S(A,B)$ and $P(A,B)$ the values of the corresponding sums. Under this convention, our aim is to find a partition minimizing $S(A,B)$, — equivalently, maximizing $P(A,B)$; call such partition *optimal*.

Let $A_- = A \cap \{1,2,\ldots,n\}$ and $A_+ = A \cap \{n+1,\ldots,2n\}$; define B_- and B_+ similarly. We have $|A_+| = n - |A_-| = |B_-|$ and, similarly, $|A_-| = |B_+|$. Thus, for every j, the numbers a_j and b_j fall in different halves of the interval $[1, 2n]$. For two sets X and Y, we say that $X < Y$ if $x < y$ for all $x \in X$ and $y \in Y$. We first show that an optimal partition has a simple structure.

Lemma. Any optimal partition necessarily satisfies $A_- < B_- < B_+ < A_+$, up to swapping A with B.

Proof. Arguing indirectly, we will show that the partition may be modified into a better one. Without loss of generality, we suppose that $1 \in A$. Then the partition starts with $1,\ldots,p \in A$, $p+1,\ldots,p+q \in B$, $p+q+1 \in A$, and ends with $2n, 2n-1, \ldots, 2n-(r-1) \in A$ (possibly, with $r = 0$), $2n-r, \ldots, 2n-r-(s-1) \in B$, $2n-r-s \in A$, where $p+r < n$, as illustrated below:

$$\overbrace{a\,a\,a}^{p}\underbrace{b\,b\,b\,b}_{q}a\ \ldots\ a\,\overbrace{b\,b\,b}^{s}(\underbrace{a\,a\,a\,a\,a}_{r})$$

We will make use of the following observation $(*)$. Assume that $a_i = k < b_j = k+1 \le n$. If $(n<)a_j < b_i$, then, by the rearrangement inequality, we have $a_ib_i + a_jb_j < a_ia_j + b_ib_j$. Hence, swapping k and $k+1$, we get a better partition. The same happens if $a_i < b_j \leq n < a_j = k < b_i = k+1$, and so on. In the sequel, we will visualize such occurrences by joining (a_i, b_i) and (a_j, b_j) with brackets.

Now we seek for such indices. Assume first that $r = 0$. Without loss of generality, we have $p \leq s$, so $p+1 = b_n$. Then we have $a_p = p < b_n = p+1 < a_n < b_p = 2n-(p-1)$, and $(*)$ applies:

$$\overbrace{a\,a\,a}^{p}\,b\,\ldots\,a\,\overbrace{b\,b\,b\,b}^{s}.$$

Therefore, $r > 0$. Assume now that $q \leq r$ (so $p+q \leq p+r < n$). Then $b_q = p+q < a_{p+1} = p+q+1 \leq n < b_{p+1} < a_q = 2n-q+1$, so $(*)$ applies:

$$\underbrace{a\,a\,a\,a\,a}_{p}\overbrace{b\,b\,b}^{q}a\,\ldots\,b\,\ldots\,\overbrace{a\,a\,a\,a}^{r}.$$

Otherwise, $q > r$ and, similarly, $s > p$. In this last case, we modify the partition by swapping the middle segment $[p+r+1, 2n-(p+r)]$: each number from this segment moves from A to B and vice versa (this is possible, since the segment contains equally many elements from A and B). This modification does not change the value of $P(A, B)$, since the sum still contains the same products. On the other hand, the new partition falls into the previous case:

$$\underbrace{a\,a\,a\,a}_{p}\overbrace{b\,b\,b\,b\,b\,b}^{q}a\,\ldots\,a\,\overbrace{b\,b\,b\,b\,b\,b\,b}^{s}\underbrace{a\,a\,a}_{r} \qquad \mapsto \qquad \underbrace{a\,a\,a\,a}_{p}\overbrace{b\,b\,b}^{r}\underbrace{a\,a\,a}_{q-r}b\,\ldots\,b\,\underbrace{a\,a\,a}_{s-p}\overbrace{b\,b\,b\,b}^{p}\underbrace{a\,a\,a}_{r}.$$

The lemma is proved. $\square$

Finally, it suffices to find an optimal partition among those satisfying $A_- < B_- < B_+ < A_+$. Denoting by S_k the value of S for such a partition with $|A_-| = |B_+| = k$ and $|A_+| = |B_-| = \ell = n-k$, we have

$$S_k = \sum_{i=1}^{k}(n+k+1-2i)^2 + \sum_{i=1}^{\ell}(n+\ell+1-2i)^2.$$

Since $x^2 = \frac{(x+2)(x+1)x}{6} - \frac{x(x-1)(x-2)}{6}$, we obtain

$$\begin{aligned} S_k &= \left(\frac{(n+k+1)(n+k)(n+k-1)}{6} - \frac{(n-k+1)(n-k)(n-k-1)}{6}\right) \\ &\qquad + \left(\frac{(n+\ell+1)(n+\ell)(n+\ell-1)}{6} - \frac{(n-\ell+1)(n-\ell)(n-\ell-1)}{6}\right) \\ &= (k+\ell)\frac{3n^2-1}{3} + \frac{k^3+\ell^3}{3} = \frac{3n^3-n}{3} + \frac{n^3-3nk\ell}{3}. \end{aligned}$$

Hence, in order to minimize S_k, one needs to maximize $k\ell$, and this is achieved when $k = \lfloor n/2 \rfloor$, $\ell = \lceil n/2 \rceil$. Therefore, the answer is

$$C_n = \frac{4n^3-n}{3} - n\left\lfloor \frac{n^2}{4} \right\rfloor.$$

A3. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + yf(x)) + f(xy) = f(x) + f(2019y),$$

for all real numbers x and y.

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Solution. There are three types of such functions: (i) $f(x) = 2019 - x$; (ii) $f(x) = c$ for an arbitrary constant c; and (iii) $f(x) = 0$ for $x \neq 0$, and $f(0)$ is arbitrary.

A straightforward check shows that all three types satisfy the equation hence we need to show that they are the only ones. Let $N = 2019$.

First of all, setting $x = Nx'$, we arrive at the equation $f(Nx' + yf(Nx')) + f(Nx'y) = f(Nx') + f(Ny)$. After a change $g(x) = f(Nx)/N$ this equation reads

$$g\bigl(x + yg(x)\bigr) + g(xy) = g(x) + g(y) \qquad (x, y \in \mathbb{R}), \tag{3}$$

which does not depend on N. Now we investigate the corresponding functions g.

Setting $x = 1$ we get $g(1 + yg(1)) = g(1)$. If $g(1) \neq 0$, then $1 + yg(1)$ attains all real values, so we arrive at the answer (ii). Otherwise, $g(1) = 0$, and by setting $y = 1$ we get $g\bigl(x + g(x)\bigr) = 0$. If $a = 1$ is the unique real number with $g(a) = 0$, then we obtain $x + g(x) = 1$, whence $g(x) = 1 - x$, which falls into (i). Hence in the sequel we assume that

$$g(1) = 0, \quad \text{and also } g(a) = 0 \text{ for some } a \neq 1. \tag{$*$}$$

We will make use of the following two arguments.

Claim 1. If b is an arbitrary zero of g, then by substituting $x = z$ we get $g(zy) = g(y)$. Recalling that $g(g(0)) = g(0 + g(0)) = 0$, we obtain also $g(g(0)y) = g(y)$. $\square$

Claim 2. Let a and b are two zeroes of g, and let s be its *non-zero*, i.e. $g(s) \neq 0$. We claim that g is p-periodic, where $p = (a - b)s$. Indeed, substituting $x = as$ and using Claim 1, we get that he expression

$$g(as + yg(s)) = g(as) + g(y) - g(asy) = g(s) + g(y) - g(sy)$$

does not de[pend on a. Hence $g(as + yg(s)) = g(bs + yg(s))$ for all y, which proves the required periodicity, since $yg(s)$ attains all real values. $\square$

Now, if $g(x) = 0$ for all $x \neq 0$, we get the remaining answer (iii). Assume now that there exists $s \neq 0$ with $g(s) \neq 0$, so by Claim 2 g is periodic with some period p. Substituting $x = p$ and using periodicity we get $g(yg(0)) + g(py) = g(0) + g(y)$. Since $g(yg(0)) = g(y)$ by Claim 1, we arrive at $g(py) = g(0)$ which shows g is constant.

Remark. After arriving at $(*)$ and obtaining Claims 1 and 2, alternative approaches are possible.

E.g., denote by $Z = \{x \in \mathbb{R}\colon g(x) = 0\}$ the set of zeros of g. Claim 1 yields that Z is *a-invariant*, i.e., $aZ = Z$. We want to show that $Z - Z = \mathbb{R}$; this will, by means of Claim 2, yield that g is periodic with *every* period, i.e., constant.

For any $\beta \in Z$, we plug in $y = \beta$ and use Claim 1 to obtain $g(x + \beta g(x)) = 0$, so $x + \beta g(x) \in Z$ for all x. Now, setting $\beta = 1$ and $\beta = a$ (from $(*)$) we get $x + g(x), x + ag(x) \in Z$. The first inclusion yields also $a(x + g(x)) \in Z$, and hence $(a - 1)x = a(x + g(x)) - (x + ag(x)) \in Z - Z$. This shows that $Z - Z = \mathbb{R}$.

COMBINATORICS

C1. Let k and N be integers such that $k > 1$ and $N > 2k + 1$. A number of N persons sit around the Round Table, equally spaced. Each person is either a knight (always telling the truth) or a liar (who always lies). Each person sees the nearest k persons clockwise, and the nearest k persons anticlockwise. Each person says: "I see equally many knights to my left and to my right." Establish, in terms of k and N, whether the persons around the Table are necessarily all knights.

Russia, Sergey Berlov

Solution. The answer is in the affirmative if $2k+1$ is coprime to N; otherwise, the answer is in the negative.

A key step is to prove that the configuration of knights and liars is always $(2k+1)$-periodic. Thus, if $2k+1$ is coprime to N, then the configuration around the Table is either all-knight or all-liar. The latter is, however, ruled out by the condition in the statement, whence the answer in the affirmative.

On the other hand, if $\gcd(2k+1, N) = d > 1$, the d-periodic configuration where each knight is followed by $d-1$ liars satisfies the condition in the statement. Indeed, the configuration is symmetric about each knight, so every knight is allowed to make the required statement. On the other hand, there are exactly $(2k+1)/d$ knights in the $(2k+1)$-string centred at each liar. Since $(2k+1)/d$ is odd, a liar cannot see equally many knights on both sides, and may therefore make the required statement as well.

To prove $(2k+1)$-periodicity, let $P_1, P_2, \ldots, P_N$ be a clockwise circular labelling of the persons around the Table; extend the labelling periodically with period N. For each index i, let $a_i = 0$ if P_i is a liar, let $a_i = 1$ if P_i is a knight, and let m_i^+ and m_i^- denote the number of knights P_i sees clockwise and anticlockwise, respectively.

Clearly, $m_i^+ = m_i^-$ whenever P_i is a knight. We now proceed to show that $m_i^+ - m_i^- = \pm 1$ whenever P_i is a liar.

Suppose, if possible, that $|m_i^+ - m_i^-| \geq 2$ for some index i. Without loss of generality, let $m_i^+ \geq m_i^- + 2$. Since P_i is clearly a liar, $a_i = 0$, so $m_{i+1}^- = m_i^- + a_i - a_{i-k} = m_i^- - a_{i-k} \leq m_i^-$. Consequently, $m_{i+1}^+ = m_i^+ + a_{i+1+k} - a_{i+1} \geq m_i^+ - 1 \geq m_i^- + 1 \geq m_{i+1}^- + 1$. Hence P_{i+1} is a liar, so $a_{i+1} = 0$, and $m_{i+1}^+ = m_i^+ + a_{i+1+k} \geq m_i^+ \geq m_i^- + 2 \geq m_{i+1}^- + 2$. Continue all the way around the Table to infer that $m_j^+ \geq m_j^- + 2$ for all j, so the P_j are all liars — a contradiction, since the all-liar configuration is ruled out by the condition in the statement.

Let now n_i be the total number of knights in the $(2k+1)$-person string centred at P_i. By the preceding, the n_i are all odd, so the $n_i - n_{i+1}$ are all even. The condition $|n_i - n_{i+1}| = |a_{i-k} - a_{i+1+k}| \leq 1$ then forces the n_i to be all equal, so $a_{i-k} = a_{i+1+k}$ for all indices i. This establishes $(2k+1)$-periodicity and concludes the proof.

Remarks. (1) The argument in the solution shows that the configuration of knights and liars is d-periodic, where $d = \gcd(2k+1, N)$. On the other hand, one can prove that a d-periodic configuration satisfies the conditions in the statement if and only if the period contains at least one knight and is *balanced with respect to knights*, i.e., the left and right halves of the d-person string centred at each knight contain equally many knights. Not all periods (even with an odd number of knights) are balanced — check out (LLKKK); on the other hand, there exist non-trivial balanced periods, e.g., (KLKLLKLLL).

(2) Assuming images through rotations and symmetries to be *distinct*, it is possible to determine the total number of such configurations — although it is perhaps too much for an olympiad purposes. It turns out that this number is the d-th Lucas number L_d.

Here is a sketch of the proof. In what follows, 'balanced' stands for what was called 'balanced with respect to knights' in the previous remark. Let $d = 2t - 1$. Given an infinite d-periodic binary sequence $\dots$, a_0, a_1, $\dots$ containing at least one unit, consider the d-periodic sequence $b_i = a_{ti}$. We claim that (a_i) is balanced if and only if $(*)$ any maximal contiguous block of zeroes in (b_i) has an even length (and there is at least one unit in this sequence).

Indeed, if condition $(*)$ is satisfied, then all zeroes in (a_i) may be split (d-periodically) into pairs so that the zeroes in each pair are t distance apart. The zeroes in every period a_{i+1}, $\dots$, a_{i+d} can therefore be split into pairs, the zeroes in each pair being either t or $t-1$ distance apart. If this period is centred at a unit, then the zeroes of each pair fall in different halves, so these halves contain equally many zeroes. Consequently, the period of (a_i) is indeed balanced.

Conversely, assume that the sequence (b_i) contains a contiguous block of an odd number of zeroes flanked by units, say, $b_0 = 1$, $b_1 = \cdots = b_{2q+1} = 0$, and $b_{2q+2} = 1$. Then $a_0 = a_{q+1} = 1$, $a_1 = \cdots = a_q = 0$, and $a_t = a_{t+1} = \cdots = a_{t+q} = 0$, so $m_0^+ - m_{q+1}^+ = (a_1 + \cdots + a_{q+1}) - (a_t + \cdots + a_{t+q}) = 1$. Consequently, the sequence is not balanced for one of the two knights at positions 0 and $q+1$.

It then suffices to determine the number of d-periodic sequences (b_i) satisfying $(*)$. Splitting all zeroes into pairs of consecutive zeroes shows that this number is equal to the number of ways to split a (discrete) circle of length d into arcs of lengths 1 and 2. This number is indeed $L_d = F_d + F_{d-2}$, where the F_n are the Fibonacci numbers.

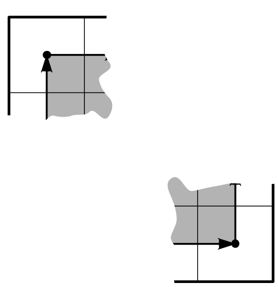
C2. Fix an integer $n \geq 2$. A fairy chess piece *leopard* may move one cell up, or one cell to the right, or one cell diagonally down-left. A leopard is placed onto some cell of a $3n \times 3n$ chequer board. The leopard makes several moves, never visiting a cell twice, and comes back to the starting cell. Determine the largest possible number of moves the leopard could have made.

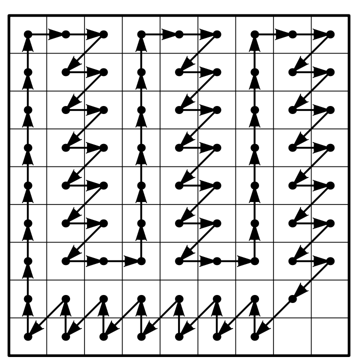
Russia, Dmitry Khramtsov

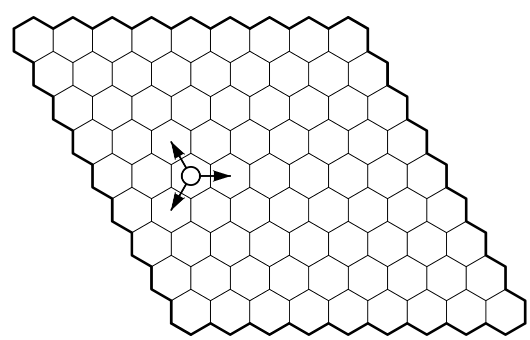
Solution. The required maximum is $9n^2 - 3$. We first show that the number of visited cells is divisible by 3. To this end, refer to a coordinate frame whose axes point upward and rightward, respectively, so that the centres of the cells have integral coordinates in the range 1 through $3n$. Assign each cell (i, j) the residue of $i + j$ modulo 3. Notice that equally many cells are assigned each residue, since this is the case for every 3×1 rectangle. Any move of the leopard is made from a cell assigned residue r to one assigned residue $r + 1$ modulo 3. Hence, the residues along any path of the leopard are $\cdots \to 0 \to 1 \to 2 \to 0 \to 1 \to \cdots$. Consequently, every cyclic path contains equally many cells assigned each residue, so the total number of cells along the cycle is divisible by 3.

Next, we show that the leopard cannot visit all cells. Argue indirectly. The leopard's cycle forms a closed non-self-intersecting polyline enclosing some region along whose boundary the leopard moves either clockwise or counter-clockwise. On the other hand, the cell $(3n, 1)$ may be visited only as $(3n - 1, 1) \to (3n, 1) \to (3n, 2)$, while the cell $(1, 3n)$ may be visited only as $(1, 3n - 1) \to (1, 3n) \to (2, 3n)$ — see the left figure below. The region bounded by the leopard's cycle lies to the left of the path in the former case, but to the right of the path in the latter case. This is a contradiction.

The argument above shows that the number of visited cells does not exceed $9n^2 - 3$. The figure in the middle below shows that this bound is indeed achieved.







Remarks. **(1)** Variations of the argument in the second paragraph above may equally well work. For instance, consider the situation around the cell $(1, 1)$. The cycle comes to that cell from $(2, 2)$, and leaves it to, say, $(1, 2)$. The previous part of the cycle should then look like $(i, 2) \to (i - 1, 1) \to (i - 1, 2) \to (i - 2, 1) \to \cdots \to (2, 2) \to (1, 1) \to (1, 2)$, say with a maximal i. Then the cell $(i, 1)$ could only be visited as $(i + 1, 2) \to (i, 1) \to (i + 1, 1)$; again, the two parts have opposite orientations with respect to the interior region.

(2) The problem becomes more symmetric under an affine transformation mapping the square lattice to the honeycomb illustrated in the right figure above.

C3. Fix an odd integer $n > 1$. For a permutation p of the set $\{1, 2, \ldots, n\}$, let S be the number of pairs of indices (i, j), $1 \le i \le j \le n$, for which $p_i + p_{i+1} + \cdots + p_j$ is divisible by n. Determine the maximum possible value of S.

CROATIA

Solution 1. For visual clarity, write the numbers $p_1, \ldots, p_n$ in a row, and put a dot between any two of them, and also to the left and to the right of the row. We say that a pair of dots is *good* is n divides the sum of the numbers between them. So we need to find the maximal possible number of good pairs.

Let $n = 2k - 1$; we claim that the answer is $\binom{k+1}{2}$. A confirming example is constructed as follows. Partition the numbers $1, 2, \ldots, n$ into $k - 1$ pairs $\{1, n-1\}$, $\{2, n-2\}$, $\ldots$, $\{k-1, k\}$ and a singleton $\{n\}$; the sum in each part is n. Arrange the numbers so that the elements of each pair are adjacent; paint red a dot between the two numbers from the same pair. Then each pair of non-red dots is good, so there are at least $\binom{k+1}{2}$ good pairs.

Now we show that this is indeed the maximal possible number of good pairs. For this purpose, notice first that removal of the number n from the row does not change *goodness* of any pair; it may, however, glue together some good pairs, which we will take care of later. Hence, we first consider a permutation $q_1, \ldots, q_{n-1}$ of $1, 2, \ldots, n-1$ and estimate the number of good pairs for such permutation.

Split all entries in the new row into $k - 1$ pairs $P_1, \ldots, P_{k-1}$ of adjacent entries. Say that a pair of dots is *of type* (i, j) if the space between them meets exactly the pairs $P_i, \ldots, P_j$. Now we estimate the number of good pairs of each type separately.

Case 1: types (i, j) *with* $i = j$. Such pair may be good only if it encloses a pair of complementary numbers $(s, n-s)$, i.e., if P_i contains these numbers in some order. Let $d \le k-1$ be the number of such pairs P_i.

Case 2: types (i, j) *with* $i < j$. There are four pairs of dots of any such fixed type. Let a be the left entry in P_i, and b be the right entry in P_j. Then the four sums have the forms s, $s+a$, $s+b$, and $s+a+b$, for some s. One of these sums may be divisible by n. However, *two* of the four sums may be divisible by n only if $a+b = n$; in this case, P_i and P_j contain complementary numbers a and b, respectively. Since the total number of complementary pairs is $k-1$, the number of such pairs (i, j) is at most $(k-1) - d$; hence the total number of good pairs of the considered types is at most $\binom{k-1}{2} + (k-1) - d$.

Combining the bounds above, we get that the number of good pairs in the new situation is at most $d + \binom{k-1}{2} + (k-1) - d = \frac{1}{2}k(k-1)$.

It remains to check how many good pairs are lost when n has been removed. These are exactly the pairs of dots such that n stands at one of the ends of the segment flanked by the dots; one of these segments consists of n alone. Assume that there are a numbers to the right of n (including n itself). Then, only every second of them can be the other end of a segment enclosed by a good pair, hence there are at most $\left\lfloor \frac{1}{2}(a+1) \right\rfloor$ such pairs. Similarly, since there are $b = n + 1 - a$ numbers to the left of n, there are at most $\left\lfloor \frac{1}{2}(b+1) \right\rfloor$ corresponding good pairs. The segment covering n alone is counted twice, so the total number of lost good segments is at most $\left\lfloor \frac{1}{2}(a+1) \right\rfloor + \left\lfloor \frac{1}{2}(b+1) \right\rfloor - 1 \le \left\lfloor \frac{1}{2}(a+1) + \frac{1}{2}(b+1) \right\rfloor - 1 = \left\lfloor \frac{1}{2}(a+b) \right\rfloor = \frac{1}{2}(n+1) = k$.

Thus, the total number of good pairs is at most $\frac{1}{2}k(k-1) + k = \frac{1}{2}k(k+1)$, as required.

Solution 2. We present a different approach for the estimate. Consider the prefix sums s_0, s_1, $\ldots$, s_n of the sequence p_1, p_2, $\ldots$, p_n, i.e., $s_0 = 0$, and $s_i = s_{i-1} + p_i$, $i = 1, 2, \ldots, n$. For each residue r modulo n, let C_r be the number of indices i such that $s_i \equiv r \pmod{n}$, so $S = \sum_{r=0}^{n-1} \binom{C_r}{2}$. Letting $c_1 \ge c_2 \ge \ldots \ge c_k$ be the non-zero elements in the list C_0, C_1, $\ldots$, C_{n-1}, we are to maximise the sum $\sum_{i=1}^k \binom{c_i}{2}$. Alternatively, but equivalently, we are to maximise $\sum_{i=1}^k c_i^2$, since the c_i add up to $n+1$.

Lemma. $c_1 + c_2 \le (n+5)/2$.

Proof. Fix residues $0 \le r < r' < n$, one of which occurs c_1 times in the list

$$s_0 \pmod n, \quad s_1 \pmod n, \quad \ldots, \quad s_n \pmod n,$$

and the other c_2 times.

There is at most one i such that $s_{i-1} \equiv r \pmod n$ and $s_i \equiv r' \pmod n$, in which case $p_i = r' - r$. Similarly, there is at most one i such that $s_{i-1} \equiv r' \pmod n$ and $s_i \equiv r \pmod n$, in which case $p_i = n + r - r'$. There is at most one i such that either $s_{i-1} \equiv s_i \equiv r \pmod n$ or $s_{i-1} \equiv s_i \equiv r' \pmod n$, in which case $p_i = n$.

Dropping 3 of the $c_1 + c_2$ occurrences of r and r', the remaining $c_1 + c_2 - 3$ occurrences are at least 2 distance apart from one another, so their number does not exceed $\lceil \frac{1}{2}\big((n+1) - 3\big) \rceil = \lceil \frac{1}{2}(n-2) \rceil = (n-1)/2$. Consequently, $c_1 + c_2 \le 3 + (n-1)/2 = (n+5)/2$. $\square$

Finally, let $c = c_1 + c_2$, to write

$$\begin{aligned} c_1^2 + c_2^2 + \cdots + c_k^2 = c_1^2 + (c_2^2 + \cdots + c_k^2) \le c_1^2 + c_2(c_2 + \cdots + c_k) = c_1^2 + (c - c_1)(n + 1 - c_1) \\ = 2c_1^2 - (c + n + 1)c_1 + c(n+1) = f(c_1). \end{aligned}$$

Since f is convex, and $c/2 \le c_1 \le c - 1$, it follows that

$$\begin{aligned} f(c_1) &\le \max\big(f(c/2), f(c-1)\big) = \max\big(c(n+1)/2, c^2 - 3c + n + 3\big) \\ &\le \max\big((n+1)(n+5)/4, (n+1)(n+7)/4\big) \quad \text{(since } c \le (n+5)/2 \text{ by the lemma)} \\ &= (n+1)(n+7)/4. \end{aligned}$$

Consequently, $\sum_{i=1}^k \binom{c_i}{2} = \frac{1}{2}\left(\sum_{i=1}^k c_i^2 - \sum_{i=1}^k c_i\right) = \frac{1}{2}\sum_{i=1}^k c_i^2 - \frac{1}{2}(n+1) \le \frac{1}{8}(n+1)(n+7) - \frac{1}{2}(n+1) = \frac{1}{8}(n+1)(n+3)$. The bound is achieved if $c_1 = (n+3)/2$, $k = (n+1)/2$ and $c_2 = \cdots = c_k = 1$, which corresponds to the example in the previous solution.

GEOMETRY

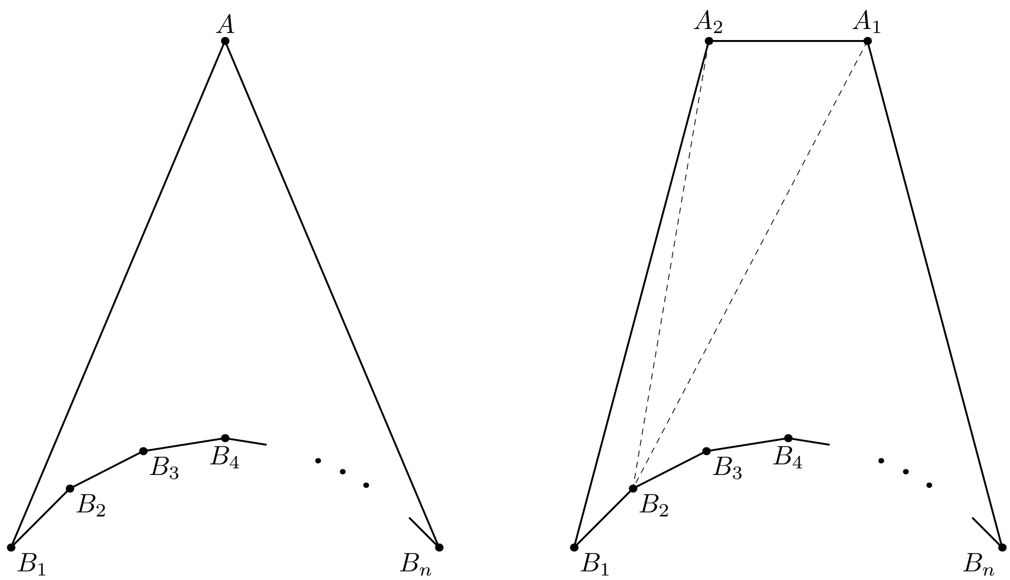
G1. Prove that for every positive integer n there exists a polygon which admits exactly n different triangulations.

(A *polygon* is a bounded region whose boundary is a non-self-intersecting polyline; we assume that no three vertices of a polygon are collinear. A *triangulation* is a dissection of the polygon into triangles by diagonals having no common interior points with each other, neither with the sides of the polygon.)

Iran, Morteza Saghafian

Solution. The left figure below shows an example of a polygon admitting a unique triangulation: the only its diagonals lying inside the polygon come from A, so all of them must be drawn. (Notice that the "exterior" polygon $B_1B_2 \dots B_n$ is convex.)

Now we prove that the right figure below shows a polygon $A_1A_2B_1B_2 \dots B_n$ with exactly n triangulations. Indeed, any triangulation must contain a triangle with side A_1A_2, and there are n possible such, namely $A_1A_2B_i$ for $i = 1, 2, \dots, n$. After such triangle has been chosen, the rest part of the polygon splits into two (or one) polygons admitting a unique triangulation. Hence the result.



G2. Let BM be a median in an acute-angled triangle ABC. A point K is chosen on the line through C tangent to the circumcircle of $\triangle BMC$ so that $\angle KBC = 90^\circ$. The segments AK and BM meet at J. Prove that the circumcenter of $\triangle BJK$ lies on the line AC.

Russia, Aleksandr Kuznetsov

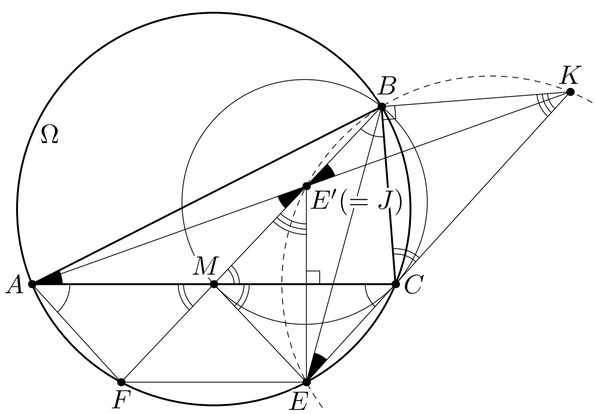
Solution 1. Let Ω be the circumcircle of the triangle ABC, and let BM and CK meet Ω again at F and E, respectively. Since CE is a tangent to the circle (BMC), we have $\angle ACE = \angle MBC = \angle FBC$, so $\overset{\frown}{AE} = \overset{\frown}{FC}$ and $EF \parallel AC$. By the symmetry in the perpendicular bisector of AC, we get $\angle EMC = \angle FMA = \angle BMC$ and $ME = MF$.

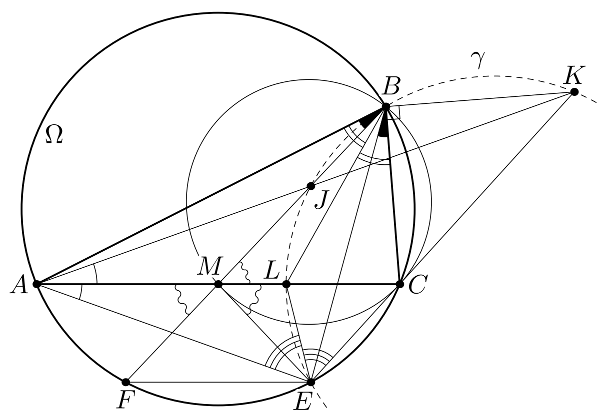
Let E' be the reflection of E in AC; by the above, E' lies on BM. Since $EE' \perp AC$, we have $\angle EE'M = \pi/2 - \angle BMC = \pi/2 - \angle BCK = \angle BKC$, so the quadrilateral $EE'BK$ is cyclic. Now, since $ME' = ME = MF$, we have $ME' \cdot MB = MF \cdot MB = MA \cdot MC = MA^2$, hence the triangles MAB and $ME'A$ are similar, and

$$\angle ME'A = \angle MAB = \angle CAB = \angle CEB = \angle KEB = \angle KE'B.$$

Therefore, E' lies on AK. So $E' = J$, and the circumcenter of the triangle BKJ lies on the perpendicular bisector of EE', i. e., on AC.

Remark. The point E' is well-known as the projection of the orthocenter of the triangle ABC onto the median BM.





Solution 2. We present a direct proof of the fact that the points E and J are symmetric to each other with respect to AC. As in the solution above, we introduce the points E and F, and prove that $EF \parallel AC$ and $\angle EMC = \angle FMA = \angle BMC$. This yields that $\angle ABM = \angle EBC$; hence BE is a symmedian in the triangle ABC, and the quadrilateral $ABCE$ is harmonic, i. e., $AB/BC = AE/EC$.

This means that the interior angle bisectors of $\angle ABC$ and $\angle AEC$ meet at a point L on AC, and that the circle $\gamma = (BLE)$ is an Apollonius circle for the segment AC. On the other hand, $\angle BLE = 2\pi - \angle BCE - (\angle CBL + \angle CEL) = 2\pi - \angle BCE - \pi/2 = \pi/2 + \angle BCK = \pi/2 - \angle BKE$, hence K also lies on γ. Therefore, $AK/KC = AE/EC$, or $AK/AE = KC/EC$. This yields that AC is an interior angle bisector in the triangle KAE. Hence, the points $J = MB \cap AK$ and $E = ME \cap AE$ are symmetric to each other in AC.

Since γ is symmetric with respect to AC, the point J lies on γ, which yields what we were to show

Solution 3. We present a yet different direct proof of the fact that the points E and J are symmetric to each other with respect to AC. Again, we introduce the points E and F, and prove that $EF \parallel AC$ and $\angle EMC = \angle FMA = \angle BMC$. This yields that the point $X = AF \cap CE$ lies on the perpendicular bisector ℓ of AC.

Let C' be the point opposite to C in the circle (BCM). Then $\angle C'BC = \angle C'MC = \pi/2$, hence C' lies on BK, as well as on ℓ. Since CK is tangent to (BCM), we have $\angle C'CK = \pi/2$. Let S be the projection of K onto ℓ; due to the right angles at C and S, the quadrilateral $KCC'S$ is cyclic. This yields that

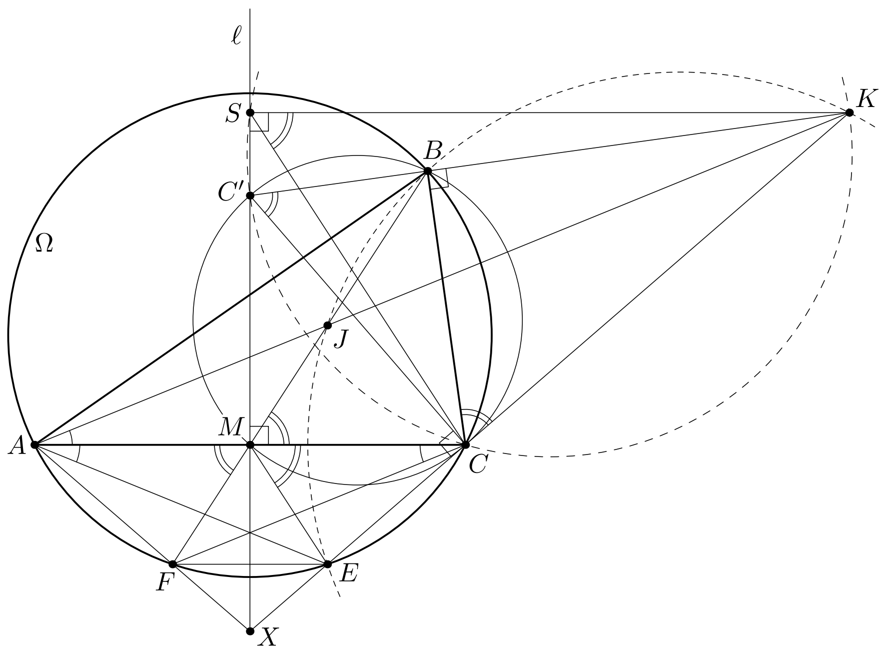
$$\angle KSC = \angle KC'C = \angle BC'C = \angle BMC = \angle CME,$$

so $SC \parallel ME$. Hence, the corresponding sides of the triangles KSC and CME are parallel, and these triangles are homothetic at X.

This yields

$$\frac{XF}{XC} = \frac{XE}{XC} = \frac{XC}{XK} = \frac{XA}{XK},$$

so $AK \parallel FC$ and hence $\angle KAC = \angle FCA = \angle CAE$. Therefore, the points $J = MB \cap AK$ and $E = ME \cap AE$ are symmetric to each other in AC.



Now one may show, as in Solution 1, that the quadrilateral $EJBK$ is cyclic, and that its circumcenter lies on AC, as desired.

G3. Let ABC be an acute-angled triangle. The line through C perpendicular to AC meets the external angle bisector of $\angle ABC$ at D. Let H be the foot of the perpendicular from D onto BC. The point K is chosen on AB so that $KH \parallel AC$. Let M be the midpoint of AK. Prove that $MC = MB + BH$.

Georgia, Giorgi Arabidze

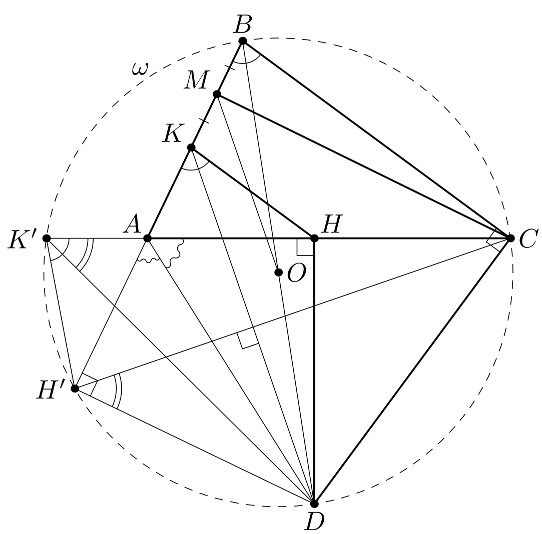
Solution. Reflect K and H across the external bisector AD, obtaining K' and H', respectively. Then we need to prove that $MC = MA + AH' = MH'$. The reflection also implies that

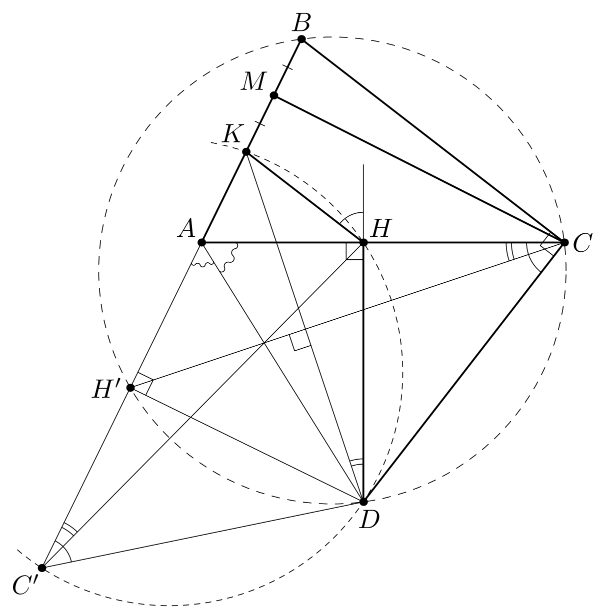
$$\angle DH'B = \angle DH'A = 90^\circ = 180^\circ - \angle BCD, \quad \text{and} \quad \angle CK'H' = \angle AK'H' = \angle AKH = \angle H'BC,$$

so the points B, C, D, H', and K' lie on a circle ω with diameter BD.

Since $HD \perp K'H$, a simple angle chasing shows that $\angle CH'D = \angle CK'D = 90^\circ - \angle K'DH$. Because of the symmetry, $\angle KDH' = \angle K'DH$, so $\angle KDH' + \angle CH'D = 90^\circ$, and we conclude that $CH' \perp DK$.

Let O be the center of ω, which is also the midpoint of diameter BD. Then OM is a midline of the triangle BDK, and $OM \parallel KD$. Hence the chord $CH'$of ω is perpendicular to OM, which proves that OM is the perpendicular bisector of CH'. Therefore $MC = MH'$ and we are done.





Remarks. (1) There are several variations of the solution above. E. g., one may reflect C and H in AD obtaining C' and H'. Since $\angle AC'D = \angle ACD = 90^\circ - \angle ACB = 180^\circ - \angle KHD$, the quartilateral $C'DHK$ is cyclic, so $\angle KDH = \angle KC'H = \angle ACH'$. Since $DH \perp AC$, this yields $KD \perp H'C$. Now one may proceed as above, with the aid of the circle $(H'DCB)$ with diameter BD.

(2) The problem can be solved with the aid of trigonometry, without constructing extra points. In fact, if $BC = a$, $CA = b$, $AB = c$, $BM = MK = x$, $AK = y$, and $AH = z$, we need to prove that $CM = x + y + z$, so by the law of cosines in $\triangle BCM$ it suffices to show that

$$(x+y+z)^2 = BC^2 + BM^2 - 2 \cdot BC \cdot BM \cos B \iff (y+z)(c+z) = a(y\cos B + b\cos C),$$

which can be verified by computing

$$y = \frac{c \sin\frac{A}{2}\cos C}{\sin\left(\frac{A}{2} + C\right)} \quad \text{and} \quad z = \frac{b \sin\frac{A}{2}\cos C}{\sin\left(\frac{A}{2} + C\right)}.$$

G4. Let ABC be an acute-angled triangle with $AB \neq AC$, and let I and O be its incenter and circumcenter, respectively. Let the incircle touch BC, CA and AB at D, E and F, respectively. Assume that the line through I parallel to EF, the line through D parallel to AO, and the altitude from A are concurrent. Prove that the concurrency point is the orthocenter of the triangle ABC.

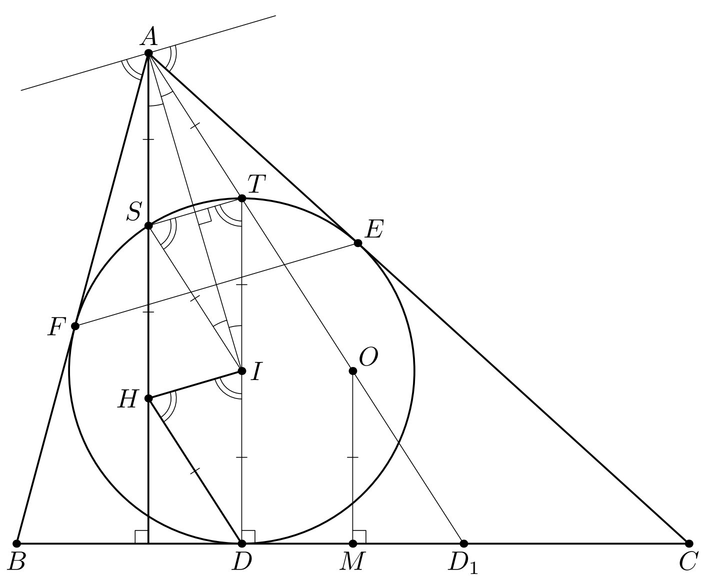
Croatia, Petar Nizić-Nikolac

Solution. Let H be the given concurrency point, and let r be the inradius. Since EF is parallel to the external bisector of $\angle BAC$, it forms equal angles with the altitude AH and the circumradius AO. The sides of the triangle IDH are parallel to the three lines mentioned above, hence its angles at H and I are equal, so $DH = DI = r$.

Choose a point S on the altitude AH so that $DISH$ is a parallelogram — hence a rhombus. Let T be the point of the incircle opposite to D. Then the isosceles triangles SIT and HDI are translates of each other. Notice now that $AI \perp EF \parallel ST$ to conclude that AI is the internal angle bisector of $\angle SIT$. This yields $\angle IAS = \angle AIT = \angle AIS$, so $AS = SI = r$ as well. Hence $SATI$ is another rhombus (congruent to $DISH$).

Therefore, AI is also the internal angle bisector of $\angle SAT$, which means that T lies on AO. On the other hand, it is well known that AT passes through the tangency point D_1 of the A-excircle with BC. Hence the four points A, T, O, and D_1 are all collinear.

Let now M be the midpoint of BC (which is also the midpoint of DD_1). Both OM and DT are perpendicular to BC, hence they are parallel, and so OM is a midline in the triangle DTD_1; therefore, $OM = DT/2 = r = AH/2$. Thus, H is the point on the altitude such that $AH = 2OM$; these conditions determine the orthocenter.



Remark. It can be seen that the assumption in the problem statement is equivalent to the fact that $OI \parallel BC$.

G5. *Version 1.* Let Ω be the circumcircle of an acute-angled triangle ABC. Let D be the midpoint of the minor arc AB of Ω. A circle ω centered at D is tangent to AB at E. The tangents to ω through C meet the segment AB at K and L, where K lies on the segment AL. A circle Ω_1 is tangent to the segments AL, CL, and also to Ω at point M. Similarly, a circle Ω_2 is tangent to the segments BK, CK, and also to Ω at point N. The lines LM and KN meet at P. Prove that $\angle KCE = \angle LCP$.

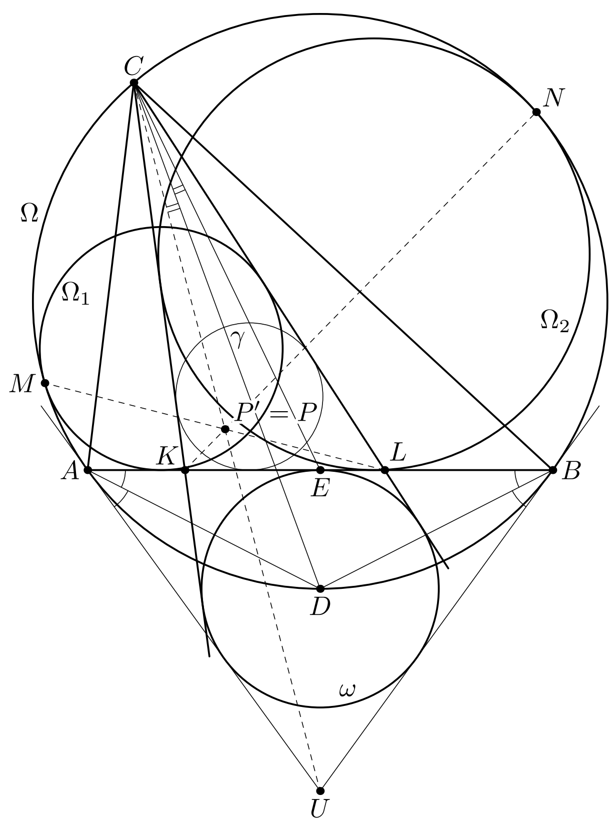
Version 2, generalized. Let Ω be the circumcircle of an acute-angled triangle ABC. A point D is chosen on the internal bisector of $\angle ACB$ so that the points D and C are separated by AB. A circle ω centered at D is tangent to the segment AB at E. The tangents to ω through C meet the segment AB at K and L, where K lies on the segment AL. A circle Ω_1 is tangent to the segments AL, CL, and also to Ω at point M. Similarly, a circle Ω_2 is tangent to the segments BK, CK, and also to Ω at point N. The lines LM and KN meet at P. Prove that $\angle KCE = \angle LCP$.

POLAND

Solution to Version 1. Let U be the meeting point of the tangents to Ω at A and B. Since $\angle UAD = \angle DBA = \angle DAB = \angle UBD$, the point D is the incenter of the triangle UAB, and hence ω is its incircle.

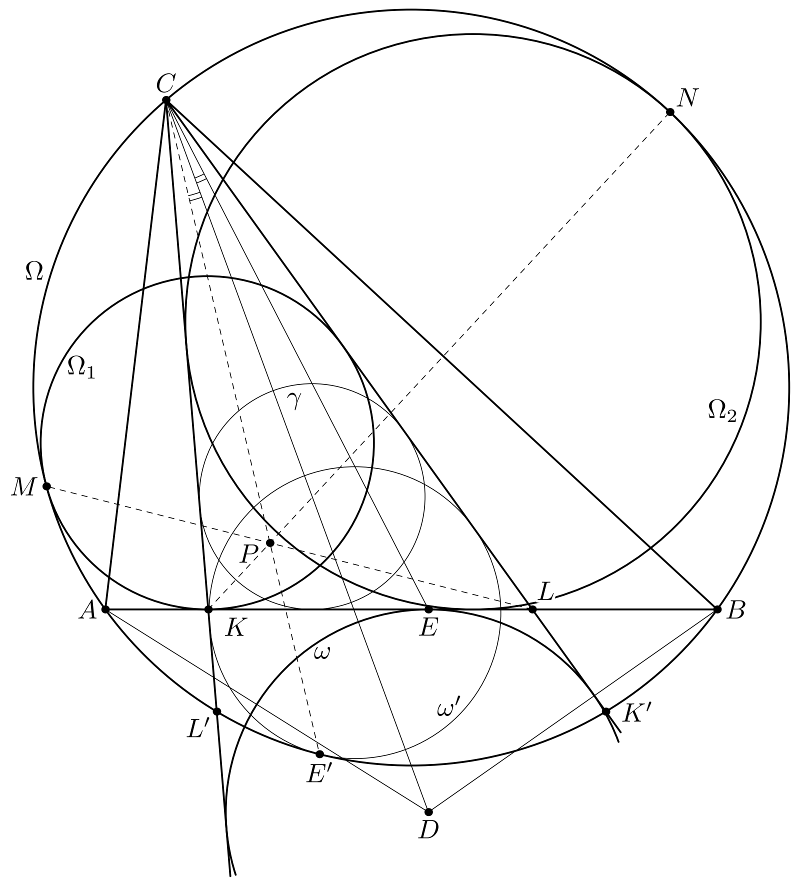
The symmetry in the internal bisector CD of $\angle ACB$ swaps the sidelines $CA \leftrightarrow CB$, the tangents $CK \leftrightarrow CL$, and also the median CE with the symmedian CU. Hence, in order to solve the problem, it suffices to show that the symmetry swaps also CE and CP — i.e., that the points C, P, and U are collinear.

Let γ be the incircle of the triangle CKL, and let P' be the positive homothetic center of the circles γ and Ω. Application of Monge's theorem to the triples $(\Omega, \Omega_1, \gamma)$ and $(\Omega, \Omega_2, \gamma)$ shows that the point P' lies on the lines LM and KN, i.e., that $P' = P$. Finally, applying Monge's theorem to Ω, γ, and ω yields that C, P', and U are collinear, as desired.



Solution to Version 2. Let φ be a composition of the inversion at C with radius $\sqrt{CA \cdot CB}$ and the symmetry in the angle bisector CD. Then φ is an involution swapping $\Omega \leftrightarrow AB$, $A \leftrightarrow B$, and $CK \leftrightarrow CL$. Let K', L'¡ and E' be the φ-images of K, L, and E, respectively; all of them lie on Ω. Moreover, the points K' and L' lie on CL and CK, respectively. Finally, the φ-image of ω is a circle ω' tangent to $CL = \varphi(CK)$, $CK = \varphi(CL)$, and to the arc $K'L'$ of Ω (which is the φ-image of the segment KL) — i.e., ω' is the C-mixtilinear circle of the triangle $CK'L'$.

We have $\angle KCE = \angle K'CE' = \angle LCE'$, so we need to prove that the points C, P, and E' are collinear — in other words, that the lines KM, LN, and CE' are concurrent. Let γ be the incircle of the triangle CKL, and let Q be the center of the homothety with positive coefficient mapping γ to Ω. Applying Monge's theorem to the triples $(\Omega, \Omega_1, \gamma)$, $(\Omega, \Omega_2, \gamma)$, and $(\Omega, \omega', \gamma)$, we obtain that the point Q lies on the lines KN, LM, and CE', which yields the required result.



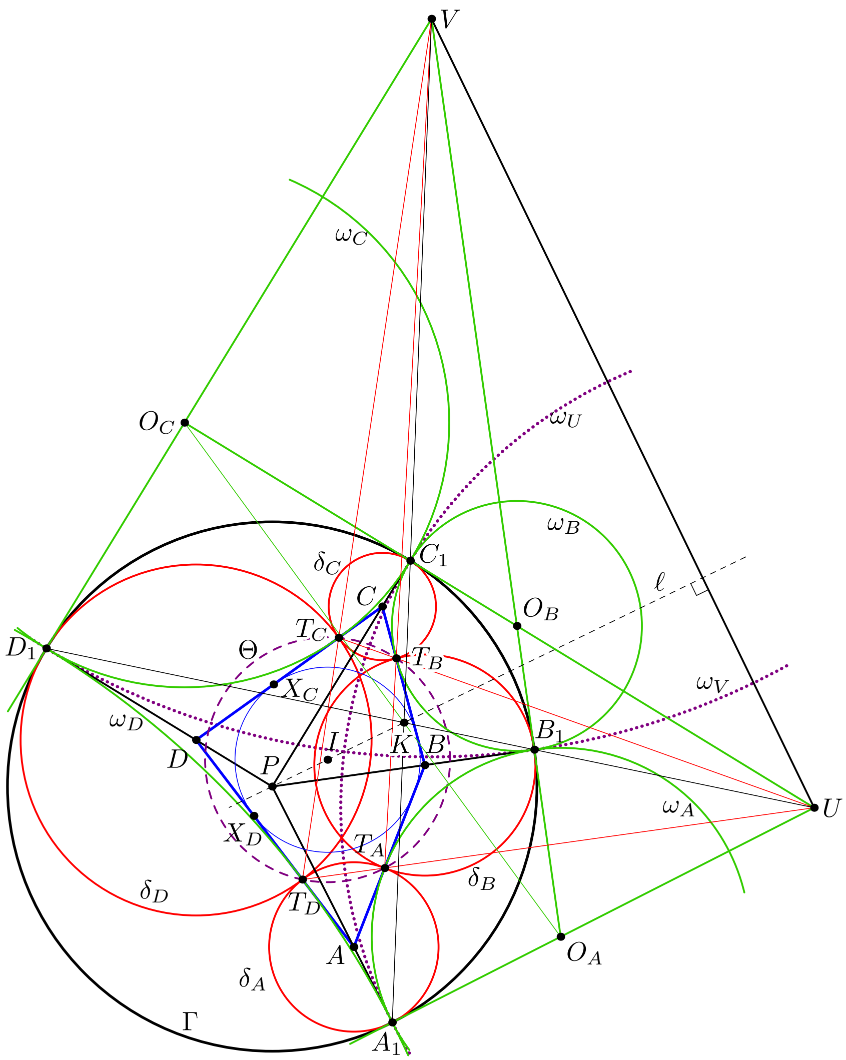
G6. A quadrilateral $ABCD$ is circumscribed about a circle with center I. A point $P \neq I$ is chosen inside $ABCD$ so that the triangles PAB, PBC, PCD, and PDA have equal perimeters. A circle Γ centered at P meets the rays PA, PB, PC, and PD at A_1, B_1, C_1, and D_1, respectively. Prove that the lines PI, A_1C_1, and B_1D_1 are concurrent.

USA, Ankan Bhattacharya

Solution. *Step 1.* Let s be the common *semiperimeter* of the triangles PAB, PBC, PCD, and PDA. By homothety at P, we assume that Γ has radius s. Let ω_A, ω_B, ω_C, and ω_D be the P-excircles of the four triangles. Then A_1, B_1, C_1, and D_1 are their tangency points with the rays PA, PB, PC, and PD, so the excircles are cyclically tangent at those points.

Let ω_A, ω_B, ω_C, and ω_D touch AB, BC, CD, and DA at T_A, T_B, T_C, and T_D, respectively. The points A_1, T_A, and T_D lie on a circle δ_A centered at A_1; introduce the circles δ_B, δ_C, and δ_D similarly. These circles form another necklace with T_A, T_B, T_C, and T_D as tangency points, and they are all internally tangent to Γ.

Let the incircle of $ABCD$ touch the sides AB, BC, CD, and DA at X_A, X_B, X_C, and X_D, respectively. Since $DT_C = DT_D$ and $DX_C = DX_D$, we have $X_CT_C = X_DT_D$, so the right triangles IX_CT_C and IX_DT_D are congruent. Therefore, $IT_C = IT_D$. Similarly, we obtain that the four points T_A, T_B, T_C, and T_D all lie on some circle Θ centered at I.



Step 2. Having constructed all those circles, we proceed by drawing lines. Let U be the positive homothetic center of δ_B and δ_D (if U did not exist, then the picture would have an axis of symmetry, in which case the problem is trivial). Each of the circles δ_A, δ_C, and Γ is tangent to both δ_B and δ_D; by Monge's theorem, we obtain that U lies on T_AT_D, T_BT_C, and B_1D_1. Next, the inversion ι with center U and radius $\sqrt{UT_A \cdot UT_D} = \sqrt{UT_B \cdot UT_C}$ preserves Θ and swaps $\delta_B \leftrightarrow \delta_D$ (and hence $B_1 \leftrightarrow D_1$); therefore, ι also preserves δ_A, δ_C, and Γ, by the tangency condition. Thus, the inversion preserves the points A_1 and C_1 of tangency of Γ with δ_A and δ_C; so UA_1 and UC_1 are common tangents to Γ, δ_A and Γ, δ_C, respectively.

Similarly, the lines T_AT_B, T_CT_D, A_1C_1, and the tangents to Γ at B_1 and D_1 are all concurrent at the positive homothetic center V of δ_A and δ_C.

Step 3. Finally, let ω_U be the circle centered at U of radius $UA_1 = UC_1$ (i.e., the circle of inversion from the previous step); define ω_V similarly. We claim that the points P, I, and K lie on the radical axis ℓ of ω_U and ω_V, which finishes the solution.

The powers of P with respect to the two circles are $PA_1^2 = PB_1^2$, so $P \in \ell$. The powers of K are $-KA_1 \cdot KC_1 = -KB_1 \cdot KD_1$ (due to the circle Γ), so $K \in \ell$. Finally, the circle Θ is preserved by the inversions in ω_U and ω_C, so Θ is orthogonal to both circles; hence the powers of I with respect to both equal IT_A^2, and $I \in \ell$.

Remark. Many parts of the solution above admit different reasoning; here we list some facts which may be helpful in such reasoning.

The arguments un Step 2 show that UV is the polar line of K with respect to Γ (n fact, the triangle UVK is autopolar). This shows that $PK \perp UV$ and reduces the problem to showing $PI \perp UV$.

A simple computation in sines shows that the point K is the negative homothetic center of ω_A and ω_C, as well as of ω_B and ω_D. On the other hand, U is the positive hompthetic center of ω_A and ω_D, as well as of ω_B and ω_C; a similar statement holds for V. Thus, e.g., the inversion ι swaps $\omega_B \leftrightarrow \omega_C$.

Step 2 yields also that the quadrilateral $A_1B_1C_1D_1$ is harmonic. In this case, there exist two circles tangent to ω_A, ω_B, ω_C, and ω_D (one of them may degenerate into a line). The centers of those two circles also lie on PK.

NUMBER THEORY

N1. Let p and q be relatively prime positive odd integers such that $1 < p < q$. Let A be a set of pairs of integers (a, b), where $0 \le a \le p-1$, $0 \le b \le q-1$, containing exactly one pair from each of the sets

$$\{(a,b),\,(a+1,b+1)\}, \quad \{(a,q-1),\,(a+1,0)\}, \quad \{(p-1,b),\,(0,b+1)\},$$

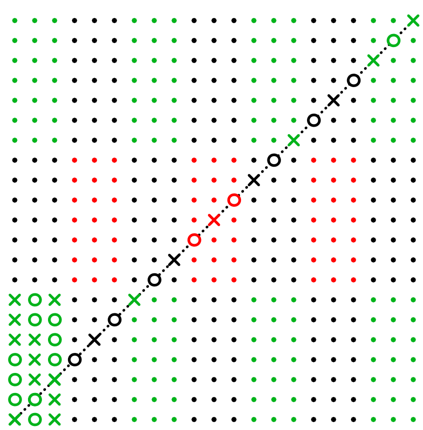
whenever $0 \le a \le p-2$ and $0 \le b \le q-2$. Show that A contains at least $(p-1)(q+1)/8$ pairs whose entries are both even.

United Kingdom, Agnijo Banerjee and Joe Benton

Solution 1. Since p and q are relatively prime, the Chinese Remainder Theorem shows that the function $f\colon \{0,1,\dots,pq-1\} \to \{0,1,\dots p-1\} \times \{0,1,\dots,q-1\}$, $f(n) = (n \bmod p,\ n \bmod q)$, is bijective. In terms of f, the condition in the statement reads: For each non-negative integer $n \le pq-2$, exactly one of the images $f(n)$ and $f(n+1)$ is a member of A. Consequently, A is one of the two sets $B_0 = \{f(n)\colon 0 \le n \le pq-1,\ 2 \mid n\}$ and $B_1 = \{f(n)\colon 0 \le n \le pq-1,\ 2 \nmid n\}$.

Visualise the situation as follows: Mark all points in B_0 and B_1 by crosses and noughts, respectively. Extend the marking (p,q)-periodically to the square $S = \{0,1,\dots pq-1\}^2$. Split S into $p \times q$ rectangles $C_{ij} = \{(a,b)\colon (a,b) \in S, \lfloor a/p \rfloor = i, \lfloor b/q \rfloor = j\}$, $i = 0,1,\dots,q-1$, $j = 0,1,\dots,p-1$; thus, C_{00} is the initial rectangle. We are interested in the points along the diagonal D from $(0,0)$ to $(pq-1,pq-1)$; they are marked with crosses and noughts alternately. Since p and q are both odd, it is easily seen that: For an even n, the cross at $f(n)$ has both coordinates even if and only if (n,n) lies in a C_{ij}, where i and j are both even; and for an odd n, the nought at $f(n)$ has both coordinates even if and only if (n,n) lies in a C_{ij}, where i and j are both odd.

Colour green (respectively, red) all points in C_{ij} if i and j are both even (respectively, odd); no other point is coloured whatsoever. We are to bound from below the number of green crosses and the number of red noughts along D.



To this end, let $g_\times$, $g_\circ$, $r_\times$, and $r_\circ$ denote the numbers of green crosses, green noughts, red crosses, and red noughts, respectively, along the diagonal D. Notice that the points in C_{00} whose coordinates are both even correspond to green crosses and red noughts along D, to write $g_\times + r_\circ = \frac{1}{2}(p+1) \cdot \frac{1}{2}(q+1)$.

Next, evaluate $g_\circ - r_\circ$. To this end, let D_1 be the set of all points along D whose coordinates are both odd. Then

$$g_\circ - r_\circ = \left| D_1 \cap \bigcup_{2|i,\,2|j} C_{ij} \right| - \left| D_1 \cap \bigcup_{2\nmid i,\,2\nmid j} C_{ij} \right| = \left| D_1 \cap \bigcup_{2|i} C_{ij} \right| - \left| D_1 \cap \bigcup_{2\nmid j} C_{ij} \right|$$
$$= \frac{p-1}{2} \cdot \frac{q+1}{2} - \frac{q+1}{2} \cdot \frac{p-1}{2} = 0.$$

Finally, notice that each non-empty intersection of D with a green rectangle C_{ij} begins and ends with a cross, so it contains by 1 more crosses than noughts. Hence $g_\times - g_\circ$ equals the number of green rectangles met by D. For any fixed even i, D meets at most one green C_{ij}, while for any fixed j, D meets at least one green C_{ij}, so $\frac{1}{2}(q+1) \geq g_\times - g_\circ \geq \frac{1}{2}(p+1)$.

Write $g_\times = \frac{1}{2}(g_\times + r_\circ) + \frac{1}{2}(g_\circ - r_\circ) + \frac{1}{2}(g_\times - g_\circ)$ and $r_\circ = \frac{1}{2}(g_\times + r_\circ) - \frac{1}{2}(g_\circ - r_\circ) - \frac{1}{2}(g_\times - g_\circ)$, to conclude, by the preceding, that $g_\times \geq \frac{1}{2} \cdot \frac{1}{2}(p+1) \cdot \frac{1}{2}(q+1) + \frac{1}{2} \cdot \frac{1}{2}(p+1) = \frac{1}{8}(p+1)(q+3) > \frac{1}{8}(p-1)(q+1)$ and $r_\circ \geq \frac{1}{2} \cdot \frac{1}{2}(p+1) \cdot \frac{1}{2}(q+1) - \frac{1}{2} \cdot \frac{1}{2}(q+1) = \frac{1}{8}(p-1)(q+1)$. The required estimate follows.

Remark. It can similarly be shown that $g_\circ + r_\times = \frac{1}{2}(p-1) \cdot \frac{1}{2}(q-1)$, $g_\times - r_\times = \frac{1}{2}(p+1) \cdot \frac{1}{2}(q+1) - \frac{1}{2}(p-1) \cdot \frac{1}{2}(q-1) = \frac{1}{2}(p+q)$, and $\frac{1}{2}(q-1) \geq r_\circ - r_\times \geq \frac{1}{2}(p-1)$. Some of these relations may equally well be used in the final estimates.

Solution 2. First label each square (a, b) with an integer n between 0 and $pq-1$ such that $n \equiv a \pmod p$ and $n \equiv b \pmod q$. Then the conditions on A are equivalent to saying that A contains either all pairs (a, b) with an even value of n or all those with an odd value of n. Thus we look for an expression for M, the number of points with two even coordinates and an even value of n.

Note that the value of n mod p is given by $n - p\lfloor n/p \rfloor$, which is congruent to $n + \lfloor n/p \rfloor$ modulo 2. Thus $\frac{1}{2}\left(1 + (-1)^{n + \lfloor n/p \rfloor}\right)$ is 1 if the remainder of n when divided by p is even and 0 otherwise. Hence:

$$M = \frac{1}{8} \sum_{n=0}^{pq-1} \left(1 + (-1)^{n+\lfloor n/p \rfloor}\right) \left(1 + (-1)^{n + \lfloor n/q \rfloor}\right) (1 + (-1)^n).$$

Expanding the product and evaluating the sum from 0 to $pq-1$ of each term individually, most terms give values that we can calculate explicitly. Certainly, $\sum_{n=0}^{pq-1} (-1)^n = 1$, since p and q are both odd. Then, rewriting the sum $\sum_{n=0}^{pq-1}$ as $\sum_{n=0}^{p-1} + \sum_{p}^{2p-1} + \cdots + \sum_{p(q-1)}^{pq-1}$, we find,

$$\sum_{n=0}^{pq-1} (-1)^{\lfloor n/p \rfloor} = p - p + p - \cdots + p = p, \quad \sum_{n=0}^{pq-1} (-1)^{n + \lfloor n/p \rfloor} = 1 + 1 + \ldots + 1 = q.$$

Then, recalling the labelling (a, b) given by the Chinese Remainder Theorem, note that $\lfloor n/p \rfloor + \lfloor n/q \rfloor \equiv 2n + a + b \equiv a + b \pmod 2$. Thus

$$\sum_{n=0}^{pq-1} (-1)^{\lfloor n/q \rfloor + \lfloor n/p \rfloor} = \sum_{a=0}^{p-1} \sum_{b=0}^{q-1} (-1)^{a+b} = 1,$$

for example by considering a chessboard colouring of $[0, p-1] \times [0, q-1]$. This allows us to reduce the above expression of M to:

$$M = \frac{1}{8} \left(pq + 2p + 2q + 2 + \sum_{n=0}^{pq-1} (-1)^{n + \lfloor n/p \rfloor + \lfloor n/q \rfloor}\right).$$

Now for $0 \leq j \leq p + q - 1$ let a_j denote the minimum value of n such that $\lfloor n/p \rfloor + \lfloor n/q \rfloor \geq j$. Then

$$M = \frac{1}{8} \left(pq + 2p + 2q + 2 + \sum_{j=0}^{p+q-2} \sum_{n=a_j}^{a_{j+1}-1} (-1)^{n+j}\right).$$

Now consider the partition of $[0, pq)$ into the $p + q - 1$ intervals $[a_j, a_{j+1})$. Between any two consecutive multiples of p there is at most one multiple of q, so in this partition, for each

$0 \le m < q$, the interval $[mp, (m+1)p)$ either appears as a part or is split into exactly two parts. The interval $[mp, (m+1)p)$ has odd length, so if it is split into two parts, one of these parts has even length and the other has odd length. Also, there are $p-1$ multiples of q between 0 and pq, so $p-1$ of these q intervals are split. Thus exactly $p-1$ of the intervals $[a_j, a_{j+1})$ have even length.

Note further that among any a_j, a_{j+1}, a_{j+2} some pair must consist of either consecutive multiples of p or consecutive multiples of q. Therefore, it is not possible that two consecutive intervals $[a_j, a_{j+1})$, $[a_{j+1}, a_{j+2})$ both have even length. Finally, note that the first and last intervals are $[0, p)$ and $[pq-p, pq)$ which are both odd. Putting this all together, we see that there are p sections of contiguous odd intervals, each separated by one even length interval.

For two consecutive odd intervals $[a_j, a_{j+1})$ and $[a_{j+1}, a_{j+2})$, we have $(-1)^{a_j+j} = (-1)^{a_{j+1}+j+1}$. While if two odd intervals $[a_j, a_{j+1})$ and $[a_{j+2}, a_{j+3})$ are separated by exactly one even interval, then $(-1)^{a_j+j} = -(-1)^{a_{j+2}+j+2}$. Thus *within* any contiguous section of odd intervals, the sums $\sum_{n=a_j}^{a_{j+1}-1}(-1)^{n+j}$ are all equal, and, furthermore, the value of these sums alternates between $\{+1, -1\}$ *between* adjacent contiguous sections of odd intervals.

For an even length interval $[a_j, a_{j+1})$, we have $\sum_{n=a_j}^{a_{j+1}-1}(-1)^{n+j} = 0$. Furthermore, we have at least $(p+1)/2$ odd intervals such that $\sum_{n=a_j}^{a_{j+1}-1}(-1)^{n+j} = 1$ (noting that this is the case for $[0,p)$) and at least $(p-1)/2$ odd intervals such that $\sum_{n=a_j}^{a_{j+1}-1}(-1)^{n+j} = -1$. Hence we get the following bounds:

$$p - q + 1 \le \sum_{j=0}^{p+q-2} \sum_{n=a_j}^{a_{j+1}-1} (-1)^{n+j} \le q - p + 1 \quad \text{and} \quad (p+1)(q+3)/8 \le M \le (p+3)(q+1)/8.$$

Finally, let N be the number of pairs in A whose entries are both even. Since the total number of pairs with even entries is $(p+1)(q+1)/4$, it follows that

$$\begin{aligned} N &\ge \min\big((p+1)(q+3)/8,\ (p+1)(q+1)/4 - (p+3)(q+1)/8\big) \\ &= \min\big((p+1)(q+3)/8,\ (p-1)(q+1/8)\big) = (p-1)(q+1)/8. \end{aligned}$$

N2. Consider a pair (c, d) of integers greater than 1. Ana and Banana are playing the following game: Ana announces a degree d monic polynomial P with integral coefficients and a prime $p > c(2c+1)$. Banana then writes at most $p(2c-1)/(2c+1)$ integers on a board and is then allowed to perform a finite sequence of operations of the following kind: Choose an integer x written on the board and write $P(x)$ on the board. She wins if she reaches a stage where the numbers form a complete residue system modulo p; otherwise Ana wins. Determine all pairs (c, d) for which Banana can win regardless of how Ana plays.

CROATIA, ADRIAN BEKER

Solution. We show that Banana has a winning strategy if and only if $d \le c$.

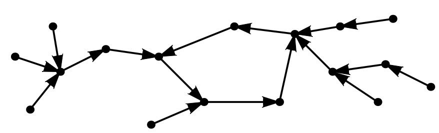
Assume first that $d \ge c+1$. Let Ana choose a large prime p (we need $p > 2c^2 + c$) congruent to 1 modulo d (such a prime exists, by a particular case of Dirichlet's theorem; this particular case is easier to prove by using the cyclotomic polynomial Φ_d). Then she announces the polynomial $P = X^d$. Since $d \mid p-1$, exactly $1 + (p-1)/d$ residues modulo p are d-th powers; all other $(d-1)(p-1)/d$ residue classes contain no values of P. Hence, in order to complete her task, Banana needs to write on the board representatives of all those classes. But this is more than she is allowed to, since

$$\frac{d-1}{d}(p-1) > \frac{2c-1}{2c+1}p \quad \Longleftarrow \quad \frac{c}{c+1}(p-1) > \frac{2c-1}{2c+1}p$$
$$\Longleftarrow \quad \frac{p}{(c+1)(2c+1)} > \frac{c}{c+1} \quad \Longleftarrow \quad p > c(2c+1).$$

We now show that Banana has a winning strategy whenever $d \le c$. To this end, usage is made of the lemma below.

Lemma. Fix an integer $d \ge 2$. Let $G = (V, E)$ be a directed graph, each vertex of which has exactly one outgoing edge and at most d incoming edges. Assume further that there are at most d loops in this graph. Then there exists a subset V' of V of cardinality $|V'| \le 1 + \frac{d-1}{d}|V|$ such that every vertex in $V \setminus V'$ is the terminus of a directed path emanating from V'.

Proof. Consider any (weak) connected component $G_1 = (V_1, E_1)$ in G — i.e., a component of the corresponding *undirected* graph. Since from each vertex emanates exactly one edge, the component contains a directed cycle (possibly a loop); and since the numbers of vertices and edges in G_1 are equal, even an undirected cycle is unique. Hence, the component is a cycle with some trees rooting out of its vertices. With reference again to uniqueness of outgoing edges, the edges of these trees are all directed towards the cycle.



Now, let V' choose exactly one vertex from each component that is just a cycle; for any other component, let V' choose all its in-degree 0 vertices, i.e., the leaves of all trees rooting out of the vertices of the core-cycle — any vertex of such a tree can be reached from some leaf, and hence so can any vertex of the core-cycle.

To bound $|V'|$ from above, let t be the number of single-vertex components in G, and notice that $t \le d$, since there are at most d loops in the graph. From each other component that is a cycle, V' chooses at most half of its vertices, so at most $\frac{d-1}{d}$-th part of them. Finally, consider a component containing some trees. Since each in-degree is at most d, at least $\frac{1}{d}$-th part of the vertices have incoming edges, hence V' chooses at most $\frac{d-1}{d}$-th part of the vertices. Consequently,

$$|V'| \le t + \frac{d-1}{d}(|V| - t) = \frac{t}{d} + \frac{d-1}{d}|V| \le 1 + \frac{d-1}{d}|V|,$$

as desired. This establishes the lemma. $\square$

Now Banana considers a graph with vertex set $\mathbb{Z}_p$. Regard P as a polynomial over $\mathbb{Z}_p$, and draw an edge $a \to P(a)$ for every a in $\mathbb{Z}_p$. Since $\deg P = d$, each b in $\mathbb{Z}_p$ has at most d preimages, so the in-degree of each vertex is at most d. Since P is monic and $d > 1$, the equation $P(x) = x$ has at most d roots in $\mathbb{Z}_p$, hence the graph has at most d loops. Thus, Banana can implement the lemma to find the set V'; she writes down all its elements on the board. The lemma shows that she may write down all other residues after that; and the implications below show that the cardinality of V' lies within the required range:

$$\begin{aligned} |V'| \leq \frac{d-1}{d}p + 1 \leq \frac{2c-1}{2c+1}p \quad &\Longleftarrow \quad \frac{c-1}{c}p + 1 \leq \frac{2c-1}{2c+1}p \\ &\Longleftarrow \quad \frac{p}{c(2c+1)} \geq 1 \quad \Longleftarrow \quad p \geq c(2c+1). \end{aligned}$$

The 11^{th} Romanian Master of Mathematics Competition

Day 2 — Solutions

Problem 4. Let P be a point in the plane of an equilateral triangle ABC such that $AP < BP < CP$. Suppose that the side length of $\triangle ABC$ can be uniquely determined from the values of AP, BP and CP alone. Prove that P lies on the circumcircle of ABC.

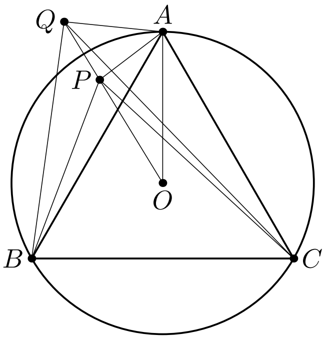
United Kingdom, Jeremy King and the UK International PSC

Solution 1. We start with the following simple lemma.

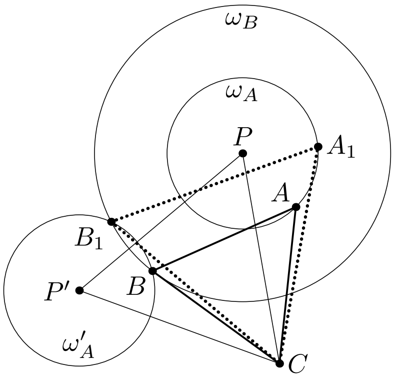
Rescaling lemma. Assume that a point $Q \neq P$ in the plane satisfies $\frac{AP}{AQ} = \frac{BP}{BQ} = \frac{CP}{CQ}$. Then the distances PA, PB, and PC do not determine the side length of ABC uniquely.

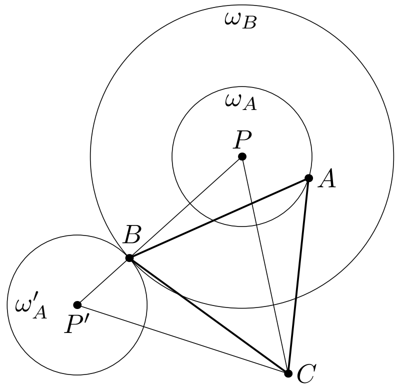
Proof. The common ratio $\mu = \frac{AP}{AQ}$ is different from 1, as otherwise P and Q would be two common points of three circles with non-collinear centers A, B, and C. Now rescale the 4-tuple (A, B, C, Q) with coefficient μ to get a 4-tuple (A', B', C', Q') such that $A'B'C'$ is an equilateral triangle (with side length different from that of ABC), but $A'Q' = AP$, $B'Q' = BP$, and $C'Q' = CQ$. $\square$

Back to the solution, suppose P is not on the circumcircle of ABC. Define O to be the circumcentre of ABC and Q to be the point on the ray OP such that $OP \cdot OQ = OA^2 = OB^2 = OC^2$ (i.e. the inverse of P with respect to the circumcircle). Then the triangles OAP and OQA are similar, hence $\frac{AQ}{AP} = \frac{OQ}{OA}$. Similarly, $\frac{BQ}{BP} = \frac{CQ}{CP} = \frac{OQ}{OA}$. Application of the Rescaling lemma finishes the solution.



Solution 2. Fix the points P and C in the plane, and then look to construct A and B such that AP and BP are the given lengths, and ABC is equilateral. Without loss of generality, we may insist that ABC is oriented anticlockwise. Now the length conditions are equivalent to A and B lying on the circles ω_A and ω_B centered at P and having the given lengths as radii; the equilateral condition is now equivalent to the statement that the anticlockwise rotation by 60° at C maps A to B.

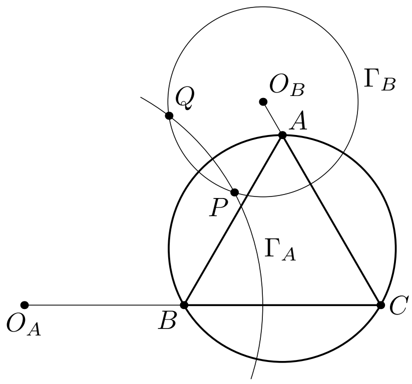


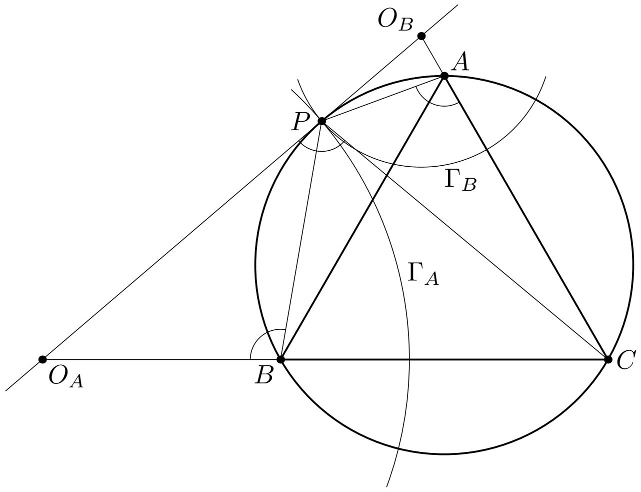


Let the rotation map ω_A to ω'_A, and P to P'; notice that the points C, P, and P' form an equilateral triangle. If the circles ω'_A and ω_B meet at two distinct points B and B_1, those points give rise to two distinct equilateral triangles ABC and A_1B_1C. As C does not belong to the perpendicular bisector PP' of BB_1, the side lengths of those two triangles are distinct.

So, under the problem conditions, the circles ω'_A and ω_B should be tangent to each other at B. This yields $PB + P'B = PP'$, or $PB + PA = PC$. Since the triangle ABC is equilateral, we get $PB \cdot AC + PA \cdot BC = PC \cdot AB$, and thus Ptolemy's theorem shows P must lie on the circumcircle of ABC.

Solution 3. Given ABC, and P such that the lengths AB, BC, CA uniquely determine the side length of ABC, let Γ_A be the locus of points Q such that $\frac{BQ}{CQ} = \frac{BP}{CP}$, and let Γ_B be the locus of points Q such that $\frac{AQ}{CQ} = \frac{AP}{CP}$. Those are Apollonius circles with foci (B, C) and (A, C), respectively. If these circles meet at two points, P and Q, then the Rescaling lemma from Solution 1 shows that the side length of $\triangle ABC$ is not uniquely determined.





Otherwise, the circles are tangent at P, thus P is collinear with the centers O_A and O_B of Γ_A and Γ_B, respectively. Since $O_AP^2 = O_AB \cdot O_AC$, the triangles O_ABP and O_APC are similar, hence $\angle(PB, BC) = \angle(PB, BO_A) = \angle(PO_A, PC)$. Similarly, $\angle(PA, AC) = \angle(PO_B, PC) = \angle(PO_A, PC)$. Therefore, $\angle(PB, BC) = \angle(PA, AC)$, which shows that the points A, B, C, and P are indeed concyclic.

Remarks. (1) Solution 3 may be finished in a different fashion. Notice that the Apollonius circles Γ_A ana Γ_B are both orthogonal to the circumcircle Ω of $\triangle ABC$. Hence, they either meet on Ω, or meet at two points inverse to each other with respect to Ω.

One may also notice that, under the problem conditions, the line O_AO_BP is tangent to the circumcircle at P.

(2) All three solutions above may be adapted in order to prove the following, more general result.

> Assume that a point P in the plane of an *arbitrary* triangle ABC satisfies the following condition: If a triangle $A'B'C'$ is similar to ABC and lies in the same plane, then the conditions $PA = PA' < PB = PB' < PC = PC'$ yield $AB = A'B'$.
>
> Then P lies on the circumcircle of the triangle ABC.

Problem 5. Let H be a convex polyhedron. Two ants are crawling along the edges of H, following two different routes. Each ant's route ends right where it begins, without ever doubling back or visiting a vertex of H twice. (That is, each ant's route is a simple cycle.) On each face F of H, write down the number of edges of F along the first ant's route and the number of edges of F along the second ant's route. Is it possible that exactly one single face of H bears a pair of distinct numbers?

Nikolai Beluhov

Solution 1. The answer is in the negative. Let E be some face of H. We show that if each face of H other than E bears a pair of equal numbers, then so does E.

Let R_1 be the first ant's route, and let R_2 be the second ant's route. Let D be the symmetric difference of the edge sets of R_1 and R_2; that is, let D be the set of all edges of H along exactly one of R_1 and R_2. Colour red all edges in D that are in R_1 but not in R_2, and colour green all edges in D that are in R_2 but not in R_1; no other edge of H is coloured. Then the boundary of every face of H, other than E, contains as many red as green edges in D.

Consider a graph G whose edge set is D, and whose vertex set consists of all endpoints of those edges. This graph is assumed to be drawn on the surface of H; so its edges partition the surface into *regions*. The degree of any vertex in G is even (in fact, it is either 2 or 4). It is well known that in this case each region of the graph can be coloured one of two colours, black or white, so that each edge of G separates regions of different colours. Recall that every region consists of several faces of H; each of those faces inherits the colour of the region.

We assume E to be black. Then the total number of red edges on the boundaries of white faces equals the total number of green edges on those boundaries, since the equality holds for every separate white face. As each coloured edge separates faces of distinct colours, a similar equality holds for the black faces. But every separate black face, other than E, also satisfies such equality; therefore, so does E.

Solution 2. (*Palmer Mebane*) Let E, R_1, and R_2 be as in the previous solution, and assume again that each face of H other than E bears a pair of equal numbers; such number on a face $F \neq E$ will be referred to as the *ant number* for F.

The closed broken line R_1 dissects the surface of H into two regions. We refer to the region that does not contain E as the *inside* of route R_1, and to the region that contains E as the *outside* of route R_1. We define the inside and the outside of route R_2 similarly.

Let S^{II} be the sum of the ant numbers for all faces inside both routes. Similarly, let S^{IO} be the sum of the ant numbers for all faces inside R_1 and outside R_2; let S^{OI} be the sum of the ant numbers for all faces outside R_1 and inside R_2; let S_1^{OO} be the sum of the first ant's numbers for all faces outside both routes; and let S_2^{OO} be the sum of the second ant's numbers for all faces outside both routes. We aim to prove that $S_1^{OO} = S_2^{OO}$; this equality yields the required result.

Now, let X be the total number of edges of H each of which lies along both routes and separates a face inside both routes from one outside both routes; and let Y be the total number of edges of H each of which lies along both routes and separates a face inside R_1 and outside R_2 from one outside R_1 and inside R_2.

Notice that $S^{II} - X = S^{IO} - Y$, since both sides count the number of edges along R_2 lying inside R_1. Similarly, $S^{II} - X = S^{OI} - Y$, since both sides count the number of edges along R_1 lying inside R_2. The two obtained relations yield $S^{IO} - Y = S^{OI} - Y$.

An analogous counting argument applied to the edges of R_2 lying outside R_1, and vice versa, yields the relations $S_2^{OO} - X = S^{OI} - Y$ and $S_1^{OO} - X = S^{IO} - Y$. The right hand sides of the obtained equalities are shown to be equal, thus the left hand sides are equal as well, so $S_1^{OO} = S_2^{OO}$. This completes the proof.

Remarks. (1) In the context of Solution 2, some other relations between the characters may be obtained in an easier way. E.g., the relation $S^{II} + S^{IO} = S^{OI} + S_1^{OO}$ follows by counting the total number of edges along R_1; similarly, one obtains $S^{II} + S^{OI} = S^{IO} + S_2^{OO}$. Such easier relations, along with their corollaries, may substitute some part of the above arguments, but they do not seem to suffice for solving the problem.

(2) The question resolved in this problem was originally posed by Professor Donald Knuth in October 2018 in private correspondence with the author, in the context of Slitherlink signatures. More on this topic is to be found in Professor Knuth's Pre-Fascicle 5C of Section 7.2.2.1, Dancing Links, of Volume 4 of The Art of Computer Programming (to appear in April 2019; as of December 2018, a preliminary version is available at `https://cs.stanford.edu/~knuth/fasc5c.ps.gz`.)

Problem 6. Let $P(x)$ be a non-constant polynomial with complex coefficients, and let $Q(x,y) = P(x) - P(y)$. Assume that $Q(x,y)$ has exactly k linear factors. Let $R(x,y)$ be a non-constant factor of $Q(x,y)$ whose degree is less than k. Prove that $R(x,y)$ is a product of linear polynomials with complex coefficients.

Iran, Navid Safaei

Solution 1. *Step 1.* We begin by describing all linear factors of $Q(x,y)$. Clearly, $x-y$ is always a factor. If it is unique, the problem statement holds trivially. If $x-y+c$ for some $c \neq 0$ is also a factor, this yields $Q(x,x+c)=0$, so $P(x)=P(x+c)$, which is impossible for a non-constant polynomial.

Now let $x+\varepsilon y+c$ with $\varepsilon \neq 1$ be another factor of $Q(x,y)$. Perform the linear shift $x \mapsto x+\frac{c}{1-\varepsilon}$, $y \mapsto y+\frac{c}{1-\varepsilon}$; the new polynomials are denoted by the same letters. The obtained polynomials are within the same set up, and now $x-\varepsilon y$ is a factor of $Q(x,y)$ (now it becomes clear that $\varepsilon \neq 0$). This yields that $P(x)=P(\varepsilon x)=P(\varepsilon^2 x)=\dots$, so $x-\varepsilon^t y$ is also a factor of Q, for every integer t. This may happen only if ε is a root of unity.

Take ε to be the root of unity with the smallest (positive) argument for which $x-\varepsilon y$ is a factor of Q; then $\varepsilon=\exp(2\pi i/\ell)$ for some ℓ, and all those factors having such form are $x-\varepsilon^t y$ for $t=0,1,\dots,\ell-1$. The product of those factors is $x^\ell - y^\ell$. In particular, this shows that the set $\mathcal{X}$ of roots of P is invariant under the rotation at 0 by $2\pi/\ell$.

Now, if Q has a linear factor which is not listed above, similar reasons show that X is also invariant under some rotation with different center r. We show that this is impossible. Without loss of generality, $r=1$. Let z_0 be a root with maximal absolute value, i.e., $|z_0-0|$. Due to rotations, we may assume that $\Re z_0 \leq 0$; hence $|z_0-1|>|z_0-0|$. Similarly, if z_1 is a root maximally distant from 1, then (after a suitable rotation) we have $|z_1-0|>|z_1-1| \geq |z_0-1|>|z_0-0|$. This is a contradiction.

Therefore, there are no extra linear factors, so $k=\ell$, and ε is a primitive root of unity of degree k.

Step 2. So, x^k-y^k divides Q and thus each its homogeneous component. All such components have the form $Q_i(x,y)=c_i(x^i-y^i)$, so all nonzero components of Q have degrees divisible by k. In particular, $\deg Q = nk$ for some integer n.

In order to solve the problem, it suffices to show that an *irreducible* factor R of Q is linear, whenever $d=\deg R<k$. Let $Q=RT$, and let

$$R(x,y)=\sum_{i=0}^{d} R_i(x,y), \qquad T(x,y)=\sum_{i=0}^{nk-d} T_i(x,y), \qquad Q(x,y)=\sum_{i=0}^{nk} Q_i(x,y),$$

where R_i, Q_i, and T_i are homogeneous of degree i. Then $R_d(x,y)$ divides $Q_{nk}(x,y)=c_{nk}(x^{nk}-y^{nk})$, so it has a factor of the form $S(x,y)=x-\mu y$. Notice that S does not divide T_{nk-d}, as $Q_{nk}=R_d T_{nk-d}$ factors into distinct linear factors.

If $R=S$, we are done. Otherwise, there exists a maximal $m<d$ such that $S \nmid R_m$. Then the component

$$Q_{nk-d+m}=\sum_{i=m}^{d} R_i T_{nk-d+m-i} = R_m T_{nk-d} + \sum_{i=m+1}^{d} R_i T_{nk-d+m-i}$$

is non-zero, because only the first summand is not divisible by S. This is impossible, since $k \nmid nk-d+m$.

Remarks. **(1)** In Step 1, after showing that $x^\ell - y^\ell$ divides $Q(x,y)$ for $\ell \geq 2$, one may apply an argument similar to that in Step 2, in order to show that $Q(x,y)$ has no extra linear factors (with nonzero constant term).

Indeed, if $R(x,y) = x + \mu y + c$ were such a factor, one may mention that $\deg Q = n\ell$ for some integer n, and write $Q = RT$, where the polynomials T and Q are expanded as above:

$$T(x,y) = \sum_{i=0}^{n\ell-1} T_i(x,y), \qquad Q(x,y) = \sum_{i=0}^{n\ell} Q_i(x,y).$$

Then $S(x,y) = x - \mu y$ does not divide $T_{n\ell-1}$, so it does not divide $Q_{n\ell-1} = cT_{n\ell-1} + ST_{n\ell-2}$ either. This is a contradiction, because $\ell \nmid n\ell - 1$.

(2) To avoid the technical step 1, the easier version below could be taken into consideration:

> Let $P(x)$ be a polynomial with complex coefficients, and let k be a positive integer. Let $R(x,y)$ be a factor of $P(x^k) - P(y^k)$ whose degree is less than k. Prove that $R(x,y)$ factors into linear factors with complex coefficients.

Solution 2. We present a different argument for Step 2 in the above solution. From Step 1, we know that $P(x) = P(\varepsilon x)$, where ε is a primitive root of unity of degree k; therefore, P is a polynomial in x^k.

Now let $R(x,y)$ be an irreducible factor of $Q(x,y)$ with $d = \deg R < k$. As above, expand R and Q into homogeneous components:

$$R(x,y) = \sum_{i=0}^{d} R_i(x,y), \qquad Q(x,y) = \sum_{i=0}^{nk} Q_i(x,y),$$

Recall that Q_{nk} has no multiple factors.

The condition $P(\varepsilon x) = P(x)$ yields now that $Q(x,y) = P(\varepsilon x) - P(\varepsilon y) = Q(\varepsilon x, \varepsilon y)$. Therefore, $R^*(x,y) = R(\varepsilon x, \varepsilon y)$ is also an (irreducible) factor of $Q(x,y)$. Hence, R^* is either coprime to R, or proportional to it.

Notice that

$$R^*(x,y) = \sum_{i=0}^{d} \varepsilon^i R_i(x,y)$$

has the same leading homogeneous component as $R(x,y)$, up to a multiplicative constant. Hence, if the factors R and R^* were coprime, then RR^* would divide Q, hence $(R_d)^2$ would divide Q_{nk}. But the latter has no multiple factors, which is a contradiction.

Thus, R^* is proportional to R, with a factor of ε^d. Now compare the expansions of R and R^* and recall that $d < k$ to conclude that R_d is the only nonzero homogeneous component of R. Since any homogeneous polynomial in two variables is a product of linear factors, R must be linear.

Remark. A version of the above argument may also serve in completing Step 1, as in Remark (1) to the previous solution.