9-th Mediterranean Mathematical Competition 2006

1. Every point of a plane is colored red or blue, not all with the same color. Can this be done in such a way that, on every circumference of radius 1,

   (a) there is exactly one blue point;

   (b) there are exactly two blue points;

2. Let P be a point inside a triangle ABC, and A_1B_2, B_1C_2, C_1A_2 be segments through P parallel to AB, BC, CA respectively, where points A_1, A_2 lie on BC, B_1, B_2 on CA, and C_1, C_2 on AB. Prove that

$$\text{Area}(A_1A_2B_1B_2C_1C_2) \geq \frac{2}{3}\text{Area}(ABC).$$

3. The side lengths a, b, c of a triangle ABC are integers with $\gcd(a, b, c) = 1$. The bisector of angle BAC meets BC at D.

   (a) Show that if triangles DBA and ABC are similar then c is a square.

   (b) If $c = n^2$ is a square ($n \geq 2$), find a triangle ABC satisfying (a).

4. Let $0 \leq x_{i,j} \leq 1$, where $i = 1, \ldots, m$, $j = 1, \ldots, n$. Prove the inequality

$$\prod_{j=1}^{n}\left(1-\prod_{i=1}^{m}x_{i,j}\right)+\prod_{i=1}^{m}\left(1-\prod_{j=1}^{n}(1-x_{i,j})\right) \geq 1.$$