X Международная Жаутыковская олимпиада по математике
Алматы, 2014

14 января 2014 года, 9.00–13.30
Первый день
(Каждая задача оценивается в 7 баллов)

1. На сторонах *BC*, *CA* и *AB* треугольника *ABC* лежат точки *M*, *N*, *K* соответственно, не совпадающие с вершинами. Треугольник *MNK* назовём *красивым*, если $\angle BAC=\angle KMN$ и $\angle ABC=\angle KNM$. Докажите, что если в треугольнике *ABC* существуют два красивых треугольника с общей вершиной, то треугольник *ABC* – прямоугольный.

2. Существует ли функция $f:\mathbf{R}\to\mathbf{R}$, удовлетворяющая следующим условиям:

(i) для каждого вещественного y существует вещественное x такое, что $f(x)=y$, и

(ii) $f(f(x))=(x-1)f(x)+2$ при всех вещественных x?

3. Даны сто различных натуральных чисел. Назовем пару чисел *хорошей*, если числа в ней отличаются в 2 или в 3 раза. Какое наибольшее число хороших пар могут образовывать эти сто чисел? (Одно и то же число может входить в несколько пар.)

Problem 1.

Let KMN and $KM'N'$ be two beautiful triangles with common vertex, $\angle KMN = \angle A = \angle KM'N'$. Without loss of generality, assume that M' lies between B and M. The segments MN and $M'N'$ have a common point, we denote it by R. Since $\angle KMR = \angle KM'R$, the points K, M, M', R are concyclic and $\angle KM'M = 180^0 - \angle KRM = \angle KRN$. Similarly, K, N, N', R are concyclic, therefore $\angle KN'N = \angle KRN$. Thus $\angle KM'C = \angle KM'M = \angle KN'N = 180^0 - \angle KN'C$. It follows that the quadrilateral $KM'CN'$ is cyclic, and $180^0 = \angle C + \angle M'KN' = 2\angle C$, so the angle C is right.

Problem 2.

Does there exist a function $f: R \to R$ satisfying the following two conditions:

1) f takes all real values;
2) $f(f(x)) = (x-1)f(x) + 2$ for all $x \in R$?

Answer: there is no such f .

Suppose that such f does exist.

1. Denote $f(1) = a$. Set $x = 1$ in
$$f(f(x)) = (x-1)f(x) + 2 \,, \qquad (1)$$

Then *f(a)*=2.

2. Now setting $x = a$ in (1) we obtain $f(2) = (a-1)\cdot 2 + 2$, then $f(2) = 2a$.
3. By condition, $\exists b \in R | f(b) = 1$. Let $x = b$ in (1), then
$$a = f(1) = f\big(f(b)\big) = (b-1)\cdot 1 + 2 = b + 1 \quad , b = a - 1.$$
4. Further, , $\exists c \in R | f(c) = 0$. Setting $x = c$ in (1) we obtain
$$f(0) = f\big(f(c)\big) = (c-1)\cdot 0 + 2 = 2,\ f(0) = 2.$$
So we have $2 = f(0) = f(a)$, whence $f\big(f(0)\big) = f(f(a))$, or
$(0-1)f(0) + 2 = (a-1)f(a) + 2$, or $-1\cdot 2 + 2 = (a-1)\cdot 2 + 2$, hence $a = 0$.

As a result: $f(0) = 2, f(2) = 2a = 0, f(1) = 0, b = -1, f(-1) = 1.$

5. Let now $d \in R$ be such that $f(d) = -1$. Set $x = d$ in (1), then
$1 = f(-1) = f\big(f(d)\big) = (d-1)\cdot(-1) + 2 = -d + 3$ whence $d = 2$. That is $f(2) = -1$, contrary to $f(2) = 0$.

Note. There exist function f satisfying (1) such that $E(f) \neq R$. For example

$$f(x) = \begin{cases} \frac{x-2}{x-1}, & x \neq 1 \\ 0, & x = 1 \end{cases} \quad \text{or} \quad f(x) = \begin{cases} 0, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

Problem 3.

The answer is 180.

We reformulate the problem as follows. Given are 100 lattice points (that is, points with integral coordinates). How many pairs of neighbours (points at distance 1) can they form?

To prove that this problem is equivalent to the original one, we assign the number $2^i 3^j$ to the point (i, j). In the set of numbers thus obtained the number of pairs in question is equal to the number of neighbouring points in the set of 100 points.

Conversely, in any set of 100 numbers we find for each number its largest divisor m not divisible by 2 or 3 and divide the set into groups of numbers with the same m. Obviously the numbers in a good pair belong to the same group. Now we can assign to each group a set of points where a point (i, j) corresponds to the number $2^i 3^j m$. If some numbers from different groups correspond to coinciding or neighbouring points, we translate the image of each group by a vector long enough to avoid that.

We can prove now that the maximum number of neighbouring pairs is attained when the points form a 10×10 square (and then the number is 180).

Let us consider rows (i.e. the set of points with the same ordinate) and columns (i.e. the set of points with the same abscissa). Suppose we have a nonempty rows and b nonempty columns. Clearly $ab \geq 100$.

If a row contains k points then its points form at most $k-1$ pairs. Denoting the numbers of points in the rows by $k_1, k_2, ..., k_a$, we have at most $(k_1-1)+(k_2-1)+...+(k_a-1)=100-a$ horizontal pairs of neighbouring points. Similarly, we have at most $100-b$ pairs of vertical pairs. Adding these inequalities we have that the total number of pairs does not exceed $200-a-b \leq 200-2\sqrt{ab} \leq 180$.

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4. Существует ли многочлен $P(x)$ с целыми коэффициентами такой, что $P\left(1+\sqrt{3}\right)=2+\sqrt{3}$ и $P\left(3+\sqrt{5}\right)=3+\sqrt{5}$?

5. Пусть $U=\{1,2,...,2014\}$. Для натуральных a, b, c обозначим через $f(a,b,c)$ количество упорядоченных наборов множеств $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$, удовлетворяющих следующим условиям:

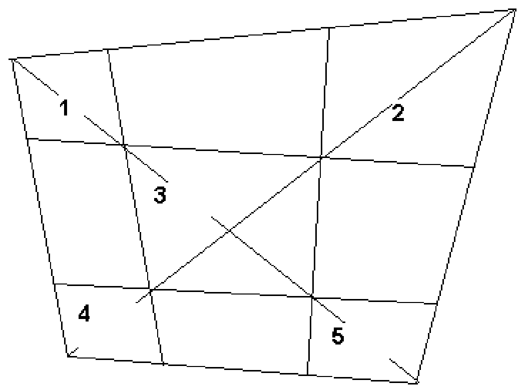
(i) $Y_1 \subseteq X_1 \subseteq U$ и $|X_1|=a$;

(ii) $Y_2 \subseteq X_2 \subseteq U \setminus Y_1$ и $|X_2|=b$;

(iii) $Y_3 \subseteq X_3 \subseteq U \setminus (Y_1 \cup Y_2)$ и $|X_3|=c$.

Докажите, что $f(a,b,c)$ не меняется при перестановке a, b и c. (Здесь $|A|$ обозначает количество элементов множества A.)

6. Выпуклый четырёхугольник поделен на девять четырехугольников четырьмя отрезками, точки пересечения которых лежат на диагоналях исходного четырехугольника (см. рисунок). Известно, что в четырехугольники 1, 2, 3, 4 можно вписать окружности. Докажите, что в четырехугольник 5 также можно вписать окружность.



Problem 4.

The answer is no.

Solution. Note that for each polynomial $P(x)$ with integral coefficients the integers a, b, c, d such that $P(1+\sqrt{3})=a+b\sqrt{3}$ and $P(3+\sqrt{5})=c+d\sqrt{5}$ are uniquely defined. We call a polynomial *regular* if $a-c$ and $b-d$ are even. If P, Q are regular and k is an integer, then $P+Q$ and kP are obviously regular. Let us prove that $R=PQ$ is also regular. Indeed, if $P(1+\sqrt{3})=a+b\sqrt{3}$, $P(3+\sqrt{5})=c+d\sqrt{5}$, $Q(1+\sqrt{3})=a'+b'\sqrt{3}$, $Q(3+\sqrt{5})=c'+d'\sqrt{5}$, then

$R(1+\sqrt{3})=(a+b\sqrt{3})(a'+b'\sqrt{3})=(aa'+3bb')+(ab'+ba')\sqrt{3}$,

$R(3+\sqrt{5})=(c+d\sqrt{5})(c'+d'\sqrt{5})=(cc'+5dd')+(cd'+dc')\sqrt{5}$.

Clearly if $a\equiv c\ (\mathrm{mod}\ 2)$, $b\equiv d\ (\mathrm{mod}\ 2)$, $a'\equiv c'(\mathrm{mod}\ 2)$, $b'\equiv d'\ (\mathrm{mod}\ 2)$, then $aa'+3bb'\equiv cc'+5dd'\ (\mathrm{mod}\ 2)$ и $ab'+ba'\equiv cd'+dc'\ (\mathrm{mod}\ 2)$.

Now the polynomial $P(x)=x$ is regular. It follows that so are all the polynomials with integral coefficients, therefore, the desired polynomial does not exist.

Solution of problem 5.

Note that it is enough to prove that $f(a,b,c) = f(b,a,c) = f(a,c,b)$. First, let us consider the following interpretation of our problem:

For every 6-tuple $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ satisfying conditions of the problem, we construct three sequences

$$A = (a_1, \ldots, a_{2014}),\ B = (b_1, \ldots, b_{2014}),\ C = (c_1, \ldots, c_{2014})$$

as follows:

for $i = 1,\ldots,2014$

$$a_i = \begin{cases} 2, \text{ if } i \in Y_1 \\ 1, \text{ if } i \in X_1 \setminus Y_1, \\ 0, \text{ otherwise.} \end{cases}$$

Similarly, we define sequences B, C. Conditions (i), (ii), (iii) imply the following conditions for sequences (A, B, C):

(P1) number of nonzero elements in A is a; number of nonzero elements in B is b; number of nonzero elements in C is c;

(P2) if $a_i = 2$ for some i, then $b_i = c_i = 0$; if $b_i = 2$, then $c_i = 0$.

Clearly, for every sequences (A, B, C) satisfying (P1), (P2) we may construct a sequence $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ that satisfies (i), (ii), (iii) of the problem.

So, $f(a,b,c)$ is a number of sequences (A, B, C) satisfying (P1), (P2).

Let us first prove that $f(a,b,c) = f(b,a,c)$. We establish the bijection Φ_1 between triples corresponding to the order (a, b, c) and (b, a, c) as follows

$$\Phi_1((A, B, C)) = (A', B', C),$$

where $A' = (a'_1, \ldots, a'_{2014})$, $B' = (b'_1, \ldots, b'_{2014})$ and for all $i = 1,\ldots,2014$

$$(a'_i, b'_i) = (b_i, a_i) \text{ if } (a_i, b_i) \neq (1,2) \text{ and } (a'_i, b'_i) = (a_i, b_i) \text{ otherwise.}$$

(Applying this transform twice we get the initial triple.)

Applying Φ_1, we get the property (P1) for (b,a): the number of entries 1,2 in A' is b and the number of entries 1,2 in B' is a. Let us check that (P2) will also be satisfied. If no, then there is i with $a'_i = 2$ and $b'_i \in \{2,1\}$; the pair (2,2) cannot occur since we interchanged (a_i, b_i); b'_i cannot be 1 since we did not interchange (1,2). As to the sequence C, if $b'_i = 2$, then a_i was equal to 2 which gives that $c_i = 0$. So, $f(a,b,c) = f(b,a,c)$.

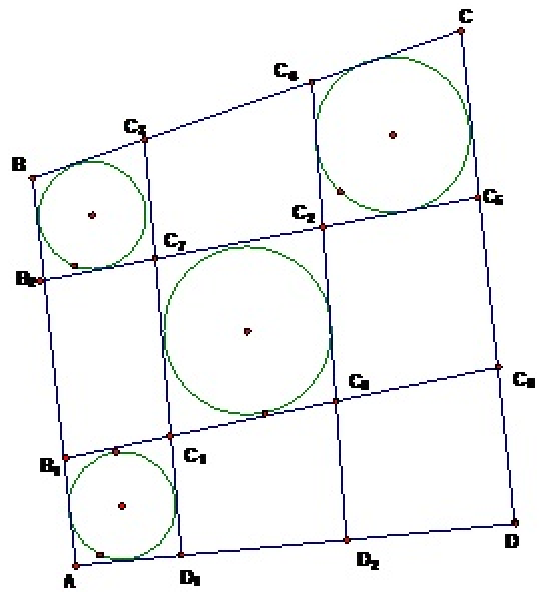
To prove now that $f(a,b,c) = f(a,c,b)$ we consider a similar bijection Φ_2:

$\Phi_2((A, B, C)) = (A, C', B')$ with $B' = (b'_1, \ldots, b'_{2014})$, $C' = (c'_1, \ldots, c'_{2014})$ and for all $i = 1,\ldots,2014$

$$(b'_i, c'_i) = (c_i, b_i) \text{ if } (b_i, c_i) \neq (1,2) \text{ and } (b'_i, c'_i) = (b_i, c_i) \text{ otherwise.}$$

Using a similar argument as explained above, conditions for (P1), (P2) hold for a pair (B', C'). To show a full check with (P2), finally note that if $a_i = 2$, then $b_i = c_i = 0$ and the same holds after Φ_2.

Problem 6.



We use the following

Lemma. A convex quadrilateral $XYZT$ has an inscribed circle if and only if

$$\tan\frac{\angle YXZ}{2} : \tan\frac{\angle TXZ}{2} = \tan\frac{\angle YZX}{2} : \tan\frac{\angle TZX}{2}.$$

Proof of the lemma. Let the incircles of triangles XYZ and XTZ touch XZ at Y_1 and T_1, respectively. It is easy to see that $XY_1 = \dfrac{XY + XZ - YZ}{2}$ and $XT_1 = \dfrac{XT + XZ - YZ}{2}$, and $XYZT$ is tangential if and only if $XY_1 = XT_1$, which is equivalent to $XY_1 : Y_1Z = XT_1 : T_1Z$ and, further, to

$$\tan\frac{\angle YXZ}{2} : \tan\frac{\angle TXZ}{2} = \tan\frac{\angle YZX}{2} : \tan\frac{\angle TZX}{2}.$$

Applying the lemma to quadrilaterals $B_1AD_1C_1$, $C_7C_1C_8C_2$, $C_4C_2C_6C$, we have

$$\tan\frac{\angle B_1AC_1}{2} : \tan\frac{\angle D_1AC_1}{2} = \tan\frac{\angle B_1C_1A}{2} : \tan\frac{\angle D_1C_1A}{2} = \tan\frac{\angle C_8C_1C_2}{2} : \tan\frac{\angle C_7C_1C_2}{2} =$$

$$= \tan\frac{\angle C_8C_2C_1}{2} : \tan\frac{\angle C_7C_2C_1}{2} = \tan\frac{\angle CC_2C_4}{2} : \tan\frac{\angle CC_2C_5}{2} = \tan\frac{\angle C_2CC_4}{2} : \tan\frac{\angle C_2CC_5}{2}.$$

that is,

$$\tan\frac{\angle BAC}{2} : \tan\frac{\angle DAC}{2} = \tan\frac{\angle BCA}{2} : \tan\frac{\angle DCA}{2}$$

and the quadrilateral $ABCD$ is circumscribed.

Applying again the lemma to quadrilaterals $BB_2C_7C_3$, $C_7C_1C_8C_2$, $ABCD$, we get

$$\tan\frac{\angle C_6C_8D}{2} : \tan\frac{\angle D_2C_8D}{2} = \tan\frac{\angle C_6DC_8}{2} : \tan\frac{\angle D_2DC_8}{2}.$$

and the quadrilateral $C_6C_8DD_2$ is circumscribed, q.e.d.