MORE SOLUTIONS OF A PROBLEM FOR MATHEMATICAL COMPETITIONS

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Abstract. The paper considers nine solutions of one important fact for effective participation in mathematical competitions – the conditional algebraic inequality.

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It is more profitable to solve one and the same problem in several different manners than to solve several problems, only in one manner. If we solve one and the same problem in different manners, we can compare the different solutions and establish which is the shortest one, which is more convincing, more elegant. Thus, one attains and builds art of problem solving.

(W.W. Sawyer, Prelude to Mathematics)

To prove inequalities in Mathematics is an important and very interesting job. The first author of the present paper has published many books and articles in prestigeous journals which have become useful instruments for sholars, including students with interest to Mathematics and teachers preparing such students. The present paper is dedicated to an algebraic inequality and various approaches to its proof. The ideas are connected with classic and well-known inequalities: inequalities of means, Cauchy-Buniakovski-Schwarz's inequality, Bernoulli's inequality, Chebishev's inequality, Jensen's inequality, Minkowski's inequality, Schur's inequality, Holder's inequality, Huygens's inequality, Muirhead's inequality and many others. No doubt, all of them will help young gifted students and lovers of Mathematics.

Problem. Let *a* and *b* be two two positive real numbers such that $a+b=1$. Prove the inequality $a^3+b^3\geq\frac{1}{4}$**.** When does the equality occur?

We propose nine different solutions of this problem.

Solution 1. Taking the two sides of $a+b=1$ squared we get $a^2+2ab+b^2=1$. The sum of this equality and the exact inequality $a^2-2ab+b^2\geq 0\left(\Leftrightarrow (a-b)^2\geq 0\right)$, we obtain the inequality:

$$a^2+b^2\geq\frac{1}{2}. \tag{1}$$

From here after multiplying (1) by $a+b=1$

$$\left(a^2+b^2\right)\left(a+b\right)\geq\frac{1}{2}\times 1$$

$$\Leftrightarrow a^3+ab\left(a+b\right)+b^3\geq\frac{1}{2}$$

$$\Leftrightarrow a^3+b^3\geq\frac{1}{2}-ab\,. \qquad (2)$$

Also, we have

$$a^2+b^2=\left(a+b\right)^2-2ab=1-2ab\,,$$

and from here and (1):

$$1-2ab\geq\frac{1}{2}$$

$$\Leftrightarrow ab\leq\frac{1}{4}\,. \qquad (3)$$

Since, $\max\left(ab\right)=\frac{1}{4}$, it follows that:

$$a^3+b^3\geq\frac{1}{2}-\frac{1}{4}\text{, i.e.}$$

$$a^3+b^3\geq\frac{1}{4}\text{, q.e.d.}$$

The equality holds if and only if $a=b=\frac{1}{2}$.

Solution 2. From $a+b=1$ it follows that $b=1-a$ and we have:

$$a^3+b^3=a^3+\left(1-a\right)^3=a^3+1-3a+3a^2-a^3=3a^2-3a+1\,. \qquad (4)$$

Now we will prove that

$$3a^2-3a+1\geq\frac{1}{4} \qquad (5)$$

$$\Leftrightarrow 12a^2-12a+3\geq 0/:3$$

$$\Leftrightarrow 4a^2-4a+1\geq 0$$

$$\Leftrightarrow\left(2a-1\right)^2\geq 0\,.$$

The last inequality holds true, i.e. the inequality (5) holds true. It follows that the equality holds if and only if $2a-1=0$, i.e. $a=\frac{1}{2}$ and $b=1-a=\frac{1}{2}$.

Solution 3. We have series of equivalent inequalities from $a+b=1$**:**

$$a^3+b^3\geq\frac{1}{4}$$

$$\Leftrightarrow\left(a+b\right)\left(a^2-ab+b^2\right)\geq\frac{1}{4}$$

$$\Leftrightarrow a^2-ab+b^2\geq\frac{1}{4}/\cdot 4$$

$$\Leftrightarrow 4a^2-4ab+4b^2\geq 1$$

$$\Leftrightarrow 4a^2-4a\left(1-a\right)+4\left(1-a\right)^2\geq 1$$

$$\Leftrightarrow 4a^2+4a^2-4a+4-8a+4a^2-1\geq 0$$

$$\Leftrightarrow 12a^2 - 12a + 3 \geq 0 / : 3$$
$$\Leftrightarrow 4a^2 - 4a + 1 \geq 0$$
$$\Leftrightarrow (2a-1)^2 \geq 0 .$$

The last is true and it follows that the given inequality is also true. The equality holds if and only if $2a - 1 = 0 \Rightarrow a = \frac{1}{2}$ and $b = 1 - a = \frac{1}{2}$.

Solution 4. Because $a + b = 1$; $(a, b > 0)$ we will use the substitution $a = \frac{1}{2} + x, b = \frac{1}{2} - x$, where $x \in \left[0, \frac{1}{2}\right)$. We get:

$$a^3 + b^3 = \left(\frac{1}{2} + x\right)^3 + \left(\frac{1}{2} - x\right)^3 = \frac{1}{8} + \frac{3}{4}x + \frac{3}{2}x^2 + x^3 + \frac{1}{8} - \frac{3}{4}x + \frac{3}{2}x^2 - x^3, \text{ i.e.}$$
$$a^3 + b^3 = \frac{1}{4} + 3x^2 \geq \frac{1}{4},$$

because $3x^2 \geq 0$, $x \in \left[0, \frac{1}{2}\right)$.

The equality holds if and only if $3x^2 = 0$, i.e. $x = 0$ and $a = b = \frac{1}{2}$.

Solution 5. We will use the well-known inequality of **Cauchy-Buniakovski-Schwarz** for $n = 2$:

$$(a_1 b_1 + a_2 b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2); \ (a_1, a_2, b_1, b_2 \in \mathbb{R}).$$

For $a_1 = a$, $a_2 = b$, $b_1 = 1$, $b_2 = 1$ we have that:

$$(a \cdot 1 + b \cdot 1)^2 \leq (a^2 + b^2)(1^2 + 1^2)$$
$$\Leftrightarrow (a+b)^2 \leq 2(a^2 + b^2)$$
$$\Leftrightarrow (\text{by } a + b = 1) \ a^2 + b^2 \geq \frac{1}{2} / (a+b)$$
$$\Leftrightarrow (a^2 + b^2)(a+b) \geq \frac{1}{2}(a+b)$$
$$\Leftrightarrow (\text{by } a + b = 1) \ a^3 + ab^2 + a^2 b + b^3 \geq \frac{1}{2}$$
$$\Leftrightarrow a^3 + b^3 + ab(a+b) \geq \frac{1}{2}$$
$$\Leftrightarrow (\text{by } a + b = 1) \ a^3 + b^3 + ab \geq \frac{1}{2}$$
$$\Leftrightarrow a^3 + b^3 \geq \frac{1}{2} - ab . \qquad (6)$$

Applying the arithmetic/geometric inequality, we have:

$$\frac{a+b}{2} \geq \sqrt{ab} ,$$

i.e. because $a + b = 1$:

$$ab \le \frac{1}{4} / (-1)$$

$$\Rightarrow -ab \ge -\frac{1}{4}.$$

(7)

It follows from (6) and (7), that:

$$a^3 + b^3 \ge \frac{1}{2} - \frac{1}{4}$$

$$\Leftrightarrow a^3 + b^3 \ge \frac{1}{4}, \text{ q.e.d.}$$

The equality holds if and only if $a = b = \frac{1}{2}$.

Solution 6. Now we will apply the inequality of the arithmetic and quadratic means:

$$\frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}}, \text{ i.e.}$$

$$a^2 + b^2 \ge \frac{(a+b)^2}{2}$$

or

$$\left(a^2+b^2\right)^2 \ge \frac{(a+b)^4}{4}.$$

(8)

Also, we will use the following inequality:

$$\frac{a^2}{x} + \frac{b^2}{y} \ge \frac{(a+b)^2}{x+y}; \ (a,b \in \mathbb{R})(x,y>0)$$

(9)

Since $xy(x+y) > 0$ and $(bx-ay)^2 \ge 0$, the equality holds if and only if $bx = ay$, i.e. $\frac{a}{x} = \frac{b}{y}$. Note, that the inequality (9) is one of the versions of the well-known inequality of Cauchy-Buniakovski-Schwarz. From (8) and (9) and the initial condition $a+b=1$ we have:

$$a^3+b^3 = \frac{a^4}{a} + \frac{b^4}{b} = \frac{\left(a^2\right)^2}{a} + \frac{\left(b^2\right)^2}{b} \overset{(9)}{\ge} \frac{\left(a^2+b^2\right)^2}{a+b} = \left(a^2+b^2\right)^2 \overset{(8)}{\ge} \frac{(a+b)^4}{4} = \frac{1}{4},$$

i.e.

$$a^3 + b^3 \ge \frac{1}{4}, \text{ q.e.d.}$$

The equality holds if and only if $\frac{a^2}{a} = \frac{b^2}{b}$, i.e. $a = b = \frac{1}{2}$.

Solution 7. We will use trigonometry, introducing the substitution $a = \sin^2 x, \quad b = \cos^2 x,$ where $x \in \left(0, \frac{\pi}{2}\right)$. We have:

$$a^3+b^3 = (a+b)\left(a^2-ab+b^2\right) = \left(\sin^2 x + \cos^2 x\right)\left(\sin^4 x - \sin^2 x.\cos^2 x + \cos^4 x\right) =$$

$$= \sin^4 x - \sin^2 x.\cos^2 x + \cos^4 x = \left(\sin^2 x + \cos^2 x\right)^2 - 3\sin^2 x.\cos^2 x = 1\text{-}\frac{3}{4}\sin^2 2x \ge \frac{1}{4},$$

because

$$\sin^2 2x \le 1, \text{ i.e. } -\sin^2 2x \ge -1, \text{ i.e. } -\frac{3}{4}\sin^2 2x \ge -\frac{3}{4},$$

and from here:

$$1\text{-}\frac{3}{4}\sin^2 2x \ge \frac{1}{4}.$$

Thus, the given inequality is proved. The equality holds if and only if $\sin^2 2x = 1$, i.e. $\sin 2x = 1$ and from here $2x = \frac{\pi}{2}$, i.e. $x = \frac{\pi}{4}$. Thus, $a = b = \sin^2 \frac{\pi}{4} = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$.

Solution 8. First, we will prove the following cubic inequality:

$$\frac{a^3 + b^3}{2} \ge \left(\frac{a+b}{2}\right)^3; \ (a,b > 0).$$

(10)

We will use the obvious inequality:

$$(a-b)^2 \ge 0$$

$$\Leftrightarrow (\text{because } a+b=1)\ (a-b)^2 (a+b) \ge 0$$

$$\Leftrightarrow (a-b)(a-b)(a+b) \ge 0$$

$$\Leftrightarrow (a-b)\left(a^2 - b^2\right) \ge 0$$

$$\Leftrightarrow a^3 - a^2 b - ab^2 + b^3 \ge 0 \,/ \cdot 3$$

$$\Leftrightarrow 3a^3 + 3b^3 \ge 3a^2 b + 3ab^2$$

$$\Leftrightarrow 4a^3 + 4b^3 \ge a^3 + 3a^2 b + 3ab^2 + b^3$$

$$\Leftrightarrow 4\left(a^3 + b^3\right) \ge (a+b)^3 \,/ : 8$$

$$\Leftrightarrow \frac{a^3 + b^3}{2} \ge \left(\frac{a+b}{2}\right)^3,$$

and this is the inequality (10). It follows now from the equality $a + b = 1$, that:

$$a^3 + b^3 \ge \frac{1}{4}, \text{ q.e.d.}$$

Th equality holds if and only if $a = b = \frac{1}{2}$.

Solution 9. We will apply the **Chebishev's inequality** for $n = 2$:

$$a_1 b_1 + a_2 b_2 \ge \frac{1}{2}(a_1 + a_2)(b_1 + b_2),$$

(11)

where $a_1 = a$, $a_2 = b$, $b_1 = a^2$, $b_2 = b^2$. Without WLOG we assume that $a \le b$. Thus:

$$0 \le a_1 \le a_2 \le 1 \text{ and } 0 \le b_1 \le b_2 \le 1, \text{ i.e.:}$$

$$0 < a \le b < 1 \text{ and } 0 < a^2 \le b^2 < 1.$$

We get now from (11):

$$a \cdot a^2 + b \cdot b^2 \geq \frac{1}{2}(a+b)(a^2+b^2)$$

$$\Leftrightarrow (\text{because } a+b=1)\ a^3+b^3 \geq \frac{1}{2}(a^2+b^2)$$

$$\Leftrightarrow a^3+b^3 \geq \frac{1}{2}\left[(a+b)^2-2ab\right]$$

$$\Leftrightarrow a^3+b^3 \geq \frac{1}{2}(1-2ab),$$

and from (7):

$$a^3+b^3 \geq \frac{1}{2}\left(1-\frac{1}{2}\right), \text{ i.e.}$$

$$a^3+b^3 \geq \frac{1}{4}, \text{ q.e.d.}$$

The equality holds if and only if $a=b=\frac{1}{2}$.

Solutions 2. and 3. are simple. The substitution $b=1-a$ or $a=1-b$ are quite logical.

Solution 4. is exceptionally simple. It is used oftenly in proofs of similar inequalities. Solution 5. applies a very important but well-known inequality, i.e. the inequality of Cauchy-Buniakovski-Schwarz in combination with the the inequality of the arithemtic/geometric means $(A \geq G)$ for two positive numbers. Solution 6. is connected with the inequality of the arithmetic/quadratic means $(A \leq Q)$. Solution 7. is based on fundamental knowledge of trigonometry: $\sin^2 x + \cos^2 x = 1\ (\forall x \in \mathbb{R})$, $\sin 2x = 2\sin x.\cos x$, $-1 \leq \sin 2x \leq 1$, $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Solution 8 uses a cubic inequality which is accompanied with a proof, for which we a proof is propwsed too. Solution 9. is based on the well-known inequality of Chebishev for two sequences of numbers which are monotonic. The inequality $A \geq G$ is used too. Concluding, the authors believe that the present paper will help students and their teachers to prepare participations in Mathematical Olympiads and competitions.

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