The 3rd Romanian Master of Mathematics Competition – Solutions
Day 1: Friday, February 26, 2010, Bucharest

Problem 1. For a finite non-empty set of primes P, let $m(P)$ be the largest possible number of consecutive positive integers, each of which is divisible by at least one member of P. (In the sequel, the number $|P|$ is the size of the set P.)

(i) Show that $|P| \le m(P)$, with equality if and only if $\min(P) > |P|$;

(ii) Show that $m(P) < (|P|+1)(2^{|P|}-1)$.

Romania, Dan Schwarz

Solution. In the sequel we will consider P being made of the primes $1 < p_1 < p_2 < \cdots < p_k$, with $k = |P| \ge 1$.

(i) By the Chinese Remainder Theorem there will exist some $a \in \mathbb{N}$ such that $a \equiv -i \pmod{p_i}$, hence $p_i \mid a+i$. Then the set $\{a+i\ ;\ i = 1,2,\ldots,k\}$ of k consecutive integers has the desired property, hence $m(P) \ge k$. When $\min P > |P|$, within any set of $|P|+1$ consecutive integers at most one is divisible by any $p \in P$, hence by the Pigeonhole Principle there will be one not divisible by any of the primes in P. On the other hand, when $\min P \le |P|$, we will make again use of the Chinese Remainder Theorem, so there will exist some $a \in \mathbb{N}$ such that $a \equiv -r_i \pmod{p_i}$, hence $p_i \mid a + r_i$, where $\{r_i\ ;\ i = 1,2,\ldots,k\} = \{1,2,\ldots,k\}$ and the extra requirement that $r_1 = k+1-p_1$. It follows that the set $\{a+i\ ;\ i = 1,2,\ldots,k,k+1\}$ of $k+1$ consecutive integers has the desired property, hence $m(P) \ge k+1 > |P|$.

(ii) Now, let a set made of m consecutive integers have the desired property. For any $\emptyset \ne I \subseteq \{1,2,\ldots,k\}$, the number $N(I)$ of its elements which are divisible by $\prod_{i\in I} p_i$ will satisfy the inequality

$$\frac{m}{\prod_{i\in I} p_i} - 1 < \left\lfloor \frac{m}{\prod_{i\in I} p_i} \right\rfloor \le N(I) \le \left\lceil \frac{m}{\prod_{i\in I} p_i} \right\rceil < \frac{m}{\prod_{i\in I} p_i} + 1.$$

Then, by the Principle of Inclusion/Exclusion, one has

$$m = \sum_{i=1}^{k} (-1)^{i+1} \sum_{|I|=i} N(I) < \sum_{i=1}^{k} \binom{k}{i} + m \sum_{i=1}^{k} (-1)^{i+1} \sum_{|I|=i} \frac{1}{\prod_{i\in I} p_i}.$$

The first term is clearly equal to $2^k - 1$, while the second is equal to

$$m\left(1 - \prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right)\right) \le m\left(1 - \prod_{i=1}^{k}\left(1 - \frac{1}{i+1}\right)\right) = m - \frac{m}{k+1},$$

therefore $m < (k+1)\left(2^k - 1\right)$, and so will be $m(P)$. ■

Remarks.[1] A simpler variant could be
(ii) Prove that $|P| \le m(P) \le \max\limits_{|P'|=|P|} m(P') < \infty$.[2]

Problem 2. For each positive integer n, find the largest real number C_n with the following property. Given any n real-valued functions $f_1(x), f_2(x), \ldots, f_n(x)$ defined on the closed interval $0 \le x \le 1$, one can find numbers $x_1, x_2, \ldots, x_n$, such that $0 \le x_i \le 1$, satisfying

$$\left|f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n) - x_1 x_2 \cdots x_n\right| \ge C_n.$$

Serbia, Marko Radovanović

Solution. First we will prove that $C_n \ge \dfrac{n-1}{2n}$, i.e. that for any n functions $f_1, f_2, \ldots, f_n : [0,1] \to \mathbb{R}$, there exist numbers $x_1, x_2, \ldots, x_n$ in $[0,1]$ such that

$$|f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n) - x_1 x_2 \cdots x_n| \ge \frac{n-1}{2n}.$$

For $n = 1$ this is trivial. For $n \ge 2$ suppose, contrariwise, that for all $x_1, x_2, \ldots, x_n$ in $[0,1]$ we have

$$|f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n) - x_1 x_2 \cdots x_n| < \frac{n-1}{2n}.$$

Plugging in $x_i = 1$ for $1 \le i \le n$, we get $\left|\sum_{i=1}^{n} f_i(1) - 1\right| < \dfrac{n-1}{2n}$.
Plugging in $x_i = 0$ for $1 \le i \le n$, we get $\left|\sum_{i=1}^{n} f_i(0)\right| < \dfrac{n-1}{2n}$.
Plugging in (for every $1 \le i \le n$) $x_i = 0$ and $x_j = 1$ for all $j \ne i$, we get $\left|f_i(0) + \sum_{j\ne i} f_j(1)\right| < \dfrac{n-1}{2n}$. Since

$$(n-1)\sum_{i=1}^{n} f_i(1) = \sum_{i=1}^{n}\left(f_i(0) + \sum_{j\ne i} f_j(1)\right) - \sum_{i=1}^{n} f_i(0),$$

by the triangle inequality we have

$$(n-1)\left|\sum_{i=1}^{n} f_i(1)\right| < (n+1)\frac{n-1}{2n}.$$

On the other hand, by again the triangle inequality we have

$$1 \le \left|\sum_{i=1}^{n} f_i(1)\right| + \left|\sum_{i=1}^{n} f_i(1) - 1\right| < \frac{n+1}{2n} + \frac{n-1}{2n} = 1,$$

which is a contradiction.

To prove that $\boxed{C_n = \dfrac{n-1}{2n}}$ is the largest constant, it will be sufficient to prove that for the n (equal) functions

$$f_i(x) = f(x) := \frac{x^n}{n} - \frac{n-1}{2n^2},\quad 1 \le i \le n,$$

and any n numbers $x_1, x_2, \ldots, x_n$ in $[0,1]$, we have

$$|f(x_1) + f(x_2) + \cdots + f(x_n) - x_1 x_2 \cdots x_n| \le \frac{n-1}{2n},$$

equivalent to

$$-\frac{n-1}{2n} \le \frac{1}{n}\sum_{i=1}^{n} x_i^n - \prod_{i=1}^{n} x_i - \frac{n-1}{2n} \le \frac{n-1}{2n}.$$

The LHS inequality follows from the AM-GM inequality. The RHS inequality is equivalent to

$$F(\boldsymbol{x}) = F(x_1, x_2, \ldots, x_n) := \frac{1}{n}\sum_{i=1}^{n} x_i^n - \prod_{i=1}^{n} x_i \le \frac{n-1}{n}$$

at all points $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ of the hypercube $[0,1]^n$. Since F is convex in every variable, its maximum is reached at some vertex $\boldsymbol{v}$ of the hypercube (point with $x_i = 0$ or $x_i = 1$, for all $1 \le i \le n$). It is easy to see that for all such points we have $F(v) \le \dfrac{n-1}{n}$, which completes the proof. ■

Remarks. The choice of the functions

$$f_i(x) := \frac{x^n}{n} - \frac{n-1}{2n^2}, \ 1 \le i \le n$$

could be justified by the fact that if we try all $f_i = f$ and all $x_i = x$, the relation becomes $|nf(x) - x^n| \ge C_n$, as tight as possible for some $x \in [0,1]$. Then $f(x) = \dfrac{x^n}{n} - \dfrac{1}{n}C_n$ is a potential candidate.

On a different note, RHS inequality above is equivalent to

$$F(\boldsymbol{x}) = F(x_1, x_2, \dots, x_n) := \frac{1}{n}\sum_{i=1}^{n} x_i^n - \prod_{i=1}^{n} x_i \ge 0$$

at all points $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ of the hypercube $[0,1]^n$, and this again may be justified by F being convex, since it can be easily seen that $F(\boldsymbol{v}) \ge 0$ at any vertex $\boldsymbol{v}$ of the hypercube, so one may use a unifying argument for both sides of that inequality.

Problem 3. Let $A_1A_2A_3A_4$ be a convex quadrilateral with no pair of parallel sides. For each $i = 1,2,3,4$, define ω_i to be the circle touching the quadrilateral externally, and which is tangent to the lines $A_{i-1}A_i$, A_iA_{i+1} and $A_{i+1}A_{i+2}$ (indices are considered modulo 4, so $A_0 = A_4$, $A_5 = A_1$ and $A_6 = A_2$). Let T_i be the point of tangency of ω_i with the side A_iA_{i+1}. Prove that the lines A_1A_2, A_3A_4 and T_2T_4 are concurrent if and only if the lines A_2A_3, A_4A_1 and T_1T_3 are concurrent.

Russia, Pavel Kozhevnikov

Solution. We start with a reformulation of a well-known statement on harmonic cyclic quadruples (K_1, K_2, K_3, K_4), also provable by polar transformation (projective methods).

Lemma. Being given four pairwise non-parallel lines ℓ_i, $i = 1,2,3,4$, tangent to a circle ω at points K_i, and such that lines ℓ_1, ℓ_3 and K_2K_4 are concurrent, then lines ℓ_2, ℓ_4 and K_1K_3 are also concurrent.

Proof. Let O be the centre of ω, $X = \ell_1 \cap \ell_3 \cap K_2K_4$, $Y = \ell_2 \cap \ell_4$. We have $OX \perp K_1K_3$ and $OY \perp K_2K_4$. Let $Z = OX \cap K_1K_3$, $T = OY \cap K_2K_4$. Notice that triangles $\triangle OK_3X$ and $\triangle OZK_3$ are similar, and also similar are triangles $\triangle OK_4Y$ and $\triangle OTK_4$, hence $OY \cdot OT = OK_4^2 = OK_3^2 = OX \cdot OZ$.

This means that triangles $\triangle OXT$ and $\triangle OYZ$ are similar, hence $YZ \perp OX$, and so $Y \in K_1K_3$. □

Suppose now lines A_2A_3, A_4A_1 and T_1T_3 are concurrent at a point P. Let T_4', T_2' be the tangency points of lines A_4A_1, respectively A_2A_3, to circle ω_1, and let T_3' be the second meeting point of line T_1T_3 and circle ω_1. Let the tangent to ω_1 at T_3' meet the lines A_4A_1, A_2A_3 at points A_4', respectively A_3'. The (direct) homothety of centre P that takes ω_1 to ω_3 maps T_3' to T_3, hence $A_3A_4 \parallel A_3'A_4'$.

Let $Q = A_1A_2 \cap A_3A_4$, $Q' = A_1A_2 \cap A_3'A_4'$. Applying the *Lemma* to circle ω_1 and lines A_2A_3, $A_3'A_4'$, A_4A_1, A_1A_2, yields that points Q', T_2', T_4' are collinear. The (inverse) homothety of centre A_1 that takes $\triangle QA_1A_4$ to $\triangle Q'A_1A_4'$ maps ω_4 to ω_1, so maps T_4 to T_4', hence $Q'T_4' \parallel QT_4$. Similarly, the (inverse) homothety of centre A_2 that takes $\triangle QA_2A_3$ to $\triangle Q'A_2A_3'$ maps ω_2 to ω_1, so maps T_2 to T_2', hence also $Q'T_2' \parallel QT_2$. Since points Q', T_2', T_4' are collinear, it follows points Q, T_2, T_4 are also collinear.

The converse implication is done in a similar way, due to the cyclic nature of the notations used (just increase each index by 1). ■

Alternative Solution. (D. Şerbănescu) Suppose Q, T_2, T_4 are collinear. We will show P, T_1, T_3 are collinear. We will use the notations of the solution above, but also let S_1', S_1'' be the tangency points of line A_1A_2 to circle ω_2, respectively ω_4, and let S_3', S_3'' be the tangency points of line A_3A_4 to circle ω_2, respectively ω_4. Let T_4'' be the (other than T_2) meeting point of line QT_2T_4 and circle ω_2, and let the tangent line to ω_2 at T_4'' (parallel to A_1A_4) meet A_2A_3 at P' (via the (direct) homothety of centre Q that takes ω_4 to ω_2).

Clearly $\triangle PT_2T_4 \sim \triangle P'T_2T_4''$ and $\triangle P'T_2T_4''$ is isosceles, so $PT_2 = PT_4$ (in other words, if Q, T_2, T_4 are collinear then $PT_2 = PT_4$; the other implication trivially also holds, but is irrelevant here).

From $PT_2 = PT_4$ and $PT_2' = PT_4'$ follows $T_2'T_2 = T_4'T_4$. As external tangents, $T_2'T_2 = T_1S_1'$ and $T_4'T_4 = T_1S_1''$, hence T_1 is the midpoint of $S_1'S_1''$. Similarly, T_3 is the midpoint of $S_3'S_3''$. It follows that P, T_1, T_3 lie on the radical axis of the circles ω_2 and ω_4, hence are collinear.

The converse implication is done in a similar way. ■

Remarks.

1. The statement remains true if points $\top_i$ are replaced by their symmetrical $\perp_i$ with respect to the midpoints of the segments A_iA_{i+1}.

2. Via some trigonometrical computations, one obtains that both conditions in the statement are equivalent to the condition $\sin\dfrac{A_1}{2}\sin\dfrac{A_3}{2} = \sin\dfrac{A_2}{2}\sin\dfrac{A_4}{2}$.

END

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[1] When the primes in P are the first $|P|$ consecutive primes $q_1, q_2, \dots, q_k$, it is easy to see that the set of integers between 1 and the next prime q_{k+1} has the desired property, so $m(P) \ge q_{k+1} - 2$. It is obvious that one can check just the positive integers less than $\prod_{1\le i\le k} q_i$ in order to verify that this is indeed the value of $m(P)$, and this conjecture seems to hold, since it is true for $k = 1,2,3,4,5$. However, for $k = 6$, there exists a larger set than (prescribed) between 1 and 13, made by the numbers between 113 and 127.

[2] The **proof** goes by simple induction on $k = |P|$. Denote $m(k) = \max\limits_{|P|=k} m(P)$; it is quite clear that $m(1) = 1$. Assume therefore that $m(k) < \infty$ and consider sets P with $|P| = k+1$. If $\max P > 2m(k)+1$, then $m(P) < 2m(k)+2$, since within any set of $2m(k)+2$ consecutive integers at most one is divisible by $\max P$, so there would exist a subset of at least $m(k)+1$ consecutive integers for $P \setminus \{\max P\}$, absurd. If $\max P \le 2m(k)+1$, then there are a finite number of such sets P, and clearly $m(P) < \prod_{p\in P} p$, so all will have a common upper bound. It follows that $m(k+1) < \infty$. □

The 3rd Romanian Master of Mathematics Competition – Solutions
Day 2: Saturday, February 27, 2010, Bucharest

Problem 4. Determine whether there exist a polynomial $f(x_1, x_2)$ in two variables, with integer coefficients, and two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ in the plane, satisfying all the following conditions:

(i) A is an integer point (i.e., a_1 and a_2 are integers);

(ii) $|a_1 - b_1| + |a_2 - b_2| = 2010$;

(iii) $f(n_1, n_2) > f(a_1, a_2)$, for all integer points (n_1, n_2) in the plane other than A;

(iv) $f(x_1, x_2) > f(b_1, b_2)$, for all points (x_1, x_2) in the plane other than B.

Italy, Massimo Gobbino

Solution. The triple $(f(x_1, x_2), A, B)$ does exist, so $\boxed{\text{YES}}$.

Let $A = O = (0,0)$, $B = (x_0, y_0) = (2009 + \frac{2}{3}, \frac{1}{3})$. The idea is to search for a polynomial f such that $f(x, y) = 0$ is the equation of an ellipse centred at B, passing through O and with tangent line $y = 0$ at O. In fact, if f is chosen like this, the ellipse $f(x, y) = 0$ is completely contained in the region $\{(x, y)\ ;\ 0 \le y \le \frac{2}{3}\}$, with O the only integer point on the ellipse or in its interior; clearly, the absolute minimum of $f(x, y)$ is attained at B and $f(x, y)$ is positive at all integer points other than O. Therefore, we consider polynomials of the type

$$f(X, Y) = 9M(X - x_0)^2 + 9N(X - x_0)(Y - y_0) + 9P(Y - y_0)^2 - Q$$

where M, N, P, Q are integers with $M, P, Q, 4MP - N^2 > 0$.

The condition that the ellipse $f(x, y) = 0$ passes through O, with tangent line $y = 0$ at O, is expressed by

$$\begin{cases} 6029^2 M + 6029N + P - Q = 0 \\ 2 \cdot 6029M + N = 0. \end{cases}$$

It is then sufficient to choose $M = 1$, $N = -2 \cdot 6029$, P any integer greater than 6029^2 and $Q = P - 6029^2$. ■

Alternative Solution.[1] (D. Schwarz)

Given any integer point $A(a_1, a_2)$, there exist infinitely many points $B(b_1, b_2)$ with $b_1, b_2 \in \mathbb{Q} \setminus \mathbb{Z}$, and such that $|a_1 - b_1| + |a_2 - b_2| = 2010$, for example $b_1 = a_1 + \alpha + r$, $b_2 = a_2 + \beta + (1 - r)$, with $\alpha, \beta \in \mathbb{Z}_+$, $r \in \mathbb{Q} \cap (0, 1)$, and $\alpha + \beta = 2009$. We now will consider polynomials of the type

$$f(X, Y) = N\big((X - a_1)^2 + (Y - a_2)^2 + \varepsilon\big)\big((X - b_1)^2 + (Y - b_2)^2\big)$$

where $\varepsilon \in \mathbb{Q}_+^*$ and $N \in \mathbb{Z}_+^*$ large enough for $f(X, Y)$ to have integer coefficients.

One then has $f(b_1, b_2) = 0$, while $f(x, y) > 0$ for all points (x, y) in the plane, other than B.

One also then has $f(a_1, a_2) = N\varepsilon\big((a_1 - b_1)^2 + (a_2 - b_2)^2\big)$, while one has, for all integer points (n_1, n_2) in the plane, other than A, $\min\big((n_1 - a_1)^2 + (n_2 - a_2)^2 + \varepsilon\big) = 1 + \varepsilon$ and $\min\big((n_1 - b_1)^2 + (n_2 - b_2)^2\big) = m$, for some $m \in \mathbb{Q} \cap (0, \frac{1}{2}]$ (for example $m = \frac{1}{2}$ when $r = \frac{1}{2}$), therefore $F(n_1, n_2) > N(1+\varepsilon)m$.

In order to have $f(n_1, n_2) > f(a_1, a_2)$ it is thus enough that $N(1+\varepsilon)m \ge N\varepsilon\big((a_1 - b_1)^2 + (a_2 - b_2)^2\big)$, therefore let us take $\varepsilon = m / \big((a_1 - b_1)^2 + (a_2 - b_2)^2 - m\big) \in \mathbb{Q}_+^*$, and then choose some appropriate N. ■

Remarks. It is a case of applying the knowledge that some closed simple curve given by $f(x, y) = 0$ separates the plane into two regions, with values of one sign in the interior region, and values of the opposite sign in the exterior region. Once this idea comes to mind, the problem turns into a simple exercise in analytic geometry of the conics.

Notice that the point $C(2x_0, 2y_0)$, the symmetrical of A with respect to B, also lies on the ellipse $F(x, y) = 0$. What in fact we have done is to take the circle given by equation $\Gamma(x, y) = (x - x_0)^2 + (y - y_0)^2 - (x_0^2 + y_0^2)$ and then stretch it on a direction perpendicular to OB, until a resulting rational ellipse contains no other integer points than A.

Problem 5. Let n be a given positive integer. Say that a set K of points with integer coordinates in the plane is *connected* if for every pair of points $R, S \in K$, there exist a positive integer ℓ and a sequence $R = T_0, T_1, \dots, T_\ell = S$ of points in K, where each T_i is distance 1 away from T_{i+1}. For such a set K, we define the set of vectors

$$\Delta(K) = \{\overrightarrow{RS} \mid R, S \in K\}.$$

What is the maximum value of $|\Delta(K)|$ over all connected sets K of $2n + 1$ points with integer coordinates in the plane?

Russia, Grigory Chelnokov

Solution. We claim the answer is $\boxed{2n^2 + 4n + 1}$. A model is $K = \{(0,0)\} \cup \{(i, 0)\ ;\ 1 \le i \le n\} \cup \{(0, i)\ ;\ 1 \le i \le n\}$, when $W = \{(a, -b)\ ;\ 0 \le a, b \le n\} \cup \{(-a, b)\ ;\ 0 \le a, b \le n\}$. It is left to prove that $|W| \le 2n^2 + 4n + 1$ for any set K.

What the statement of the problem describes is a connected graph $G = (K, E)$ of order $2n + 1$, whose vertices are the points in K, while the edges are the horizontal/vertical segments of length 1 that connect (some of) these points. The key to the proof is to sequence the elements of K as $A_0, A_1, \dots, A_{2n}$ such that the graph $G_k := G[A_0, A_1, \dots, A_k]$ is connected for every $1 \le k \le 2n$; this can be done through

Lemma.[2] The vertices of a finite connected graph G can always be enumerated, say as a sequence $v_0, \dots, v_{|G|-1}$, so that $G_k := G[v_0, \dots, v_k]$ is connected for every $1 \le k \le |G| - 1$.

Proof. Pick any vertex as v_0, and assume inductively that $v_0, \dots, v_k$ have been chosen for some $0 \le k < |G| - 1$. Now pick a vertex $v \in G - G_k$. As G is connected, it contains a $v - v_0$ path P. Choose as v_{k+1} the last vertex of P in $G - G_k$; then v_{k+1} has as neighbour in G_k the next vertex of P. The connectedness of every G_k follows by induction on k. □

Moreover, if we just keep the edges through which A_{k+1} has the (selected) neighbour in G_k, then G_k is a tree, and so G_{2n} is a spanning tree of G. Call the vertex *horizontal* (*vertical*) if the edge that connects him is horizontal (vertical). Denote by h, respectively v, the number of horizontal, respectively vertical vertices; since G_{2n} is a tree, it follows $h + v = 2n$. The point A_0 contributes $2n + 1$ vectors $\overrightarrow{A_0 A_i}$. Now, for $0 \le k \le 2n$, each point A_{k+1} contributes at most $(2n + 1) - x$ new vectors, where $x = h$ if the vertex is horizontal, respectively $x = v$ if the vertex is vertical, since those vectors $\overrightarrow{A_{k+1} A_i}$, with ends at the corresponding edges of same direction, will be duplicated by the vectors determined by the opposite parallel sides of the parallelograms created, which have already been accounted for.

Therefore the total number of distinct vectors will be $|W| \le (2n+1)^2 - h^2 - v^2$. But $h^2+v^2 \ge \frac{1}{2}(h+v)^2 = 2n^2$, hence $|W| \le (2n+1)^2 - 2n^2 = 2n^2+4n+1$.[3] ■

Problem 6. Given a polynomial $f(x)$ with rational coefficients, of degree $d \ge 2$, we define the sequence of sets $f^0(\mathbb{Q}), f^1(\mathbb{Q}), \dots$ by $f^0(\mathbb{Q}) = \mathbb{Q}$ and $f^{n+1}(\mathbb{Q}) = f(f^n(\mathbb{Q}))$ for $n \ge 0$. (Given a set S, we write $f(S)$ for the set $\{f(x) \mid x \in S\}$.)

Let $f^\omega(\mathbb{Q}) = \bigcap_{n=0}^\infty f^n(\mathbb{Q})$ be the set of numbers that are in all of the sets $f^n(\mathbb{Q})$. Prove that $f^\omega(\mathbb{Q})$ is a finite set.

Romania, Dan Schwarz

Solution. For any function f, denote its n-th iterate f^n. Take $d = \deg f \ge 2$. One can write $f(x) = \frac{1}{N}(ax^d + g(x))$ for some $N \in \mathbb{Z}_+^*$, $a \in \mathbb{Z}^*$, and some $g \in \mathbb{Z}[x]$, with $\deg g \le d-1$, $g(x) = \sum_{i=0}^{d-1} a_i x^i$, $a_i \in \mathbb{Z}$, for all $0 \le i \le d-1$.

Finally, $f^\omega(\mathbb{Q}) \subset f^n(\mathbb{Q}) \subset f^{n-1}(\mathbb{Q}) \subseteq \mathbb{Q}$, for $n \ge 1$.

For any $x \in \mathbb{Q}$, one can uniquely write $x = \frac{\mu(x)}{\nu(x)}$, with $\mu(x), \nu(x) \in \mathbb{Z}$, and $\nu(x) > 0$, $\gcd(\mu(x), \nu(x)) = 1$. Take now

$$M = \frac{\sum_{i=0}^{d-1} |a_i| + 2N}{|a|} + 1, \text{ and}$$

$$\mathcal{M} := \{x \in \mathbb{Q}\,;\, |x| > M\}, \quad \mathcal{F} := \{x \in \mathbb{Q}\,;\, \nu(x) > a^2\}.$$

Now, $(\mathbb{Q} \setminus \mathcal{F}) \cap (\mathbb{Q} \setminus \mathcal{M})$ is obviously finite; take $\mathfrak{m}$ to be its cardinality.

For $x \in \mathcal{M}$ one has $|f(x)| \ge |x| + 1 > M$ (see *Lemma 1*), hence $f(x) \in \mathcal{M}$. Then $|f^n(x)| \ge |x| + n$. For $x \in \mathcal{F}$ one has $\nu(f(x)) \ge \nu(x) + 1 > a^2$ (see *Lemma 2*), hence $f(x) \in \mathcal{F}$. Then $\nu(f^n(x)) \ge \nu(x) + n$.

Take $x \in f^\omega(\mathbb{Q})$, and take n large enough. Then we will have $x \in f^n(\mathbb{Q})$, hence there will exist $x_n \in \mathbb{Q}$ such that $f^n(x_n) = x$. If $f^k(x_n) \in \mathcal{M}$ for $n-k > |x|$, then $|x| = |f^n(x_n)| = |f^{n-k}(f^k(x_n))| \ge |f^k(x_n)| + (n-k) > |x|$, absurd. If $f^k(x_n) \in \mathcal{F}$ for $n-k > \nu(x)$, then $\nu(x) = \nu(f^n(x_n)) = \nu(f^{n-k}(f^k(x_n))) \ge \nu(f^k(x_n)) + (n-k) > \nu(x)$, absurd.

Take n large enough so that $n > \mathfrak{m} + \max(|x|, \nu(x))$. One then has $f^k(x_n) \in (\mathbb{Q} \setminus \mathcal{F}) \cap (\mathbb{Q} \setminus \mathcal{M})$, for $0 \le k \le \mathfrak{m}$, therefore there will exist $0 \le i < j \le \mathfrak{m}$ such that $f^i(x_n) = f^j(x_n)$, therefore $f^n(x_n) = f^k(x_n)$ for some $i \le k \le j$, hence $x = f^n(x_n) = f^k(x_n) \in (\mathbb{Q} \setminus \mathcal{F}) \cap (\mathbb{Q} \setminus \mathcal{M})$.

This implies $f^\omega(\mathbb{Q}) \subseteq (\mathbb{Q} \setminus \mathcal{F}) \cap (\mathbb{Q} \setminus \mathcal{M})$, thus a finite set. ■

Lemma 1. For $x \in \mathcal{M}$ one has $|f(x)| \ge |x| + 1 > M$.

Proof. Clearly then $|x| > M > 1$. Now $\frac{|a||x|^d}{N} > \frac{\sum_{i=0}^{d-1} |a_i||x|^{d-1}}{N} + |x|^d + |x|^d > \frac{\sum_{i=0}^{d-1} |a_i||x|^i}{N} + |x| + 1 \ge \frac{|g(x)|}{N} + |x| + 1$. It follows that $|f(x)| = \left|\frac{ax^d + g(x)}{N}\right| \ge \left|\frac{|a||x|^d}{N} - \frac{|g(x)|}{N}\right| > |x| + 1 > M$. □

Lemma 2. For $x \in \mathcal{F}$ one has $\nu(f(x)) \ge \nu(x) + 1 > a^2$.

Proof. For $x \in \mathcal{F}$ one can write $f(x) = \frac{1}{N\nu(x)^d}(a\mu(x)^d + \nu(x)z) = \frac{\mu(f(x))}{\nu(f(x))}$, with $z = \nu(x)^{d-1}g(x) \in \mathbb{Z}$. Now, for $e = \gcd(\nu(x), a)$, one has $\nu(x) = er$, $a = eb$, with $\gcd(r, b) = 1$.

Then it follows that $\delta = \gcd(a\mu(x)^d + \nu(x)z, N\nu(x)^d) \le N \cdot \gcd(eb\mu(x)^d + erz, e^d r^d) = Ne \cdot \gcd(b\mu(x)^d + rz, e^{d-1} r^d) = Ne \cdot \gcd(b\mu(x)^d + rz, e^{d-1})$, since from previous relations $\gcd(b\mu(x), r) = 1$. Lastly $\delta \le Ne \cdot \gcd(b\mu(x)^d + rz, e^{d-1}) \le Nee^{d-1} = Ne^d \le N|a|^d$.

Therefore $\nu(f(x)) = \frac{N\nu(x)^d}{\delta} \ge \frac{N\nu(x)^d}{N|a|^d} > \nu(x)$, since $\nu(x) > a^2 \ge |a|^{\frac{d}{d-1}}$. It follows that $\nu(f(x)) \ge \nu(x) + 1 > a^2$. □

While *Lemma 1* is classical, *Lemma 2* is somewhat more computational, even if readily intuitive (it may be shown, with not more trouble, that $x \in f^\omega(\mathbb{Q})$ implies $\nu(x) \mid a$, thus allowing for easier computation of $f^\omega(\mathbb{Q})$).

Remarks.

1. For $\deg f = 1$, one has $f(\mathbb{Q}) = \mathbb{Q}$, therefore $f^\omega(\mathbb{Q}) = \mathbb{Q}$. On the other hand, replacing $\mathbb{Q}$ with $\mathbb{Z}$ allover, results in a much simpler statement (use *Lemma 1* only, or see point 3).

2. If we use the (quite readily established) result that $f(f^\omega(\mathbb{Q})) = f^\omega(\mathbb{Q})$, it follows that the restriction of f to $f^\omega(\mathbb{Q})$ is a permutation (hence a product of cycles) of this finite set. On the other hand, any such cycle clearly belongs to $f^\omega(\mathbb{Q})$; thus the only orbits for f are finitely many, among the elements of $f^\omega(\mathbb{Q})$.

Using a Lagrange interpolation polynomial, one can build as large a finite orbit as wanted. However, all orbits then turn out to be fixed points for some iterate $f^{|f^\omega(\mathbb{Q})|!}$. Conversely, for any nonempty finite set $Q \subset \mathbb{Q}$, we can build the polynomial $f(x) = \prod_{q \in Q}(x-q)^2 + x$, for which $f^\omega(\mathbb{Q}) = Q$, since $Q \subseteq f^\omega(\mathbb{Q})$, while for $x \in f^\omega(\mathbb{Q})$ one has $x \le f(x) \le \cdots \le f^{|f^\omega(\mathbb{Q})|!}(x) = x$, hence $x = f(x)$, whence $x \in Q$.

3. The problem 5 at IMO 2006 (Slovenia) proved that, for $f \in \mathbb{Z}[x]$ with $\deg f > 1$, f^n has at most $\deg f$ fixed points for $n = 1$ or 2 (and no new fixed points for $n > 2$), which under this interpretation translates into $|f^\omega(\mathbb{Z})| \le \deg f$.

According with the above, if f is monic, then $f^\omega(\mathbb{Q}) \subset \mathbb{Z}$, hence the same result holds.[4]

However, polynomial $f(x) = \frac{1}{2}(x-2)(x-3)$ has fixed points 1 and 6, and length-2 orbit $(0,3)$ (computable to $f^\omega(\mathbb{Q}) = \{0,1,3,6\}$), showing the fact that the above result for $\mathbb{Z}$ stands no more. Also, polynomial $f(x) = \frac{1}{4}(x^2 - 29)$ has a length-3 orbit $(5,-1,-7)$ (example by TIMO ERKAMA).[5]

4. As for the situation at hand, a simple corollary states *A sequence $(x_n)_{n\ge1}$ of rational numbers, such that $x_n = f(x_{n+1})$, is periodic.*[6] (The proof is that clearly then the terms of the sequence all belong to the finite set $f^\omega(\mathbb{Q})$, whence the claim of its periodicity.)

END

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[1] Certainly all computations are irrelevant – the idea matters.

[2] To be found in [DIESTEL, R., *Graph Theory*, Springer-Verlag, (2000)], Proposition 1.4.1. We chose to include its proof.

[3] The exactly same argument works in the d-dimensional space, for a set of $dn+1$ latticeal points; then the maximal possible number of vectors will be $(d^2-d)n^2 + 2dn + 1$, with a model made by the origin O, and n points, at distances $1, 2, \dots, n$ from origin, on each axis of the coordinate system.

[4] This is a corollary to a theorem by NARKIEWICZ (see also the seminal theorems by NORTHCOTT).

[5] Such topics are studied by discrete dynamic systems theory, closely related with the study of MANDELBROT and JULIA sets; also FEIGENBAUM constant, attractors, fractals, chaos theory.

[6] For $f(x) = x^3$ it is said to be an old China TST problem.