THE 1996 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let $ABCD$ be a quadrilateral $AB = BC = CD = DA$. Let MN and PQ be two segments perpendicular to the diagonal BD and such that the distance between them is $d > BD/2$, with $M \in AD$, $N \in DC$, $P \in AB$, and $Q \in BC$. Show that the perimeter of hexagon $AMNCQP$ does not depend on the position of MN and PQ so long as the distance between them remains constant.

Question 2

Let m and n be positive integers such that $n \leq m$. Prove that

$$2^n n! \leq \frac{(m+n)!}{(m-n)!} \leq (m^2+m)^n \ .$$

Question 3

Let P_1, P_2, P_3, P_4 be four points on a circle, and let I_1 be the incentre of the triangle $P_2P_3P_4$; I_2 be the incentre of the triangle $P_1P_3P_4$; I_3 be the incentre of the triangle $P_1P_2P_4$; I_4 be the incentre of the triangle $P_1P_2P_3$. Prove that I_1, I_2, I_3, I_4 are the vertices of a rectangle.

Question 4

The National Marriage Council wishes to invite n couples to form 17 discussion groups under the following conditions:

1. All members of a group must be of the same sex; i.e. they are either all male or all female.
2. The difference in the size of any two groups is 0 or 1.
3. All groups have at least 1 member.
4. Each person must belong to one and only one group.

Find all values of n, $n \leq 1996$, for which this is possible. Justify your answer.

Question 5

Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c} \ ,$$

and determine when equality occurs.

Solutions

Problem 1. Let M'N' and P'Q' be two segments perpendicular to BD at a distance d. We need to prove that the perimeters AMNCQP and AM'N'CQ'P' are equal. Denote by S' the projection of N into M'N' and by S the projection of P' into PQ. The triangles NS'N' and P'SP are equal. (right triangles such that NS' = P'S and equal angles $\angle$P'PS = $\angle$NN'S'.) So PP' = NN'. Since it is clear that MM' = NN' , we have MM' = NN' = PP' = QQ'. On the other hand, since SP = S'N', using T' and T the projections of M and Q' into M'N' and PQ, respectively, we have

T'M' = S'N' = PS = TQ.

So P'Q' + M'N' = ST + M'T' + T'S' + S'N'
= ST + PS + TQ + MN
= PQ + MN.

(Without loss of generality M' lies between M and A)
So the perimeter of AMNCQP is:

AM + MN + NC + CQ + QP + PA =
AM' + M'M + MN + NN' + N'C + CQ + QP + PA =
AM' + PP' + MN + QQ' + N'C + CQ + QP + PA =
AM' + MN + QP + PP' + PA + QQ' + CQ + N'C =
AM' + MN + QT + TS + SP + PP' + PA + QQ' + CQ + N'C =
AM' + M'T '+ T'S' + S'N' + N'C + CQ + QQ' + Q'P' + P'P + PA =
AM' + M'N' + N'C + CQ' + Q'P' + P'A,

which is the perimeter of AM'N'CQ'P'.

Points to be given for showing that:

MM' = NN' = PP' = QQ'	1 point
T'M' = S'N' = PS = TQ.	1 point
P'Q' + M'N' = PQ + MN.	2 points
The perimeter of AMNCQP isthe same as the perimeter of AM'N'CQ'P'	3 points

Problem 2. We first prove by induction on n that:

$$\frac{(m+n)!}{(m-n)!} = \prod_{i=1}^{n} (m^2 + m - i^2 + i).$$

1. $\dfrac{(m+1)!}{(m-1)!} = m(m+1) = m^2 + m.$

2. $\dfrac{(m+n+1)!}{(m-n-1)!} = \left(\prod_{i=1}^{n} (m^2 + m - i^2 + i)\right)(m+n+1)(m-n)$ (by induction)

$$= \left(\prod_{i=1}^{n} (m^2 + m - i^2 + i)\right) (m^2 + m - n^2 - n)$$

$$= \prod_{i=1}^{n+1} (m^2 + m - i^2 + i)$$

But $m^2 + m \geq m^2 + m - i^2 + i \geq i^2 + i - i^2 + i = 2i$, for $i \geq m$.
Therefore

$$2^n n! \leq \frac{(m+n)!}{(m-n)!} \leq (m^2 + m)^n.$$

Points to be given for showing that:

the innequlities hold for special values	up to 1 point
$\frac{(m+n)!}{(m-n)!} = \prod_{i=1}^{n} (m^2 + m - i^2 + i)$ for all n	4 points
$m^2 + m \geq m^2 + m - i^2 + i \geq i^2 + i - i^2 + i = 2i$, for $i \geq m$, and therefore the innequalities hold	2 points

Problem 3. Let C be the given circle. Draw four circles C_{12}, C_{23}, C_{34}, C_{41} with centers O_{12}, O_{23}, O_{34}, O_{41}, respectively, on the circle C such that C_{12} passes through P_1 and P_2, C_{23} passes through P_2 and P_3, C_{34} passes through P_3 and P_4, C_{41} passes through P_4 and P_1. Let the other point of intersection of C_{12} and C_{23} be Q_4, the other point of intersection of C_{23} and C_{34} be Q_1, the other point of intersection of C_{34} and C_{41} be Q_2, and the other point of intersection of C_{41} and C_{12} be Q_3. Then

$$\angle Q_4P_1P_2 = \frac{1}{2}\angle Q_4O_{12}P_2 \text{ and } \angle O_{23}P_1P_2 = \frac{1}{2}\angle P_3O_{12}P_2.$$

Clearly, O_{23}, Q_4 and P_1 are collinear.
It follows that $\angle Q_4Q_3P_2 = \angle Q_4P_1P_2 = \angle O_{23}P_1P_2\ \angle O_{23}O_{41}P_2$. Since also O_{41}, Q_3 and P_2 are collinear, it follows that Q_3Q_4 and $O_{41}O_{23}$ are parallel.
Since O_{ij} bisects The arcs P_iP_j, for (i, j) = (1, 2), (2, 3), (3, 4), (4, 1) we conclude that $O_{41}\ O_{23}$ and $O_{12}\ O_{34}$ are perpendicular, and hence $Q_3\ Q_4$ and $O_{12}\ O_{34}$ are perpendicular.
Since both (O_{12}, Q_3, P_4) and (O_{12}, Q_4, P_3) are collinear triples of points, we have $\angle P_4O_{12}P_3 = \angle Q_3O_{12}Q_4$, and this angle is bisected by $O_{12}\ O_{34}$.
Thus Q_3 and Q_4 are reflections through the axis $O_{12}O_{34}$, and so are, by a similar argument Q_1 and Q_2.
We have thus shown: Q_1, Q_2, Q_3, Q_4 form a rectangle. But as Q_4 lies on both the angle bisector $O_{12}P_3$ and the angle bisector $O_{23}P_1$ of the triangle $P_1P_2P_3$, the point Q_4 must coincide with the incenter I_4 of the triangle $P_1P_2P_3$, and by a similar argument, $Q_1 = I_1$, $Q_2 = I_2$ and $Q_3 = I_3$.

Points to be given for showing that

Q_3Q_4 and $O_{41}O_{23}$ are parallel 2 points

Q_3 Q_4 and O_{12} O_{34} are perpendicular 1 points

Q_1, Q_2, Q_3, Q_4 form a rectangle 2 points

$Q_4 = I_4$, $Q_1 = I_1$, $Q_2 = I_2$ and $Q_3 = I_3$ 1 point

Problem 4. . We may assume that the n couples will form x male groups and y female groups. Without loss of generality, let $x \geq y$, and

(*) $$x + y = 17.$$

Then, by the pigeonhole theorem, there exists a male group of size $\leq \left[\frac{n}{x}\right]$ and a female group of size $\geq \left[\frac{n}{y}\right]$. By condition (2), we have

(**) $$[\tfrac{n}{y}] - [\tfrac{n}{x}] \leq 1$$

From the conditions $x + y = 17$ and $x \geq y$ follows that $x \geq 9$, and $y \leq 8$, which in turn implies

$$[\tfrac{n}{y}] - [\tfrac{n}{x}] > [\tfrac{n}{8}] - [\tfrac{n}{9}]$$

Therefore, we only need to exclude those n such that $[\frac{n}{8}] - [\frac{n}{9}] > 1$.Let $n = 9u + s$, $0 \leq s < 9$. Then

$$[\tfrac{n}{8}] - [\tfrac{n}{9}] > 1 \Leftrightarrow [\tfrac{u+s}{8}] > 1.$$

By analyzing this condition it is clear that the only values of n that are allowed are

n = 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24, 27, 28, 29, 30, 31, 32, 36, 37, 38, 39, 40, 45, 46, 46, 47, 48, 54, 55, 56, 63, 63, 72.

Conversely, conditions (*) and (**) give rise to a set of discussing groups according to the following description:

Let $[\frac{n}{x}] = p$ and $[\frac{n}{y}] = q$, then $p \leq q \leq p + 1$

We have

$$n = px + \alpha,\ 0 \leq \alpha < x \qquad \text{and } n = qy - \beta \quad 0 \leq \beta < y.$$

So we can arrange the males in α discussing groups of size p + 1 and x - α groups of p elements. The females are distributed in β discussing groups of size q - 1, and (y - β) discussing groups of size q.

Points to be given for showing that

$x \geq 9$, and $y \leq 8$	1 points
one only needs to exclude those n such that $[\frac{n}{8}] - [\frac{n}{9}] > 1$	2 points
analyzing the condition ian giving the values of n that are allowed	2 points
given p and q as described in the solution one can arrange the males in α discussion groups of size p + 1 and x - α groups of p elements and the females in β discussion groups of size q - 1, and (y - β) groups of size q.	2 points

Problem 5. Without loss of generality we can assume that $a \geq b \geq c$.Note that if $x \geq y > 0$, then $\sqrt{y} \leq 1/2(\sqrt{x} + \sqrt{y})$, i.e.,

$$\frac{1}{2\sqrt{y}}(\sqrt{x} + \sqrt{y}) \geq 1$$

Similarly, if $y \geq x > 0$, then

$$\frac{1}{2\sqrt{y}}(\sqrt{x} + \sqrt{y}) \leq 1$$

Multiplying both inequalities by $\sqrt{x} - \sqrt{y}$ we obtain

$$\sqrt{x} - \sqrt{y} \leq \frac{1}{2\sqrt{y}}(x - y),$$

for every x, y > 0. Moreover, it is easily seen that equality occurs if and only if x = y. By applying this last inequality we obtain

$$\sqrt{a + b - c} - \sqrt{a} \leq \frac{1}{2\sqrt{a}}(b - c)$$

(**)

$$\sqrt{c + a - b} - \sqrt{b} \le \frac{1}{2\sqrt{b}}(c + a - 2b)$$

$$\sqrt{b + c - a} - \sqrt{c} \le \frac{1}{2\sqrt{c}}(b - a)$$

and by adding up the left hand and the right hand sides of these inequalities we have

$$\sqrt{a + b - c} + \sqrt{b + c - a} + \sqrt{c + a - b} - \left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)$$

$$\frac{1}{2}\left(\frac{1}{\sqrt{a}}(b - c) + \frac{1}{\sqrt{b}}(c + a - 2b) + \frac{1}{\sqrt{c}}(b - a)\right)$$

$$\frac{1}{2}\left((b - c)\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right) + (a - b)\left(\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{c}}\right)\right) \le 0,$$

i.e.,

$$\sqrt{a + b - c} + \sqrt{b + c - a} + \sqrt{c + a - b} \le \sqrt{a} + \sqrt{b} + \sqrt{c},$$

and equality occurs if and only if the three relations in (**) are equalities, i.e., if and only if $a = b = c$.

Points to be given for showing that

$\sqrt{x} - \sqrt{y} \le \frac{1}{2\sqrt{y}}(x - y)$ — 2 points

the relations in (**) hold true — 1 point each

$\sqrt{a + b - c} + \sqrt{b + c - a} + \sqrt{c + a - b} - \left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \le$

1 point

equality occurs if and only if the three relations in (**) are equalities, i.e., if and only $a = b = c$ — 1 point