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Izračunavanje nekih konačnih suma

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Sažetak: U matematici se često, na svim nivoima obrazovanja, ukaže potreba za nalaženje nekih konačnih suma. Želja nam je da u ovom radu pokažemo kako se izračunavaju neke konačne sume, što je ilustrovano nizom zanimljivih zadataka prilagođenih učenicima srednjih škola.

1. Uvod

Često u testovima logičkog tipa i različitim enigmatskim časopisima možemo naći zadatak sličan sljedećem.

Zadatak. *Napisati sljedeći član u nizu brojeva*

$$3, \frac{5}{2}, \frac{7}{3}, \frac{9}{4}, \frac{11}{5}, \cdots$$

Rješenje: Problem se svodi na određivanje općeg člana datog niza. Kako je

$$\begin{aligned} a_1 &= 3 = \frac{2\cdot 1+1}{1}, \\ a_2 &= \frac{5}{2} = \frac{2\cdot 2+1}{2}, \\ a_3 &= \frac{7}{3} = \frac{2\cdot 3+1}{3}, \\ a_4 &= \frac{9}{4} = \frac{2\cdot 4+1}{4}, \\ a_5 &= \frac{11}{5} = \frac{2\cdot 5+1}{5}, \\ &\vdots \end{aligned}$$

opći član niza $3, \frac{5}{2}, \frac{7}{3}, \frac{9}{4}, \frac{11}{5}, \cdots$ je $a_n = \frac{2n+1}{n}$, pa je $a_6 = \frac{13}{6}$. □

Ovaj postupak možemo uspješno primijeniti na izračunavanje nekih konačnih suma. U nastavku navodimo nekoliko primjera u kojima ćemo koristiti i poznate sume potencija prirodnih brojeva:

$$S_1 = 1+2+3+\cdots+n = \frac{n(n+1)}{2}, \tag{1}$$

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$$S_2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}, \tag{2}$$

$$S_3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}, \tag{3}$$

$$S_4 = \frac{n(n+1)(2n+1)\left(3n^2+3n-1\right)}{30}, \tag{4}$$

te sumu prvih n članova geometrijskog niza

$$S_n = a_1 + a_1 q + \cdots + a_1 q^{n-1} = a_1 \frac{1-q^n}{1-q}. \tag{5}$$

2. Izračunavanje nekih konačnih suma

Primjer 2.1. *Izračunati sumu:* $S = -1 + 2 + 7 + 14 + 23 + \cdots + 1598$.

Rješenje: Posmatrajmo niz $-1, 2, 7, 14, 23, \cdots$. Tada je

$$\begin{aligned}
a_1 &= -1 = 1^2 - 2,\\
a_2 &= 2 = 2^2 - 2,\\
a_3 &= 7 = 3^2 - 2,\\
a_4 &= 14 = 4^2 - 2,\\
a_5 &= 23 = 5^2 - 2,\\
&\vdots
\end{aligned}$$

Opći član niza $-1, 2, 7, 14, 23, \cdots$ je $a_n = n^2 - 2$, pa traženu sumu S možemo predstaviti u obliku

$$\begin{aligned}
S &= -1 + 2 + 7 + 14 + 23 + \cdots + 1598\\
&= \left(1^2 - 2\right) + \left(2^2 - 2\right) + \left(3^2 - 2\right) + \left(4^2 - 2\right) + \cdots + \left(40^2 - 2\right)\\
&= 1^2 + 2^2 + 3^2 + 4^2 + \cdots + 40^2 - 40 \cdot 2\ ,
\end{aligned}$$

odnosno, koristeći (2)

$$S = \frac{40 \cdot 41 \cdot 81}{6} - 40 \cdot 2 = 22060\ .$$

□

Primjer 2.2. *Izračunati sumu:* $S = 1 + 12 + 45 + 112 + \cdots + 3312$.

Rješenje: Posmatrajmo niz $1, 12, 45, 112, \cdots$. Primjetimo da je

$$\begin{aligned}
a_1 &= 1 = 1 \cdot 1^2,\\
a_2 &= 12 = 3 \cdot 2^2,\\
a_3 &= 45 = 5 \cdot 3^2,\\
a_4 &= 112 = 7 \cdot 4^2,\\
&\vdots
\end{aligned}$$

Opći član ovog niza je $a_n = (2n-1)\,n^2 = 2n^3 - n^2$, pa traženu sumu S možemo napisati u obliku

$$\begin{aligned}
S &= 1 + 12 + 45 + 112 + \cdots + 3312\\
&= 2 \cdot 1^3 - 1^2 + 2 \cdot 2^3 - 2^2 + 2 \cdot 3^3 - 3^2 + 2 \cdot 4^3 - 4^2 + \cdots + 2 \cdot 12^3 - 12^2\\
&= 2\left(1^3 + 2^3 + 3^3 + 4^3 + \cdots + 12^3\right) - \left(1^2 + 2^2 + 3^2 + 4^2 + \cdots + 12^2\right)\ .
\end{aligned}$$

Sada, koristeći (2) i (3) dobijamo da je

$$S = 12168 - 650 = 11\,518\ .$$

□

Primjer 2.3. *Izračunati sumu:* $S = \dfrac{3}{2} + \dfrac{9}{4} + \dfrac{33}{8} + \dfrac{129}{16} + \cdots + \dfrac{131073}{512}$.

Rješenje: U nizu $\frac{3}{2}, \frac{9}{4}, \frac{33}{8}, \frac{129}{16}, \cdots$ je:

$$\begin{aligned}
a_1 &= \frac{3}{2} = \frac{2+1}{2} = \frac{2^1+1}{2^1},\\
a_2 &= \frac{9}{4} = \frac{8+1}{4} = \frac{2^3+1}{2^2},\\
a_3 &= \frac{33}{8} = \frac{32+1}{8} = \frac{2^5+1}{2^3},\\
a_4 &= \frac{129}{16} = \frac{128+1}{16} = \frac{2^7+1}{2^4},\\
&\vdots
\end{aligned}$$

Uočavamo da je opći član ovog niza $a_n = \frac{2^{2n-1}+1}{2^n}$. Koristeći to i (5) sumu S možemo predstaviti u obliku

$$\begin{aligned}
S &= \frac{3}{2} + \frac{9}{4} + \frac{33}{8} + \frac{129}{16} + \cdots + \frac{131073}{512}\\
&= \frac{2+1}{2} + \frac{2^3+1}{2^2} + \frac{2^5+1}{2^3} + \frac{2^7+1}{2^4} + \cdots + \frac{2^{17}+1}{2^9}\\
&= 1 + \frac{1}{2} + \frac{2^3}{2^2} + \frac{1}{2^2} + \frac{2^5}{2^3} + \frac{1}{2^3} + \frac{2^7}{2^4} + \frac{1}{2^4} + \cdots + \frac{2^{17}}{2^9} + \frac{1}{2^9}\\
&= \left(1 + 2 + 2^2 + 2^3 + \cdots + 2^8\right) + \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^9}\right)\\
&= \frac{1\left(1-2^9\right)}{1-2} + \frac{\frac{1}{2}\left(1-\frac{1}{2^9}\right)}{1-\frac{1}{2}} = 2^9 - 1 + 1 - \frac{1}{2^9} = \frac{2^{18}-1}{2^9}\ .
\end{aligned}$$

□

Primjer 2.4. *Izračunati sumu:* $S = \dfrac{1}{5} + \dfrac{1}{45} + \dfrac{1}{117} + \dfrac{1}{221} + \cdots + \dfrac{1}{4076357}$.

Rješenje: Niz $\frac{1}{5}, \frac{1}{45}, \frac{1}{117}, \frac{1}{221}, \cdots$ možemo napisati u obliku

$$\frac{1}{1\cdot 5}, \frac{1}{5\cdot 9}, \frac{1}{9\cdot 13}, \frac{1}{13\cdot 17}, \cdots\ .$$

Opći član ovog niza je $a_n = \frac{1}{(4n-3)(4n+1)}$. Rastavljajući na parcijalne sabirke dobijamo

$$\frac{1}{(4n-3)\,(4n+1)} = \frac{1}{4}\left(\frac{1}{(4n-3)} - \frac{1}{(4n+1)}\right)\ ,$$

pa je

$$\begin{aligned}
\frac{1}{1\cdot 5} &= \frac{1}{4}\left(1-\frac{1}{5}\right),\\
\frac{1}{5\cdot 9} &= \frac{1}{4}\left(\frac{1}{5}-\frac{1}{9}\right),\\
\frac{1}{9\cdot 13} &= \frac{1}{4}\left(\frac{1}{9}-\frac{1}{13}\right),\\
\frac{1}{13\cdot 17} &= \frac{1}{4}\left(\frac{1}{13}-\frac{1}{17}\right),\\
&\vdots\\
\frac{1}{2017\cdot 2021} &= \frac{1}{4}\left(\frac{1}{2017}-\frac{1}{2021}\right).
\end{aligned}$$

Dakle, traženu sumu S možemo napisati u obliku

$$\begin{aligned}
S &= \frac{1}{1\cdot 5}+\frac{1}{5\cdot 9}+\frac{1}{9\cdot 13}+\frac{1}{13\cdot 17}+\cdots+\frac{1}{2017\cdot 2021}\\
&= \frac{1}{4}\left(1-\frac{1}{5}+\frac{1}{5}-\frac{1}{9}+\frac{1}{9}-\frac{1}{13}+\frac{1}{13}-\frac{1}{17}+\cdots+\frac{1}{2017}-\frac{1}{2021}\right)=\frac{1}{4}\left(1-\frac{1}{2021}\right)=\frac{505}{2021}.
\end{aligned}$$

□

Primjer 2.5. *Izračunati sumu:* $S=\dfrac{3}{4}+\dfrac{5}{36}+\dfrac{7}{144}+\dfrac{9}{400}+\cdots+\dfrac{21}{12100}$.

Rješenje: Posmatrajmo niz $\frac{3}{4},\frac{5}{36},\frac{7}{144},\frac{9}{400},\ldots$. U njemu je:

$$\begin{aligned}
a_1 &= \frac{3}{4}=\frac{2\cdot 1+1}{1^2\cdot 2^2},\\
a_2 &= \frac{5}{36}=\frac{2\cdot 2+1}{2^2\cdot 3^2},\\
a_3 &= \frac{7}{144}=\frac{2\cdot 3+1}{3^2\cdot 4^2},\\
a_4 &= \frac{9}{400}=\frac{2\cdot 4+1}{4^2\cdot 5^2},\\
&\vdots
\end{aligned}$$

Uočavamo da je opći član datog niza $a_n=\frac{2n+1}{n^2(n+1)^2}$. Rastavljajući na parcijalne sabirke imamo

$$\frac{2n+1}{n^2\left(n+1\right)^2}=\frac{1}{n^2}-\frac{1}{\left(n+1\right)^2},$$

pa je tražena suma

$$\begin{aligned}
S &= \frac{3}{4}+\frac{5}{36}+\frac{7}{144}+\frac{9}{400}+\cdots+\frac{21}{12100}\\
&= \frac{2\cdot 1+1}{1^2\cdot 2^2}+\frac{2\cdot 2+1}{2^2\cdot 3^2}+\frac{2\cdot 3+1}{3^2\cdot 4^2}+\frac{2\cdot 4+1}{4^2\cdot 5^2}+\cdots+\frac{2\cdot 10+1}{10^2\cdot 11^2}\\
&= \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{4^2}-\frac{1}{5^2}+\cdots+\frac{1}{10^2}-\frac{1}{11^2}\\
&= 1-\frac{1}{121}=\frac{120}{121}.
\end{aligned}$$

□

Primjer 2.6. *Izračunati sumu:* $S = \dfrac{1}{3\sqrt{1}+1\sqrt{3}} + \dfrac{1}{5\sqrt{3}+3\sqrt{5}} + \dfrac{1}{7\sqrt{5}+5\sqrt{7}} + \cdots + \dfrac{1}{81\sqrt{79}+79\sqrt{81}}$.

Rješenje: Prvo primijetimo da je

$$S = \sum_{n=1}^{40} a_n \ ,$$

za $a_n = \dfrac{1}{(2n+1)\sqrt{2n-1}+(2n-1)\sqrt{2n+1}}$. Racionalisanjem dobijamo da je

$$\begin{aligned}
a_n &= \frac{1}{(2n+1)\sqrt{2n-1}+(2n-1)\sqrt{2n+1}} \cdot \frac{(2n+1)\sqrt{2n-1}-(2n-1)\sqrt{2n+1}}{(2n+1)\sqrt{2n-1}-(2n-1)\sqrt{2n+1}} \\
&= \frac{(2n+1)\sqrt{2n-1}-(2n-1)\sqrt{2n+1}}{\left[(2n+1)\sqrt{2n-1}\right]^2-\left[(2n-1)\sqrt{2n+1}\right]^2} = \frac{(2n+1)\sqrt{2n-1}-(2n-1)\sqrt{2n+1}}{(2n+1)^2(2n-1)-(2n-1)^2(2n+1)} \\
&= \frac{(2n+1)\sqrt{2n-1}-(2n-1)\sqrt{2n+1}}{2(2n+1)(2n-1)} = \frac{1}{2}\left(\frac{\sqrt{2n-1}}{2n-1}-\frac{\sqrt{2n+1}}{2n+1}\right) \ ,
\end{aligned}$$

odakle je

$$\begin{aligned}
S &= \frac{1}{3\sqrt{1}+1\sqrt{3}} + \frac{1}{5\sqrt{3}+3\sqrt{5}} + \frac{1}{7\sqrt{5}+5\sqrt{7}} + \cdots + \frac{1}{81\sqrt{79}+79\sqrt{81}} \\
&= \frac{1}{2}\left(\frac{\sqrt{1}}{1}-\frac{\sqrt{3}}{3}+\frac{\sqrt{3}}{3}-\frac{\sqrt{5}}{5}+\frac{\sqrt{5}}{5}-\frac{\sqrt{7}}{7}+\cdots+\frac{\sqrt{79}}{79}-\frac{\sqrt{81}}{81}\right) \\
&= \frac{1}{2}\left(1-\frac{1}{9}\right) = \frac{4}{9} \ .
\end{aligned}$$

□

Primjer 2.7. *Izračunati sumu:* $S_n = 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)\,x^n$.

Rješenje: 1
Ako je $x = 1$, suma je

$$S_n = \frac{(n+1)(n+2)}{2} \ .$$

Neka je $x \neq 1$. Sumu S_n možemo napisati u obliku

$$\begin{aligned}
S_n &= \left(1+x+x^2+x^3+\cdots+x^n\right)+\left(x+2x^2+3x^3+\cdots+nx^n\right) \\
&= \left(1+x+\cdots+x^n\right)+\left(x+x^2+\cdots+x^n\right)+\left(x^2+2x^3+\cdots+(n-1)\,x^n\right) \\
&= \left(1+x+\cdots+x^n\right)+\left(x+x^2+\cdots+x^n\right)+ \\
&\quad \left(x^2+x^3+\cdots+x^n\right)+\cdots+\left(x^{n-1}+x^n\right)+x^n \ .
\end{aligned}$$

Sada, koristeći (5) dobijamo da je

$$\begin{aligned}
S_n &= \frac{1\left(x^{n+1}-1\right)}{x-1}+\frac{x\left(x^n-1\right)}{x-1}+\frac{x^2\left(x^{n-1}-1\right)}{x-1}+\cdots+\frac{x^{n-1}\left(x^2-1\right)}{x-1}+\frac{x^n(x-1)}{x-1} \\
&= \frac{x^{n+1}-1+x^{n+1}-x+x^{n+1}-x^2+\cdots+x^{n+1}-x^{n-1}+x^{n+1}-x^n}{x-1} \\
&= \frac{(n+1)\,x^{n+1}-\left(1+x+x^2+\cdots+x^{n-1}+x^n\right)}{x-1} = \frac{(n+1)\,x^{n+1}-\frac{\left(x^{n+1}-1\right)}{x-1}}{x-1} \ ,
\end{aligned}$$

odnosno

$$S_n = \frac{(nx - n + x - 2)\, x^{n+1} + 1}{(x-1)^2} \ .$$

Rješenje: 2
Tražena suma S_n je izvod sume

$$1 + x + x^2 + x^3 + x^4 + \cdots + x^n + x^{n+1} \overset{(5)}{=} \frac{x^{n+2} - 1}{x - 1} \ .$$

Derivacijom dobijamo

$$\left(1 + x + x^2 + x^3 + x^4 + \cdots + x^n + x^{n+1}\right)' = \left(\frac{x^{n+2} - 1}{x - 1}\right)' \ ,$$

odnosno

$$\begin{aligned} S_n &= 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + (n+1)\, x^n \\ &= \frac{(n+2)\, x^{n+1}(x-1) - x^{n+2} + 1}{(x-1)^2} = \frac{(nx - n + x - 2)\, x^{n+1} + 1}{(x-1)^2} \ . \end{aligned}$$

□

Primjer 2.8. *Izračunati sumu:* $S_n = 1 + 3x + 5x^2 + 7x^3 + \cdots + (2n+1)\, x^n$.

Rješenje: 1
Ako je $x = 1$, onda je

$$S_n = (n+1)^2 \ .$$

Neka je $x \neq 1$. Množenjem tražene sume S_n sa x $(x \neq 0)$ i koristeći (5) dobijamo da je

$$\begin{aligned} S_n - xS_n &= 1 + 2x + 2x^2 + 2x^3 + \cdots + 2x^n - (2n+1)\, x^{n+1} \\ &= 1 - (2n+1)\, x^{n+1} + 2x\left(1 + x + x^2 + \cdots + x^{n-1}\right) \\ &= 1 - (2n+1)\, x^{n+1} + 2x\frac{1 - x^n}{1 - x} \ . \end{aligned}$$

Dakle,

$$(1 - x)\, S_n = 1 - (2n+1)\, x^{n+1} + 2x\frac{1 - x^n}{1 - x} \ ,$$

pa je

$$S_n = \frac{1 + x - (2n+3)\, x^{n+1} + (2n+1)\, x^{n+2}}{(1-x)^2} \ .$$

Rješenje: 2
Sumu S_n možemo napisati u obliku

$$\begin{aligned} S_n &= 1 + 3x + 5x^2 + 7x^3 + \cdots + (2n+1)\, x^n \\ &= 1 + x + x^2 + x^3 + \cdots + x^n + 2x + 4x^2 + 6x^3 + \cdots + 2nx^n \\ &= \underbrace{1 + x + x^2 + x^3 + \cdots + x^n}_{S_1} + 2x\underbrace{\left(1 + 2x + 3x^2 + \cdots + nx^{n-1}\right)}_{S_2} . \end{aligned}$$

Primijetimo da je

$$S_1 \stackrel{(5)}{=} \frac{x^{n+1}-1}{x-1} \quad \text{i} \quad S_2 = (S_1)' \ ,$$

pa je

$$\begin{aligned} S_n &= \frac{x^{n+1}-1}{x-1} + 2x\left(\frac{x^{n+1}-1}{x-1}\right)' = \frac{x^{n+1}-1}{x-1} + 2x\frac{(n+1)\,x^n\,(x-1) - x^{n+1}+1}{(x-1)^2} \\ &= \frac{1+x-(2n+3)\,x^{n+1} + (2n+1)\,x^{n+2}}{(1-x)^2} \ . \end{aligned}$$

□

Primjer 2.9. *Izračunati sumu:* $S_n = x + x^2\,(1+x) + x^3\left(1+x+x^2\right) + \cdots + x^n\left(1+x+x^2+\cdots+x^{n-1}\right)$.

Rješenje: Neka je $x \neq 1$. Koristeći (5) traženu sumu S_n možemo napisati u obliku

$$\begin{aligned} S_n &= x\cdot\frac{x-1}{x-1} + x^2\cdot\frac{x^2-1}{x-1} + x^3\cdot\frac{x^3-1}{x-1} + \cdots + x^n\cdot\frac{x^n-1}{x-1} \\ &= \frac{x}{x-1}\left[x-1+x\left(x^2-1\right)+x^2\left(x^3-1\right)+\cdots+x^{n-1}\left(x^n-1\right)\right] \\ &= \frac{x}{x-1}\left[\left(x+x^3+x^5+\cdots+x^{2n-1}\right)-\left(1+x+x^2+\cdots+x^{n-1}\right)\right] \\ &= \frac{x}{x-1}\left(\frac{x^{2n+1}-x}{x^2-1}-\frac{x^n-1}{x-1}\right) = \frac{x^{n+1}\left(x^{n+1}-x-1\right)+x}{(x-1)^2\,(x+1)} \ . \end{aligned}$$

□

Primjer 2.10.

a) *Izračunati sumu* $S_n = \frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \frac{7}{2^4} + \cdots + \frac{2n-1}{2^n}$;

b) *Odrediti* $\lim\limits_{n\to\infty} S_n$.

Rješenje: a) Posmatrajmo razliku

$$\begin{aligned} S_n - 2S_n &= \frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\frac{7}{2^4}+\cdots+\frac{2n-1}{2^n} - 1 - \frac{3}{2} - \frac{5}{2^2} - \frac{7}{2^3} - \cdots - \frac{2n-1}{2^{n-1}} \\ &= -1 - \frac{2}{2} - \frac{2}{2^2} - \frac{2}{2^3} - \cdots - \frac{2}{2^{n-1}} + \frac{2n-1}{2^n} \\ &= -1 - \left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-2}}\right) + \frac{2n-1}{2^n} = -3 + \frac{2n+3}{2^n} \ . \end{aligned}$$

Dakle, $S_n = 3 - \dfrac{2n+3}{2^n}$.

b) Sada je

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(3 - \frac{2n+3}{2^n}\right) = 3 \ .$$

□

Primjer 2.11. *Izračunati:* $\lim\limits_{n\to\infty}\left(\frac{1}{2^0}+\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}+\cdots+\frac{n}{2^{n-1}}\right)$.

Rješenje: Neka je

$$S_n = \frac{1}{2^0} + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \cdots + \frac{n}{2^{n-1}} \ .$$

Tada je

$$\frac{1}{2}S_n = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots + \frac{n-1}{2^{n-1}} + \frac{n}{2^n} \ .$$

Koristeći (5) dobijamo da je

$$\begin{aligned} S_n - \frac{1}{2}S_n &= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} - \frac{n}{2^n} \\ &= \frac{\frac{1}{2^n} - 1}{\frac{1}{2} - 1} - \frac{n}{2^n} = 2 - \frac{2}{2^n} - \frac{n}{2^n} = 2 - \frac{n+2}{2^n} \ . \end{aligned}$$

Dakle, $\frac{1}{2}S_n = 2 - \frac{n+2}{2^n}$, pa je

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(4 - \frac{n+2}{2^{n-1}}\right) = 4 \ .$$

$\square$

Primjer 2.12. *Izračunati sumu:* $S_n = 1 + 11 + 111 + 1111 + \cdots + \underbrace{111\ldots1}_{n \text{ jedinica}}$.

Rješenje: Broj $\underbrace{111\ldots1}_{n \text{ jedinica}}$ možemo napisati u obliku

$$\underbrace{111\ldots1}_{n \text{ jedinica}} = \frac{\overbrace{999\ldots9}^{n devetki}}{9} = \frac{10^n - 1}{9} \ ,$$

pa imamo

$$\begin{aligned} S_n &= \frac{10-1}{9} + \frac{10^2-1}{9} + \frac{10^3-1}{9} + \cdots + \frac{10^n-1}{9} = \frac{1}{9}\left(10 + 10^2 + 10^3 + \cdots + 10^n\right) - \frac{n}{9} \\ &= \frac{1}{9} \cdot 10 \cdot \frac{10^n-1}{9} - \frac{n}{9} = \frac{10\left(10^n - 1\right)}{81} - \frac{n}{9} \ . \end{aligned}$$

$\square$

Primjer 2.13. *Brojevi* $49, 4489, 4444889, \cdots$ *se dobiju tako što se "u sredini" svakog prethodnog broja ubaci broj* 48. *Dokazati da su svi takvi brojevi potpuni kvadrati prirodnog broja.*

Rješenje: Broj $444\ldots888\ldots9$ možemo napisati u obliku

$$4 \cdot \underbrace{111\ldots1}_{n \text{ jedinica}} \cdot 10^n + 8 \cdot \underbrace{111\ldots1}_{n-1 \text{ jedinica}} \cdot 9 = 4 \cdot \underbrace{111\ldots1}_{n \text{ jedinica}} \cdot 10^n + 8 \cdot \underbrace{111\ldots1}_{n \text{ jedinica}} + 1 \ .$$

Kako je $\underbrace{111\ldots1}_{n \text{ jedinica}} = \frac{10^n - 1}{9}$, imamo

$$\begin{aligned} &4 \cdot \underbrace{111\ldots1}_{n \text{ jedinica}} \cdot 10^n + 8 \cdot \underbrace{111\ldots1}_{n \text{ jedinica}} + 1 = 4 \cdot \frac{10^n-1}{9} \cdot 10^n + 8 \cdot \frac{10^n-1}{9} + 1 \\ &= \frac{4 \cdot 10^{2n} - 4 \cdot 10^n + 8 \cdot 10^n - 8 + 9}{9} = \frac{4 \cdot 10^{2n} + 4 \cdot 10^n + 1}{9} = \left(\frac{2 \cdot 10^n + 1}{3}\right)^2 \ . \end{aligned}$$

$\square$

3. Zadaci za vježbu

Izračunati sume:

1. $S = 2 + 9 + 28 + 65 + 126 + \cdots + 8001$.
2. $S = 4 + 18 + 48 + 100 + \cdots + 8820$.
3. $S = \dfrac{1}{10} + \dfrac{1}{40} + \dfrac{1}{88} + \dfrac{1}{154} + \cdots + \dfrac{1}{8188}$.
4. $S = \dfrac{1}{2\sqrt{1} + 1\sqrt{2}} + \dfrac{1}{3\sqrt{2} + 2\sqrt{3}} + \dfrac{1}{4\sqrt{3} + 3\sqrt{4}} + \cdots + \dfrac{1}{225\sqrt{224} + 224\sqrt{225}}$.
5. $S_n = \dfrac{5}{2} + 5 + \dfrac{19}{2} + 18 + \cdots + \dfrac{n + 2^{n+1}}{2}$.
6. $S_n = \left(x + \dfrac{1}{x}\right)^2 + \left(x^2 + \dfrac{1}{x^2}\right)^2 + \left(x^3 + \dfrac{1}{x^3}\right)^3 + \cdots + \left(x^n + \dfrac{1}{x^n}\right)^n, x \neq \pm 1, n \in N$.
7. Neka je $\overline{aaa\ldots a}$ broj čije su sve cifre a, gdje je $a \neq 0$. Izračunati

$$S = \overline{a} + \overline{aa} + \overline{aaa} + \cdots + \underbrace{\overline{aaa\ldots a}}_{n} .$$

Zahvalnost

Zahvaljujemo se profesoru Mehmedu Nurkanoviću za motivaciju i sugestije kojima je pomogao uobličenje ovog rada.

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