XIV APMO: Solutions and Marking Schemes

1. Let $a_1, a_2, a_3, \ldots, a_n$ be a sequence of non-negative integers, where n is a positive integer. Let

$$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Prove that

$$a_1!a_2!\ldots a_n! \geq (\lfloor A_n \rfloor!)^n,$$

where $\lfloor A_n \rfloor$ is the greatest integer less than or equal to A_n, and $a! = 1 \times 2 \times \cdots \times a$ for $a \geq 1$ (and $0! = 1$). When does equality hold?

Solution 1.

Assume without loss of generality that $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, and let $s = \lfloor A_n \rfloor$. Let k be any (fixed) index for which $a_k \geq s \geq a_{k+1}$.

Our inequality is equivalent to proving that

$$\frac{a_1!}{s!} \cdot \frac{a_2!}{s!} \cdot \cdots \cdot \frac{a_k!}{s!} \geq \frac{s!}{a_{k+1}!} \cdot \frac{s!}{a_{k+2}!} \cdot \cdots \cdot \frac{s!}{a_n!}. \tag{1}$$

Now for $i = 1, 2, \ldots, k$, $a_i!/s!$ is the product of $a_i - s$ factors. For example, $9!/5! = 9 \cdot 8 \cdot 7 \cdot 6$. The left side of inequality (1) therefore is the product of $A = a_1 + a_2 + \cdots + a_k - ks$ factors, all of which are greater than s. Similarly, the right side of (1) is the product of $B = (n-k)s - (a_{k+1} + a_{k+2} + \cdots + a_n)$ factors, all of which are at most s. Since $\sum_{i=1}^{n} a_i = nA_n \geq ns$, $A \geq B$. This proves the inequality. [**5 marks to here.**]

Equality in (1) holds if and only if either:

(i) $A = B = 0$, that is, both sides of (1) are the empty product, which occurs if and only if $a_1 = a_2 = \cdots = a_n$; or

(ii) $a_1 = 1$ and $s = 0$, that is, the only factors on either side of (1) are 1's, which occurs if and only if $a_i \in \{0, 1\}$ for all i. [**2 marks for both (i) and (ii), no marks for (i) only.**]

Solution 2.

Assume without loss of generality that $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$. Let $d = a_n - a_1$ and $m = |\{i : a_i = a_1\}|$. Our proof is by induction on d.

We first do the case $d = a_n - a_1 = 0$ or 1 separately. Then $a_1 = a_2 = \cdots = a_m = a$ and $a_{m+1} = \cdots = a_n = a + 1$ for some $1 \leq m \leq n$ and $a \geq 0$. In this case we have $\lfloor A_n \rfloor = a$, so the inequality to be proven is just $a_1!a_2!\ldots a_n! \geq (a!)^n$, which is obvious. Equality holds if and only if either $m = n$, that is, $a_1 = a_2 = \cdots = a_n = a$; or if $a = 0$, that is, $a_1 = \cdots = a_m = 0$ and $a_{m+1} = \cdots = a_n = 1$. [**2 marks to here.**]

So assume that $d = a_n - a_1 \geq 2$ and that the inequality holds for all sequences with smaller values of d, or with the same value of d and smaller values of m. Then the sequence

$$a_1 + 1,\ a_2,\ a_3,\ \ldots,\ a_{n-1},\ a_n - 1,$$

though not necessarily in non-decreasing order any more, does have either a smaller value of d, or the same value of d and a smaller value of m, but in any case has the same value of A_n. Thus, by induction and since $a_n > a_1 + 1$,

$$\begin{aligned} a_1!a_2!\dots a_n! &= (a_1+1)!a_2!\dots a_{n-1}!(a_n-1)!\cdot\frac{a_n}{a_1+1} \\ &\geq (\lfloor A_n\rfloor!)^n\cdot\frac{a_n}{a_1+1} \\ &> (\lfloor A_n\rfloor!)^n, \end{aligned}$$

which completes the proof. Equality cannot hold in this case.

2. Find all positive integers a and b such that

$$\frac{a^2+b}{b^2-a} \quad\text{and}\quad \frac{b^2+a}{a^2-b}$$

are both integers.

Solution.

By the symmetry of the problem, we may suppose that $a \leq b$. Notice that $b^2 - a \geq 0$, so that if $\dfrac{a^2+b}{b^2-a}$ is a positive integer, then $a^2+b \geq b^2-a$. Rearranging this inequality and factorizing, we find that $(a+b)(a-b+1) \geq 0$. Since $a, b > 0$, we must have $a \geq b-1$. [**3 marks to here.**] We therefore have two cases:

Case 1: $a = b$. Substituting, we have

$$\frac{a^2+a}{a^2-a} = \frac{a+1}{a-1} = 1 + \frac{2}{a-1},$$

which is an integer if and only if $(a-1)|2$. As $a > 0$, the only possible values are $a-1=1$ or 2. Hence, $(a,b) = (2,2)$ or $(3,3)$. [**1 mark.**]

Case 2: $a = b-1$. Substituting, we have

$$\frac{b^2+a}{a^2-b} = \frac{(a+1)^2+a}{a^2-(a+1)} = \frac{a^2+3a+1}{a^2-a-1} = 1 + \frac{4a+2}{a^2-a-1}.$$

Once again, notice that $4a+2 > 0$, and hence, for $\dfrac{4a+2}{a^2-a-1}$ to be an integer, we must have $4a+2 \geq a^2-a-1$, that is, $a^2-5a-3 \leq 0$. Hence, since a is an integer, we can bound a by $1 \leq a \leq 5$. Checking all the ordered pairs $(a,b) = (1,2), (2,3), \dots, (5,6)$, we find that only $(1,2)$ and $(2,3)$ satisfy the given conditions. [**3 marks.**]

Thus, the ordered pairs that work are

$$(2,2), (3,3), (1,2), (2,3), (2,1), (3,2),$$

where the last two pairs follow by symmetry. [**2 marks if these solutions are found without proof that there are no others.**]

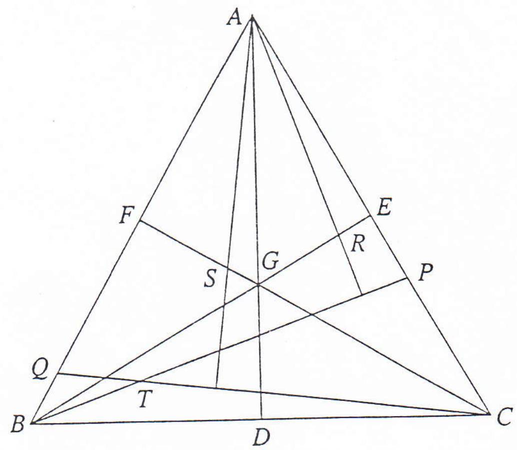
3. Let ABC be an equilateral triangle. Let P be a point on the side AC and Q be a point on the side AB so that both triangles ABP and ACQ are acute. Let R be the orthocentre of triangle ABP and S be the orthocentre of triangle ACQ. Let T be the point common to the segments BP and CQ. Find all possible values of $\angle CBP$ and $\angle BCQ$ such that triangle TRS is equilateral.

Solution.
We are going to show that this can only happen when

$$\angle CBP = \angle BCQ = 15°.$$

Lemma. If $\angle CBP > \angle BCQ$, then $RT > ST$.

Proof. Let AD, BE and CF be the altitudes of triangle ABC concurrent at its centre G. Then P lies on CE, Q lies on BF, and thus T lies in triangle BDG.



Note that

$$\angle FAS = \angle FCQ = 30° - \angle BCQ > 30° - \angle CBP = \angle EBP = \angle EAR.$$

Since $AF = AE$, we have $FS > ER$ so that

$$GS = GF - FS < GE - ER = GR.$$

Let T_x be the projection of T onto BC and T_y be the projection of T onto AD, and similarly for R and S. We have

$$R_xT_x = DR_x + DT_x > |DS_x - DT_x| = S_xT_x$$

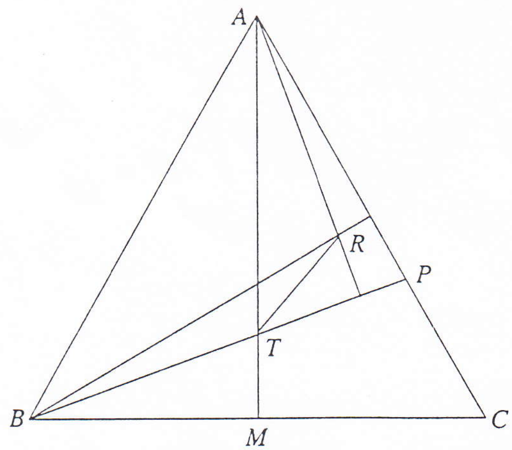
and

$$R_yT_y = GR_y + GT_y > GS_y + GT_y = S_yT_y.$$

It follows that $RT > ST$. □

[1 mark for stating the Lemma, 3 marks for proving it.]

Thus, if $\triangle TRS$ is equilateral, we must have $\angle CBP = \angle BCQ$.



It is clear from the symmetry of the figure that $TR = TS$, so $\triangle TRS$ is equilateral if and only if $\angle RTA = 30°$. Now, as BR is an altitude of the triangle ABC, $\angle RBA = 30°$. So $\triangle TRS$ is equilateral if and only if $RTBA$ is a cyclic quadrilateral. Therefore, $\triangle TRS$ is equilateral if and only if $\angle TBR = \angle TAR$. But

$$\begin{aligned} 90° &= \angle TBA + \angle BAR \\ &= (\angle TBR + \angle RBA) + (\angle BAT + \angle TAR) \\ &= (\angle TBR + 30°) + (30° + \angle TAR) \end{aligned}$$

and so

$$30° = \angle TAR + \angle TBR.$$

But these angles must be equal, so $\angle TAR = \angle TBR = 15°$. Therefore $\angle CBP = \angle BCQ = 15°$. [**3 marks for finishing the proof with the assumption that** $\angle CBP = \angle BCQ$.]

4. Let x, y, z be positive numbers such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$$

Show that

$$\sqrt{x + yz} + \sqrt{y + zx} + \sqrt{z + xy} \geq \sqrt{xyz} + \sqrt{x} + \sqrt{y} + \sqrt{z}.$$

Solution 1.

$$\begin{aligned} \sum_{\text{cyclic}} \sqrt{x + yz} &= \sqrt{xyz} \sum_{\text{cyclic}} \sqrt{\frac{1}{x} + \frac{1}{yz}} \\ &= \sqrt{xyz} \sum_{\text{cyclic}} \sqrt{\frac{1}{x}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) + \frac{1}{yz}} && \text{[1 mark.]} \\ &= \sqrt{xyz} \sum_{\text{cyclic}} \sqrt{\left(\frac{1}{x} + \frac{1}{y}\right)\left(\frac{1}{x} + \frac{1}{z}\right)} && \text{[1 mark.]} \end{aligned}$$

$$= \sqrt{xyz}\sum_{\text{cyclic}}\sqrt{\left(\frac{1}{x}+\frac{1}{\sqrt{yz}}\right)^2+\frac{(\sqrt{y}-\sqrt{z})^2}{xyz}} \qquad \text{[2 marks.]}$$

$$\geq \sqrt{xyz}\sum_{\text{cyclic}}\left(\frac{1}{x}+\frac{1}{\sqrt{yz}}\right) \qquad \text{[1 mark.]}$$

$$= \sqrt{xyz}\left(1+\sum_{\text{cyclic}}\frac{1}{\sqrt{yz}}\right) \qquad \text{[1 mark.]}$$

$$= \sqrt{xyz}+\sum_{\text{cyclic}}\sqrt{x}. \qquad \text{[1 mark.]}$$

Note. It is easy to check that equality holds if and only if $x = y = z = 3$.

Solution 2.

Squaring both sides of the given inequality, we obtain

$$\sum_{\text{cyclic}} x+\sum_{\text{cyclic}} yz+2\sum_{\text{cyclic}}\sqrt{x+yz}\sqrt{y+zx}$$

$$\geq xyz+2\sqrt{xyz}\sum_{\text{cyclic}}\sqrt{x}+\sum_{\text{cyclic}} x+2\sum_{\text{cyclic}}\sqrt{xy}. \qquad \text{[1 mark.]}$$

It follows from the given condition $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ that $xyz=\sum_{\text{cyclic}} xy$. Therefore, the given inequality is equivalent to

$$\sum_{\text{cyclic}}\sqrt{x+yz}\sqrt{y+zx}\geq\sqrt{xyz}\sum_{\text{cyclic}}\sqrt{x}+\sum_{\text{cyclic}}\sqrt{xy}. \qquad \text{[2 marks.]}$$

Using the Cauchy–Schwarz inequality [or just $x^2+y^2\geq 2xy$], we see that

$$(x+yz)(y+zx)\geq(\sqrt{xy}+\sqrt{xyz^2})^2, \qquad \text{[1 mark.]}$$

or

$$\sqrt{x+yz}\sqrt{y+zx}\geq\sqrt{xy}+\sqrt{z}\sqrt{xyz}. \qquad \text{[1 mark.]}$$

Taking the cyclic sum of this inequality over x, y and z, we get the desired inequality. [2 marks.]

Solution 3.

This is another way of presenting the idea in the first solution.

Using the condition $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ and the AM-GM inequality, we have

$$\begin{aligned} x+yz-\left(\sqrt{\frac{yz}{x}}+\sqrt{x}\right)^2 &= yz\left(1-\frac{1}{x}\right)-2\sqrt{yz} \\ &= yz\left(\frac{1}{y}+\frac{1}{z}\right)-2\sqrt{yz}=y+z-2\sqrt{yz}\geq 0, \end{aligned}$$

which gives

$$\sqrt{x+yz}\geq\sqrt{\frac{yz}{x}}+\sqrt{x}. \qquad \text{[3 marks.]}$$

Similarly, we have

$$\sqrt{y+zx} \geq \sqrt{\frac{zx}{y}} + \sqrt{y} \quad \text{and} \quad \sqrt{z+xy} \geq \sqrt{\frac{xy}{z}} + \sqrt{z}.$$

Addition yields

$$\sqrt{x+yz} + \sqrt{y+zx} + \sqrt{z+xy} \geq \sqrt{\frac{yz}{x}} + \sqrt{\frac{zx}{y}} + \sqrt{\frac{xy}{z}} + \sqrt{x} + \sqrt{y} + \sqrt{z}.$$

[2 marks.] Using the condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ again, we have

$$\sqrt{\frac{yz}{x}} + \sqrt{\frac{zx}{y}} + \sqrt{\frac{xy}{z}} = \sqrt{xyz}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \sqrt{xyz}, \qquad \textbf{[1 mark.]}$$

and thus

$$\sqrt{x+yz} + \sqrt{y+zx} + \sqrt{z+xy} \geq \sqrt{xyz} + \sqrt{x} + \sqrt{y} + \sqrt{z}. \qquad \textbf{[1 mark.]}$$

Solution 4.
This is also another way of presenting the idea in the first solution.
We make the substitution $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$. Then it is enough to show that

$$\sqrt{\frac{1}{a} + \frac{1}{bc}} + \sqrt{\frac{1}{b} + \frac{1}{ca}} + \sqrt{\frac{1}{c} + \frac{1}{ab}} \geq \sqrt{\frac{1}{abc}} + \sqrt{\frac{1}{a}} + \sqrt{\frac{1}{b}} + \sqrt{\frac{1}{c}},$$

where $a+b+c=1$. Multiplying this inequality by $\sqrt{abc}$, we find that it can be written

$$\sqrt{a+bc} + \sqrt{b+ca} + \sqrt{c+ab} \geq 1 + \sqrt{bc} + \sqrt{ca} + \sqrt{ab}. \qquad \textbf{[1 mark.]}$$

This is equivalent to

$$\sqrt{a(a+b+c)+bc} + \sqrt{b(a+b+c)+ca} + \sqrt{c(a+b+c)+ab}$$
$$\geq a+b+c+\sqrt{bc} + \sqrt{ca} + \sqrt{ab}, \qquad \textbf{[1 mark.]}$$

which in turn is equivalent to

$$\sqrt{(a+b)(a+c)} + \sqrt{(b+c)(b+a)} + \sqrt{(c+a)(c+b)} \geq a+b+c+\sqrt{bc} + \sqrt{ca} + \sqrt{ab}.$$

[1 mark.] (This is a *homogeneous* version of the original inequality.) By the Cauchy–Schwarz inequality (or since $b+c \geq 2\sqrt{bc}$), we have

$$[(\sqrt{a})^2 + (\sqrt{b})^2][(\sqrt{a})^2 + (\sqrt{c})^2] \geq (\sqrt{a}\sqrt{a} + \sqrt{b}\sqrt{c})^2$$

or

$$\sqrt{(a+b)(a+c)} \geq a + \sqrt{bc}. \qquad \textbf{[2 marks.]}$$

Taking the cyclic sum of this inequality over a, b, c, we get the desired inequality. **[2 marks.]**

5. Let $\mathbf{R}$ denote the set of all real numbers. Find all functions f from $\mathbf{R}$ to $\mathbf{R}$ satisfying:
(i) there are only finitely many s in $\mathbf{R}$ such that $f(s) = 0$, and
(ii) $f(x^4 + y) = x^3 f(x) + f(f(y))$ for all x, y in $\mathbf{R}$.

Solution 1.

The only such function is the identity function on $\mathbf{R}$.

Setting $(x, y) = (1, 0)$ in the given functional equation (ii), we have $f(f(0)) = 0$. Setting $x = 0$ in (ii), we find

$$f(y) = f(f(y)) \tag{1}$$

[1 mark.] and thus $f(0) = f(f(0)) = 0$ **[1 mark.]**. It follows from (ii) that $f(x^4 + y) = x^3 f(x) + f(y)$ for all $x, y \in \mathbf{R}$. Set $y = 0$ to obtain

$$f(x^4) = x^3 f(x) \tag{2}$$

for all $x \in \mathbf{R}$, and so

$$f(x^4 + y) = f(x^4) + f(y) \tag{3}$$

for all $x, y \in \mathbf{R}$. The functional equation (3) suggests that f is *additive*, that is, $f(a + b) = f(a) + f(b)$ for all $a, b \in \mathbf{R}$. **[1 mark.]** We now show this.

First assume that $a \geq 0$ and $b \in \mathbf{R}$. It follows from (3) that

$$f(a + b) = f((a^{1/4})^4 + b) = f((a^{1/4})^4) + f(b) = f(a) + f(b).$$

We next note that f is an *odd* function, since from (2)

$$f(-x) = \frac{f(x^4)}{(-x)^3} = \frac{f(x^4)}{-x^3} = -f(x), \quad x \neq 0.$$

Since f is odd, we have that, for $a < 0$ and $b \in \mathbf{R}$,

$$\begin{aligned} f(a + b) &= -f((-a) + (-b)) = -(f(-a) + f(-b)) \\ &= -(-f(a) - f(b)) = f(a) + f(b). \end{aligned}$$

Therefore, we conclude that $f(a + b) = f(a) + f(b)$ for all $a, b \in \mathbf{R}$. **[2 marks.]**

We now show that $\{s \in \mathbf{R} | f(s) = 0\} = \{0\}$. Recall that $f(0) = 0$. Assume that there is a nonzero $h \in \mathbf{R}$ such that $f(h) = 0$. Then, using the fact that f is additive, we inductively have $f(nh) = 0$ or $nh \in \{s \in \mathbf{R} | f(s) = 0\}$ for all $n \in \mathbf{N}$. However, this is a contradiction to the given condition (i). **[1 mark.]**

It's now easy to check that f is *one-to-one*. Assume that $f(a) = f(b)$ for some $a, b \in \mathbf{R}$. Then, we have $f(b) = f(a) = f(a - b) + f(b)$ or $f(a - b) = 0$. This implies that $a - b \in \{s \in \mathbf{R} | f(s) = 0\} = \{0\}$ or $a = b$, as desired. From (1) and the fact that f is one-to-one, we deduce that $f(x) = x$ for all $x \in \mathbf{R}$. **[1 mark.]** This completes the proof.

Solution 2.

Again, the only such function is the identity function on $\mathbf{R}$.

As in Solution 1, we first show that $f(f(y)) = f(y)$, $f(0) = 0$, and $f(x^4) = x^3 f(x)$. **[2 marks.]** From the latter follows

$$f(x) = 0 \Longrightarrow f(x^4) = 0,$$

and from condition (i) we get that $f(x) = 0$ only possibly for $x \in \{0, 1, -1\}$. [**1 mark.**]

Next we prove

$$f(a) = b \Longrightarrow f\left(\sqrt[4]{|a-b|}\right) = 0.$$

This is clear if $a = b$. If $a > b$ then

$$\begin{aligned} f(a) &= f((a-b)+b) = (a-b)^{3/4} f(\sqrt[4]{a-b}) + f(f(b)) \\ &= (a-b)^{3/4} f(\sqrt[4]{a-b}) + f(b) \\ &= (a-b)^{3/4} f(\sqrt[4]{a-b}) + f(f(a)) \\ &= (a-b)^{3/4} f(\sqrt[4]{a-b}) + f(a), \end{aligned}$$

so $(a-b)^{3/4} f(\sqrt[4]{a-b}) = 0$ which means $f\left(\sqrt[4]{|a-b|}\right) = 0$. If $a < b$ we get similarly

$$\begin{aligned} f(b) &= f((b-a)+a) = (b-a)^{3/4} f(\sqrt[4]{b-a}) + f(f(a)) \\ &= (b-a)^{3/4} f(\sqrt[4]{b-a}) + f(b), \end{aligned}$$

and again $f\left(\sqrt[4]{|a-b|}\right) = 0$. [**2 marks.**]

Thus $f(a) = b \Longrightarrow |a-b| \in \{0, 1\}$. Suppose that $f(x) = x + b$ for some x, where $|b| = 1$. Then from $f(x^4) = x^3 f(x)$ and $f(x^4) = x^4 + a$ for some $|a| \leq 1$ we get $x^3 = a/b$, so $|x| \leq 1$. Thus $f(x) = x$ for all x except possibly $x = \pm 1$. [**1 mark.**] But for example,

$$f(1) = f(2^4 - 15) = 2^3 f(2) + f(f(-15)) = 2^3 \cdot 2 - 15 = 1$$

and

$$f(-1) = f(2^4 - 17) = 2^3 f(2) + f(f(-17)) = 2^3 \cdot 2 - 17 = -1.$$

[**1 mark.**] This finishes the proof.