23rd Junior Balkan Mathematical Olympiad

Shortlisted Problems with Solutions

June 20 - 25 2019, Agros, Cyprus

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until JBMO 2020

Contributing countries

The Organising Committee and the Problem Selection Committee of the JBMO 2019 wish to thank the following countries for contributing problem proposals:

- Albania (A7, G3, N5)
- Bulgaria (C3, C4, N3, N4)
- Greece (A3, G1, G7, N1)
- North Macedonia (A2, G2)
- Romania (A5, G5)
- Saudi Arabia (A4, G6, N7)
- Serbia (A1, C2, C5, G4, N6)
- Tajikistan (A6, C1, N2)

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PROBLEMS

ALGEBRA

A1. Real numbers a and b satisfy $a^3+b^3-6ab=-11$. Prove that $-\frac{7}{3}<a+b<-2$.

A2. Let a,b,c be positive real numbers such that $abc=\frac{2}{3}$. Prove that

$$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\geqslant\frac{a+b+c}{a^3+b^3+c^3}\,.$$

A3. Let A and B be two non-empty subsets of $X=\{1,2,\ldots,11\}$ with $A\cup B=X$. Let P_A be the product of all elements of A and let P_B be the product of all elements of B. Find the minimum and maximum possible value of P_A+P_B and find all possible equality cases.

A4. Let a,b be two distinct real numbers and let c be a positive real number such that

$$a^4-2019a=b^4-2019b=c\,.$$

Prove that $-\sqrt{c}<ab<0$.

A5. Let a,b,c,d be positive real numbers such that $abcd=1$. Prove the inequality

$$\frac{1}{a^3+b+c+d}+\frac{1}{a+b^3+c+d}+\frac{1}{a+b+c^3+d}+\frac{1}{a+b+c+d^3}\leqslant\frac{a+b+c+d}{4}\,.$$

A6. Let a,b,c be positive real numbers. Prove the inequality

$$(a^2+ac+c^2)\left(\frac{1}{a+b+c}+\frac{1}{a+c}\right)+b^2\left(\frac{1}{b+c}+\frac{1}{a+b}\right)>a+b+c\,.$$

A7. Show that for any positive real numbers a,b,c such that $a+b+c=ab+bc+ca$, the following inequality holds

$$3+\sqrt[3]{\frac{a^3+1}{2}}+\sqrt[3]{\frac{b^3+1}{2}}+\sqrt[3]{\frac{c^3+1}{2}}\leqslant 2(a+b+c)\,.$$

COMBINATORICS

C1. Let S be a set of 100 positive integer numbers having the following property:

"Among every four numbers of S, there is a number which divides each of the other three or there is a number which is equal to the sum of the other three."

Prove that the set S contains a number which divides all other 99 numbers of S.

C2. In a certain city there are n straight streets, such that every two streets intersect, and no three streets pass through the same intersection. The City Council wants to organize the city by designating the main and the side street on every intersection. Prove that this can be done in such way that if one goes along one of the streets, from its beginning to its end, the intersections where this street is the main street, and the ones where it is not, will apear in alternating order.

C3. In a 5×100 table we have coloured black n of its cells. Each of the 500 cells has at most two adjacent (by side) cells coloured black. Find the largest possible value of n.

C4. We have a group of n kids. For each pair of kids, at least one has sent a message to the other one. For each kid A, among the kids to whom A has sent a message, exactly 25% have sent a message to A. How many possible two-digit values of n are there?

C5. An economist and a statistician play a game on a calculator which does only one operation. The calculator displays only positive integers and it is used in the following way: Denote by n an integer that is shown on the calculator. A person types an integer, m, chosen from the set $\{1, 2, \ldots, 99\}$ of the first 99 positive integers, and if $m\%$ of the number n is again a positive integer, then the calculator displays $m\%$ of n. Otherwise, the calculator shows an error message and this operation is not allowed. The game consists of doing alternatively these operations and the player that cannot do the operation looses. How many numbers from $\{1, 2, \ldots, 2019\}$ guarantee the winning strategy for the statistician, who plays second?

For example, if the calculator displays 1200, the economist can type 50, giving the number 600 on the calculator, then the statistician can type 25 giving the number 150. Now, for instance, the economist cannot type 75 as 75% of 150 is not a positive integer, but can choose 40 and the game continues until one of them cannot type an allowed number.

GEOMETRY

G1. Let ABC be a right-angled triangle with $\hat{A} = 90^\circ$ and $\hat{B} = 30^\circ$. The perpendicular at the midpoint M of BC meets the bisector BK of the angle $\hat{B}$ at the point E. The perpendicular bisector of EK meets AB at D. Prove that KD is perpendicular to DE.

G2. Let ABC be a triangle and let ω be its circumcircle. Let ℓ_B and ℓ_C be two parallel lines passing through B and C respectively. The lines ℓ_B and ℓ_C intersect with ω for the second time at the points D and E respectively, with D belonging on the arc AB, and E on the arc AC. Suppose that DA intersects ℓ_C at F, and EA intersects ℓ_B at G. If O, O_1 and O_2 are the circumcenters of the triangles ABC, ADG and AEF respectively, and P is the center of the circumcircle of the triangle OO_1O_2, prove that OP is parallel to ℓ_B and ℓ_C.

G3. Let ABC be a triangle with incenter I. The points D and E lie on the segments CA and BC respectively, such that $CD = CE$. Let F be a point on the segment CD. Prove that the quadrilateral $ABEF$ is circumscribable if and only if the quadrilateral $DIEF$ is cyclic.

G4. Let ABC be a triangle such that $AB \neq AC$, and let the perpendicular bisector of the side BC intersect lines AB and AC at points P and Q, respectively. If H is the orthocenter of the triangle ABC, and M and N are the midpoints of the segments BC and PQ respectively, prove that HM and AN meet on the circumcircle of ABC.

G5. Let P be a point in the interior of a triangle ABC. The lines AP, BP and CP intersect again the circumcircles of the triangles PBC, PCA, and PAB at D, E and F respectively. Prove that P is the orthocenter of the triangle DEF if and only if P is the incenter of the triangle ABC.

G6. Let ABC be a non-isosceles triangle with incenter I. Let D be a point on the segment BC such that the circumcircle of BID intersects the segment AB at $E \neq B$, and the circumcircle of CID intersects the segment AC at $F \neq C$. The circumcircle of DEF intersects AB and AC at the second points M and N respectively. Let P be the point of intersection of IB and DE, and let Q be the point of intersection of IC and DF. Prove that the three lines EN, FM and PQ are parallel.

G7. Let ABC be a right-angled triangle with $\hat{A} = 90^\circ$. Let K be the midpoint of BC, and let $AKLM$ be a parallelogram with centre C. Let T be the intersection of the line AC and the perpendicular bisector of BM. Let ω_1 be the circle with centre C and radius CA and let ω_2 be the circle with centre T and radius TB. Prove that one of the points of intersection of ω_1 and ω_2 is on the line LM.

NUMBER THEORY

N1. Find all prime numbers p for which there are non-negative integers x, y and z such that the number

$$A = x^p + y^p + z^p - x - y - z$$

is a product of exactly three distinct prime numbers.

N2. Find all triples (p, q, r) of prime numbers such that all of the following numbers are integers

$$\frac{p^2 + 2q}{q + r}, \qquad \frac{q^2 + 9r}{r + p}, \qquad \frac{r^2 + 3p}{p + q}.$$

N3. Find all prime numbers p and nonnegative integers $x \neq y$ such that $x^4 - y^4 = p(x^3 - y^3)$.

N4. Find all integers x, y such that

$$x^3(y + 1) + y^3(x + 1) = 19.$$

N5. Find all positive integers x, y, z such that

$$45^x - 6^y = 2019^z.$$

N6. Find all triples (a, b, c) of nonnegative integers that satisfy

$$a! + 5^b = 7^c.$$

N7. Find all perfect squares n such that if the positive integer $a \geqslant 15$ is some divisor of n then $a + 15$ is a prime power.

SOLUTIONS

ALGEBRA

A1. Real numbers a and b satisfy $a^3+b^3-6ab=-11$. Prove that $-\frac{7}{3}<a+b<-2$.

Solution. Using the identity

$$x^3+y^3+z^3-3xyz=\frac{1}{2}(x+y+z)\left((x-y)^2+(y-z)^2+(z-x)^2\right),$$

we get

$$-3=a^3+b^3+2^3-6ab=\frac{1}{2}(a+b+2)\left((a-b)^2+(a-2)^2+(b-2)^2\right).$$

Since $S=(a-b)^2+(a-2)^2+(b-2)^2$ must be positive, we conclude that $a+b+2<0$, i.e. that $a+b<-2$. Now S can be bounded by

$$S\geqslant(a-2)^2+(b-2)^2=a^2+b^2-4(a+b)+8\geqslant\frac{(a+b)^2}{2}-4(a+b)+8>18.$$

Here, we have used the fact that $a+b<-2$, which we have proved earlier. Since $a+b+2$ is negative, it immediately implies that $a+b+2<-\frac{2\cdot 3}{18}=-\frac{1}{3}$, i.e. $a+b<-\frac{7}{3}$ which we wanted.

Alternative Solution by PSC. Writing $s=a+b$ and $p=ab$ we have

$$a^3+b^3-6ab=(a+b)(a^2-ab+b^2)-6ab=s(s^2-3p)-6p=s^3-3ps-6p.$$

This gives $3p(s+2)=s^3+11$. Thus $s\neq-2$ and using the fact that $s^2\geqslant 4p$ we get

$$p=\frac{s^3+11}{3(s+2)}\leqslant\frac{s^2}{4}. \tag{1}$$

If $s>-2$, then (1) gives $s^3-6s^2+44\leqslant 0$. This is impossible as

$$s^3-6s^2+44=(s+2)(s-4)^2+8>0.$$

So $s<-2$. Then from (1) we get $s^3-6s^2+44\geqslant 0$. If $s<-\frac{7}{3}$ this is again impossible as $s^3-6s^2=s^2(s-6)<-\frac{49}{9}\cdot\frac{25}{3}<-44$. (Since $49\cdot 25=1225>1188=44\cdot 27$.) So $-\frac{7}{3}<s<-2$ as required.

A2. Let a, b, c be positive real numbers such that $abc = \frac{2}{3}$. Prove that

$$\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a} \geqslant \frac{a+b+c}{a^3+b^3+c^3}\,.$$

Solution. The given inequality is equivalent to

$$(a^3+b^3+c^3)\left(\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}\right) \geqslant a+b+c\,. \tag{1}$$

By the AM-GM Inequality it follows that

$$a^3+b^3 = \frac{a^3+a^3+b^3}{3} + \frac{b^3+b^3+a^3}{3} \geqslant a^2b + b^2a = ab(a+b)\,.$$

Similarly we have

$$b^3+c^3 \geqslant bc(b+c) \quad \text{and} \quad c^3+a^3 \geqslant ca(c+a)\,.$$

Summing the three inequalities we get

$$2(a^3+b^2+c^3) \geqslant (ab(a+b)+bc(b+c)+ca(c+a))\,. \tag{2}$$

From the Cauchy-Schwarz Inequality we have

$$(ab(a+b)+bc(b+c)+ca(c+a))\left(\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}\right) \geqslant (ab+bc+ca)^2\,. \tag{3}$$

We also have

$$(ab+bc+ca)^2 \geqslant 3(ab\cdot bc + bc\cdot ca + ca\cdot ab) = 3abc(a+b+c) = 2(a+b+c)\,. \tag{4}$$

Combining together (2),(3) and (4) we obtain (1) which is the required inequality.

Alternative Solution by PSC. By the Power Mean Inequality we have

$$\frac{a^3+b^3+c^3}{3} \geqslant \left(\frac{a+b+c}{3}\right)^3\,.$$

So it is enough to prove that

$$(a+b+c)^2\left(\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}\right) \geqslant 9\,,$$

or equivalently, that

$$(a+b+c)^2\left(\frac{1}{ac+bc} + \frac{1}{ba+ca} + \frac{1}{cb+ab}\right) \geqslant \frac{27}{2}\,. \tag{5}$$

Since $(a+b+c)^2 \geqslant 3(ab+bc+ca) = \frac{3}{2}((ac+bc)+(ba+ca)+(cb+ac))$, then (5) follows by the Cauchy-Schwarz Inequality.

Alternative Solution by PSC. We have

$$(a^3+b^3+c^3)\frac{ab}{a+b} = ab(a^2-ab+b^2) + \frac{abc^3}{a+b} \geqslant a^2b^2 + \frac{2}{3}\frac{c^2}{a+b}\,.$$

So the required inequality follows from

$$(a^2b^2 + b^2c^2 + c^2a^2) + \frac{2}{3}\left(\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b}\right) \geqslant a + b + c\,. \tag{6}$$

By applying the AM-GM Inequality three times we get

$$a^2b^2 + b^2c^2 + c^2a^2 \geqslant abc(a + b + c) = \frac{2}{3}(a + b + c)\,. \tag{7}$$

By the Cauchy-Schwarz Inequality we also have

$$((b+c) + (c+a) + (a+b))\left(\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b}\right) \geqslant (a + b + c)^2\,.$$

which gives

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geqslant \frac{a+b+c}{2}\,. \tag{8}$$

Combining (7) and (8) we get (6) as required.

A3. Let A and B be two non-empty subsets of $X = \{1, 2, \ldots, 11\}$ with $A \cup B = X$. Let P_A be the product of all elements of A and let P_B be the product of all elements of B. Find the minimum and maximum possible value of $P_A + P_B$ and find all possible equality cases.

Solution. For the maximum, we use the fact that $(P_A - 1)(P_B - 1) \geqslant 0$, to get that $P_A + P_B \leqslant P_A P_B + 1 = 11! + 1$. Equality holds if and only if $A = \{1\}$ or $B = \{1\}$.

For the minimum observe, first that $P_A \cdot P_B = 11! = c$. Without loss of generality let $P_A \leqslant P_B$. In this case $P_A \leqslant \sqrt{c}$. We write $P_A + P_B = P_A + \frac{c}{P_A}$ and consider the function $f(x) = x + \frac{c}{x}$ for $x \leqslant \sqrt{c}$. Since

$$f(x) - f(y) = x - y + \frac{c(y-x)}{yx} = \frac{(x-y)(xy-c)}{xy},$$

then f is decreasing for $x \in (0, c]$.

Since x is an integer and cannot be equal with $\sqrt{c}$, the minimum is attained to the closest integer to $\sqrt{c}$. We have $\lfloor\sqrt{11!}\rfloor = \lfloor\sqrt{2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11}\rfloor = \lfloor 720\sqrt{77}\rfloor = 6317$ and the closest integer which can be a product of elements of X is $6300 = 2 \cdot 5 \cdot 7 \cdot 9 \cdot 10$.

Therefore the minimum is $f(6300) = 6300 + 6336 = 12636$ and it is achieved for example for $A = \{2, 5, 7, 9, 10\}, B = \{1, 3, 4, 6, 8, 11\}$.

Suppose now that there are different sets A and B such that $P_A + P_B = 402$. Then the pairs of numbers $(6300, 6336)$ and (P_A, P_B) have the same sum and the same product, thus the equality case is unique for the numbers 6300 and 6336. It remains to find all possible subsets A with product $6300 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7$. It is immediate that $5, 7, 10 \in A$ and from here it is easy to see that all posibilities are $A = \{2, 5, 7, 9, 10\}, \{1, 2, 5, 7, 9, 10\}, \{3, 5, 6, 7, 10\}$ and $\{1, 3, 5, 6, 7, 10\}$.

Alternative Solution by PSC. We have $P_A + P_B \geqslant 2\sqrt{P_A P_B} = 2\sqrt{11!} = 1440\sqrt{77}$. Since $P_A + P_B$ is an integer, we have $P_A + P_B \geqslant \left\lceil 1440\sqrt{77} \right\rceil = 12636$. One can then follow the approach of the first solution to find all equality cases.

Remark by PSC. We can increase the difficulty of the alternative solution by taking $X = \{1, 2, \ldots, 9\}$. Following the first solution we have $\lfloor\sqrt{9!}\rfloor = \lfloor 72\sqrt{70}\rfloor = 602$ and the closest integer which can be a product of elements of X is $2 \cdot 4 \cdot 8 \cdot 9 = 576$. The minimum is $f(576) = 576 + 630 = 1206$ achieved by $A = \{1, 2, 4, 8, 9\}$ and $B = \{3, 5, 6, 7\}$. For equality, the set with product 630 must contain 5 and 7, either 2 and 9 or 3 and 6, and finally it is allowed to either contain 1 or not.

Our alternative solution would give $P_A + P_B \geqslant \left\lceil 144\sqrt{70} \right\rceil = 1205$. One would then need to find a way to show that $P_A + P_B \neq 1205$. To do this we can assume without loss of generality that $5 \in A$. Then the last digit of P_A is either 5 or 0. In the first case the last digit of P_B would be 0 and so P_B would also be a multiple of 5 which is impossible. The second case is analogous.

The computation of the expresions here might be a bit simpler. For example $9! = 362880$ so one expects $\sqrt{9!}$ to be slightly larger than 600.

A4. Let a, b be two distinct real numbers and let c be a positive real number such that

$$a^4 - 2019a = b^4 - 2019b = c\,.$$

Prove that $-\sqrt{c} < ab < 0$.

Solution. Firstly, we see that

$$2019(a-b) = a^4 - b^4 = (a-b)(a+b)(a^2+b^2)\,.$$

Since $a \neq b$, we get $(a+b)(a^2+b^2) = 2019$, so $a+b \neq 0$. Thus

$$\begin{aligned} 2c &= a^4 - 2019a + b^4 - 2019b \\ &= a^4 + b^4 - 2019(a+b) \\ &= a^4 + b^4 - (a+b)^2(a^2+b^2) \\ &= -2ab(a^2+ab+b^2)\,. \end{aligned}$$

Hence $ab(a^2+ab+b^2) = -c < 0$. Note that

$$a^2+ab+b^2 = \frac{1}{2}\left(a^2+b^2+(a+b)^2\right) \geqslant 0\,,$$

thus $ab < 0$. Finally, $a^2+ab+b^2 = (a+b)^2 - ab > -ab$ (the equality does not occur since $a+b \neq 0$). So

$$-c = ab(a^2+ab+b^2) < -(ab)^2 \Longrightarrow (ab)^2 < c \Rightarrow -\sqrt{c} < ab < \sqrt{c}\,.$$

Therefore, we have $-\sqrt{c} < ab < 0$.

Alternative Solution by PSC. By Descartes' Rule of Signs, the polynomial $p(x) = x^4 - 2019x - c$ has exactly one positive root and exactly one negative root. So a, b must be its two real roots. Since one of them is positive and the other is negative, then $ab < 0$. Let $r \pm is$ be the two non-real roots of $p(x)$.

By Vieta, we have

$$ab(r^2+s^2) = -c\,, \tag{1}$$

$$a+b+2r = 0\,, \tag{2}$$

$$ab + 2ar + 2br + r^2 + s^2 = 0\,. \tag{3}$$

Using (2) and (3), we have

$$r^2+s^2 = -2r(a+b) - ab = (a+b)^2 - ab \geqslant -ab\,. \tag{4}$$

If in the last inequality we actually have an equality, then $a+b=0$. Then (2) gives $r=0$ and (3) gives $s^2 = -ab$. Thus the roots of $p(x)$ are $a, -a, ia, -ia$. This would give that $p(x) = x^4 + a^4$, a contradiction.

So the inequality in (4) is strict and now from (1) we get

$$c = -(r^2+s^2)ab > (ab)^2\,,$$

which gives that $ab > -\sqrt{c}$.

A5. Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove the inequality

$$\frac{1}{a^3+b+c+d}+\frac{1}{a+b^3+c+d}+\frac{1}{a+b+c^3+d}+\frac{1}{a+b+c+d^3} \leqslant \frac{a+b+c+d}{4}.$$

Solution. From the Cauchy-Schwarz Inequality, we obtain

$$(a+b+c+d)^2 \leqslant (a^3+b+c+d)\left(\frac{1}{a}+b+c+d\right).$$

Using this, together with the other three analogous inequalities, we get

$$\begin{aligned}\frac{1}{a^3+b+c+d}&+\frac{1}{a+b^3+c+d}+\frac{1}{a+b+c^3+d}+\frac{1}{a+b+c+d^3} \\ &\leqslant \frac{3(a+b+c+d)+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)}{(a+b+c+d)^2}.\end{aligned}$$

So it suffices to prove that

$$(a+b+c+d)^3 \geqslant 12(a+b+c+d)+4\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right),$$

or equivalently, that

$$\begin{aligned}(a^3+b^3+c^3+d^3)&+3\sum a^2b+6(abc+abd+acd+bcd) \\ &\geqslant 12(a+b+c+d)+4(abc+abd+acd+bcd).\end{aligned}$$

(Here, the sum is over all possible x^2y with $x, y \in \{a, b, c, d\}$ and $x \neq y$.) From the AM-GM Inequality we have

$$a^3+a^2b+a^2b+a^2c+a^2c+a^2d+a^2d+b^2a+c^2a+d^2a+bcd+bcd \geqslant 12\sqrt[12]{a^{18}b^6c^6d^6}=12a.$$

Similarly, we get three more inequalities. Adding them together gives the inequality we wanted. Equality holds if and only if $a = b = c = d = 1$.

Remark by PSC. Alternatively, we can finish off the proof by using the following two inequalities: Firstly, we have $a+b+c+d \geqslant 4\sqrt[4]{abcd} = 4$ by the AM-GM Inequality, giving

$$\frac{3}{4}(a+b+c+d)^3 \geqslant 12(a+b+c+d).$$

Secondly, by Mclaurin's Inequality, we have

$$\left(\frac{a+b+c+d}{4}\right)^3 \geqslant \frac{bcd+acd+abd+abc}{4},$$

giving

$$\frac{1}{4}(a+b+c+d)^3 \geqslant 4(bcd+acd+abd+abc).$$

Adding those inequlities we get the required result.

A6. Let a, b, c be positive real numbers. Prove the inequality

$$(a^2+ac+c^2)\left(\frac{1}{a+b+c}+\frac{1}{a+c}\right)+b^2\left(\frac{1}{b+c}+\frac{1}{a+b}\right)>a+b+c\,.$$

Solution. By the Cauchy-Schwarz Inequality, we have

$$\frac{1}{a+b+c}+\frac{1}{a+c}\geqslant\frac{4}{2a+b+2c}\,,$$

and

$$\frac{1}{b+c}+\frac{1}{a+b}\geqslant\frac{4}{a+2b+c}\,.$$

Since

$$a^2+ac+c^2=\frac{3}{4}(a+c)^2+\frac{1}{4}(a-c)^2\geqslant\frac{3}{4}(a+c)^2\,,$$

then, writing L for the Left Hand Side of the required inequality, we get

$$L\geqslant\frac{3(a+c)^2}{2a+b+2c}+\frac{4b^2}{a+2b+c}\,.$$

Using again the Cauchy-Schwarz Inequality, we have:

$$L\geqslant\frac{(\sqrt{3}(a+c)+2b)^2}{3a+3b+3c}>\frac{(\sqrt{3}(a+c)+\sqrt{3}b)^2}{3a+3b+3c}=a+b+c\,.$$

Alternative Question by Proposers. Let a, b, c be positive real numbers. Prove the inequality

$$\frac{a^2}{a+c}+\frac{b^2}{b+c}>\frac{ab-c^2}{a+b+c}+\frac{ab}{a+b}\,.$$

Note that both this inequality and the original one are equivalent to

$$\left(c+\frac{a^2}{a+c}\right)+\left(a-\frac{ab-c^2}{a+b+c}\right)+\frac{b^2}{b+c}+\left(b-\frac{ab}{a+b}\right)>a+b+c\,.$$

Alternative Solution by PSC. The required inequality is equivalent to

$$\left[\frac{b^2}{a+b}-(b-a)\right]+\frac{b^2}{b+c}+\left[\frac{a^2+ac+c^2}{a+c}-a\right]+\left[\frac{a^2+ac+c^2}{a+b+c}-(a+c)\right]>0\,,$$

or equivalently, to

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}>\frac{ab+bc+ca}{a+b+c}\,.$$

However, by the Cauchy-Schwarz Inequality we have

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\geqslant\frac{(a+b+c)^2}{2(a+b+c)}\geqslant\frac{3(ab+bc+ca)}{2(a+b+c)}>\frac{ab+bc+ca}{a+b+c}\,.$$

A7. Show that for any positive real numbers a, b, c such that $a+b+c=ab+bc+ca$, the following inequality holds

$$3+\sqrt[3]{\frac{a^3+1}{2}}+\sqrt[3]{\frac{b^3+1}{2}}+\sqrt[3]{\frac{c^3+1}{2}} \leqslant 2(a+b+c)\,.$$

Solution. Using the condition we have

$$a^2-a+1=a^2-a+1+ab+bc+ca-a-b-c=(c+a-1)(a+b-1)\,.$$

Hence we have

$$\sqrt[3]{\frac{a^3+1}{2}}=\sqrt[3]{\frac{(a+1)(a^2-a+1)}{2}}=\sqrt[3]{\left(\frac{a+1}{2}\right)(c+a-1)(a+b-1)}\,.$$

Using the last equality together with the AM-GM Inequality, we have

$$\begin{aligned}\sum_{\text{cyc}}\sqrt[3]{\frac{a^3+1}{2}}&=\sum_{\text{cyc}}\sqrt[3]{\left(\frac{a+1}{2}\right)(c+a-1)(a+b-1)}\\&\leqslant\sum_{\text{cyc}}\frac{\frac{a+1}{2}+c+a-1+a+b-1}{3}\\&=\sum_{cyc}\frac{5a+2b+2c-3}{6}\\&=\frac{3(a+b+c-1)}{2}\,.\end{aligned}$$

Hence it is enough to prove that

$$3+\frac{3(a+b+c-1)}{2}\leqslant 2(a+b+c)\,,$$

or equivalently, that $a+b+c\geqslant 3$. From a well- known inequality and the condition, we have

$$(a+b+c)^2\geqslant 3(ab+bc+ca)=3(a+b+c)\,,$$

thus $a+b+c\geqslant 3$ as desired.

Alternative Proof by PSC. Since $f(x)=\sqrt[3]{x}$ is concave for $x\geqslant 0$, by Jensen's Inequality we have

$$\sqrt[3]{\frac{a^3+1}{2}}+\sqrt[3]{\frac{b^3+1}{2}}+\sqrt[3]{\frac{c^3+1}{2}}\leqslant 3\sqrt[3]{\frac{a^3+b^3+c^3+3}{6}}\,.$$

So it is enough to prove that

$$\sqrt[3]{\frac{a^3+b^3+c^3+3}{6}}\leqslant\frac{2(a+b+c)-3}{3}\,. \qquad (1)$$

We now write $s=a+b+c=ab+bc+ca$ and $p=abc$. We have

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)=s^2-2s\,,$$

and

$$r = a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 = (ab + bc + ca)(a + b + c) - 3abc = s^2 - 3p\,.$$

Thus,

$$a^3 + b^3 + c^3 = (a + b + c)^3 - 3r - 6abc = s^3 - 3s^2 + 3p\,.$$

So to prove (1), it is enough to show that

$$\frac{s^3 - 3s^2 + 3p + 3}{6} \leqslant \frac{(2s - 3)^3}{27}\,.$$

Expanding, this is equivalent to

$$7s^3 - 45s^2 + 108s - 27p - 81 \geqslant 0\,.$$

By the AM-GM Inequality we have $s^3 \geqslant 27p$. So it is enough to prove that $p(s) \geqslant 0$, where

$$p(s) = 6s^3 - 45s^2 + 108s - 81 = 3(s - 3)^2(2s - 3)\,.$$

It is easy to show that $s \geqslant 3$ (e.g. as in the first solution) so $p(s) \geqslant 0$ as required.

COMBINATORICS

C1. Let S be a set of 100 positive integers having the following property:

"Among every four numbers of S, there is a number which divides each of the other three or there is a number which is equal to the sum of the other three."

Prove that the set S contains a number which divides each of the other 99 numbers of S.

Solution. Let $a < b$ be the two smallest numbers of S and let d be the largest number of S. Consider any two other numbers $x < y$ of S. For the quadruples (a, b, x, d) and (a, b, y, d) we cannot get both of $d = a+b+x$ and $d = a+b+y$, since $a+b+x < a+b+y$. From here, we get $a|b$ and $a|d$.

Consider any number s of S different from a, b, d. From the condition of the problem, we get $d = a + b + s$ or a divides b, s and d. But since we already know that a divides b and d anyway, we also get that $a|s$, as in the first case we have $s = d - a - b$. This means that a divides all other numbers of S.

Alternative Solution by PSC. Order the elements of S as $x_1 < x_2 < \cdots < x_{100}$.

For $2 \leqslant k \leqslant 97$, looking at the quadruples $(x_1, x_k, x_{k+1}, x_{k+2})$ and $(x_1, x_k, x_{k+1}, x_{k+3})$, we get that $x_1|x_k$ as alternatively, we would have $x_{k+2} = x_1 + x_k + x_{k+1} = x_{k+3}$, a contradiction.

For $5 \leqslant k \leqslant 100$, looking at the quadruples $(x_1, x_{k-2}, x_{k-1}, x_k)$ and $(x_1, x_{k-3}, x_{k-1}, x_k)$ we get that $x_1|x_k$ as alternatively, we would have $x_k = x_1 + x_{k-2} + x_{k-1} = x_1 + x_{k-3} + x_{k-1}$, a contradiction.

So x_1 divides all other elements of S.

Alternative Solution by PSC. The condition that one element is the sum of the other three cannot be satisfied by all quadruples. So we have four elements such that one divides the other three. Suppose inductively that we have a subset S' of S with $|S'| = k \geqslant 4$ such that there is $x \in S'$ with $x|y$ for every $y \in S'$. Pick $s \in S \setminus S'$ and $y, z \in S'$ different from x. Considering (s, x, y, z) either $s|x$, or $x|s$ or one of the four is a sum of the other three. In the last case we have $s = \pm x \pm y \pm z$ and so $x|s$. In any case either x or s divides all elements of $S' \cup \{s\}$.

Remark by PSC. The last solution shows that the condition that the elements of S are positive can be ignored.

C2. In a certain city there are n straight streets, such that every two streets intersect, and no three streets pass through the same intersection. The City Council wants to organize the city by designating the main and the side street on every intersection. Prove that this can be done in such way that if one goes along one of the streets, from its beginning to its end, the intersections where this street is the main street, and the ones where it is not, will apear in alternating order.

Solution. Pick any street s and organize the intersections along s such that the intersections of the two types alternate, as in the statement of the problem.

On every other street s_1, exactly one intersection has been organized, namely the one where s_1 intersects s. Call this intersection I_1. We want to organize the intersections along s_1 such that they alternate between the two types. Note that, as I_1 is already organized, we have exactly one way to organize the remaining intersections along s_1.

For every street $s_1 \neq s$, we can apply the procedure described above. Now, we only need to show that every intersection not on s is well-organized. More precisely, this means that for every two streets $s_1, s_2 \neq s$ intersecting at $s_1 \cap s_2 = A$, s_1 is the main street on A if and only if s_2 is the side street on A.

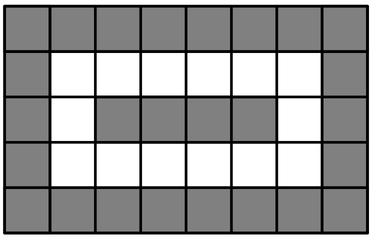
Consider also the intersections $I_1 = s_1 \cap s$ and $I_2 = s_2 \cap s$. Now, we will define the "role" of the street t at the intersection X as "main" if this street t is the main street on X, and "side" otherwise. We will prove that the roles of s_1 and s_2 at A are different.

Consider the path $A \to I_1 \to I_2 \to A$. Let the number of intersections between A and I_1 be u_1, the number of these between A and I_2 be u_2, and the number of these between I_1 and I_2 be v. Now, if we go from A to I_1, we will change our role $u_1 + 1$ times, as we will encounter $u_1 + 1$ new intersections. Then, we will change our street from s_1 to s, changing our role once more. Then, on the segment $I_1 \to I_2$, we have $v + 1$ new role changes, and after that one more when we change our street from s_1 to s_2. The journey from I_2 to A will induce $u_2 + 1$ new role changes, so in total we have changed our role $u_1+1+1+v+1+1+u_2+1 = u_1+v+u_2+5$, As we try to show that roles of s_1 and s_2 differ, we need to show that the number of role changes is odd, i.e. that $u_1 + v + u_2 + 5$ is odd.

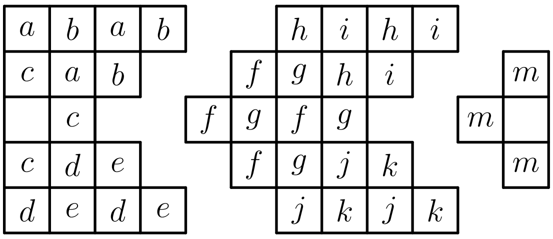
Obviously, this claim is equivalent to $2|u_1 + v + u_2$. But u_1, v and u_2 count the number of intersections of the triangle AI_1I_2 with streets other than s, s_1, s_2. Since every street other than s, s_1, s_2 intersects the sides of AI_1I_2 in exactly two points, the total number of intersections is even. As a consequence, $2|u_1 + v + u_2$ as required.

C3. In a 5×100 table we have coloured black n of its cells. Each of the 500 cells has at most two adjacent (by side) cells coloured black. Find the largest possible value of n.

Solution. If we colour all the cells along all edges of the board together with the entire middle row except the second and the last-but-one cell, the condition is satisfied and there are 302 black cells. The figure below exhibits this colouring for the 5×8 case.



We can cover the table by one fragment like the first one on the figure below, 24 fragments like the middle one, and one fragment like the third one.



In each fragment, among the cells with the same letter, there are at most two coloured black, so the total number of coloured cells is at most $(5 + 24 \cdot 6 + 1) \cdot 2 + 2 = 302$.

Alternative Solution by PSC. Consider the cells adjacent to all cells of the second and fourth row. Counting multiplicity, each cell in the first and fifth row is counted once, each cell in the third row twice, while each cell in the second and fourth row is also counted twice apart from their first and last cells which are counted only once.

So there are 204 cells counted once and 296 cells counted twice. Those cells contain, counting multiplicity, at most 400 black cells. Suppose a of the cells have multiplicity one and b of them have multiplicity 2. Then $a + 2b \leqslant 400$ and $a \leqslant 204$. Thus

$$2a + 2b \leqslant 400 + a \leqslant 604\,,$$

and so $a + b \leqslant 302$ as required.

Remark by PSC. The alternative solution shows that if we have equality, then all cells in the perimeter of the table except perhaps the two cells of the third row must be coloured black. No other cell in the second or fourth row can be coloured black as this will give a cell in the first or fifth row with at least three neighbouring black cells. For similar reasons we cannot colour black the second and last-but-one cell of the third row. So we must colour black all other cells of the third row and therefore the colouring is unique.

C4. We have a group of n kids. For each pair of kids, at least one has sent a message to the other one. For each kid A, among the kids to whom A has sent a message, exactly 25% have sent a message to A. How many possible two-digit values of n are there?

Solution. If the number of pairs of kids with two-way communication is k, then by the given condition the total number of messages is $4k + 4k = 8k$. Thus the number of pairs of kids is $\frac{n(n-1)}{2} = 7k$. This is possible only if $n \equiv 0, 1 \bmod 7$.

- In order to obtain $n = 7m + 1$, arrange the kids in a circle and let each kid send a message to the first $4m$ kids to its right and hence receive a message from the first $4m$ kids to its left. Thus there are exactly m kids to which it has both sent and received messages.

- In order to obtain $n = 7m$, let kid X send no messages (and receive from every other kid). Arrange the remaining $7m - 1$ kids in a circle and let each kid on the circle send a message to the first $4m-1$ kids to its right and hence receive a message from the first $4m - 1$ kids to its left. Thus there are exactly m kids to which it has both sent and received messages.

There are 26 two-digit numbers with remainder 0 or 1 modulo 7. (All numbers of the form $7m$ and $7m + 1$ with $2 \leqslant m \leqslant 14$.)

Alternative Solution by PSC. Suppose kid x_i sent $4d_i$ messages. (Guaranteed by the conditions to be a multiple of 4.) Then it received d_i messages from the kids that it has sent a message to, and another $n - 1 - 4d_i$ messages from the rest of the kids. So it received a total of $n - 1 - 3d_i$ messages. Since the total number of messages sent is equal to the total number of mesages received, we must have:

$$d_1 + \cdots + d_n = (n - 1 - 3d_1) + \cdots + (n - 1 - 3d_n)\,.$$

This gives $7(d_1 + \cdots + d_n) = n(n - 1)$ from which we get $n \equiv 0, 1 \bmod 7$ as in the first solution.

We also present an alternative inductive construction (which turns out to be different from the construction in the first solution).

For the case $n \equiv 0 \bmod 7$, we start with a construction for $7k$ kids, say $x_1, \ldots, x_{7k}$, and another construction with 7 kids, say $y_1, \ldots, y_7$. We merge them by demanding that in addition, each kid x_i sends and receives gifts according to the following table:

i mod 7	Sends	Receives
0	y_1, y_2, y_3, y_4	y_4, y_5, y_6, y_7
1	y_2, y_3, y_4, y_5	y_5, y_6, y_7, y_1
2	y_3, y_4, y_5, y_6	y_6, y_7, y_1, y_2
3	y_4, y_5, y_6, y_7	y_7, y_1, y_2, y_3
4	y_5, y_6, y_7, y_1	y_1, y_2, y_3, y_4
5	y_6, y_7, y_1, y_2	y_2, y_3, y_4, y_5
6	y_7, y_1, y_2, y_3	y_3, y_4, y_5, y_6

So each kid x_i sends an additional four messages and receives a message from only one of those four additional kids. Also, each kid y_j sends an additional $4k$ messages and receives from exactly k of those additional kids. So this is a valid construction for $7(k + 1)$ kids.

For the case $n \equiv 1 \bmod 7$, we start with a construction for $7k+1$ kids, say $x_1, \ldots, x_{7k+1}$, and we take another 7 kids, say $y_1, \ldots, y_7$ for which we do not yet mention how they exchange gifts. The kids $x_1, \ldots, x_{7k+1}$ exchange gifts with the kids $y_1, \ldots, y_7$ according to the previous table. As before, each kid x_i satisfies the conditions. We now put $y_1, \ldots, y_7$ on a circle and demand that each of $y_1, \ldots, y_3$ sends gifts to the next four kids on the circle and each of $y_4, \ldots, y_7$ sends gifts to the next three kids on the circle. It is each to check that the condition is satisfied by each y_i as well.

C5. An economist and a statistician play a game on a calculator which does only one operation. The calculator displays only positive integers and it is used in the following way: Denote by n an integer that is shown on the calculator. A person types an integer, m, chosen from the set $\{1, 2, \ldots, 99\}$ of the first 99 positive integers, and if $m\%$ of the number n is again a positive integer, then the calculator displays $m\%$ of n. Otherwise, the calculator shows an error message and this operation is not allowed. The game consists of doing alternatively these operations and the player that cannot do the operation looses. How many numbers from $\{1, 2, \ldots, 2019\}$ guarantee the winning strategy for the statistician, who plays second?

For example, if the calculator displays 1200, the economist can type 50, giving the number 600 on the calculator, then the statistician can type 25 giving the number 150. Now, for instance, the economist cannot type 75 as 75% of 150 is not a positive integer, but can choose 40 and the game continues until one of them cannot type an allowed number.

Solution. First of all, the game finishes because the number on the calculator always decreases. By picking $m\%$ of a positive integer n, players get the number

$$\frac{m \cdot n}{100} = \frac{m \cdot n}{2^2 5^2}.$$

We see that at least one of the powers of 2 and 5 that divide n decreases after one move, as m is not allowed to be 100, or a multiple of it. These prime divisors of n are the only ones that can decrease, so we conclude that all the other prime factors of n are not important for this game. Therefore, it is enough to consider numbers of the form $n = 2^k 5^\ell$ where $k, \ell \in \mathbb{N}_0$, and to draw conclusions from these numbers.

We will describe all possible changes of k and ℓ in one move. Since $5^3 > 100$, then ℓ cannot increase, so all possible changes are from ℓ to $\ell + b$, where $b \in \{0, -1, -2\}$. For k, we note that $2^6 = 64$ is the biggest power of 2 less than 100, so k can be changed to $k + a$, where $a \in \{-2, -1, 0, 1, 2, 3, 4\}$. But the changes of k and ℓ are not independent. For example, if ℓ stays the same, then m has to be divisible by 25, giving only two possibilities for a change $(k, \ell) \to (k - 2, \ell)$, when $m = 25$ or $m = 75$, or $(k, \ell) \to (k - 1, \ell)$, when $m = 50$. Similarly, if ℓ decreases by 1, then m is divisible exactly by 5 and then the different changes are given by $(k, \ell) \to (k + a, \ell - 1)$, where $a \in \{-2, -1, 0, 1, 2\}$, depending on the power of 2 that divides m and it can be from 2^0 to 2^4. If ℓ decreases by 2, then m is not divisible by 5, so it is enough to consider when m is a power of two, giving changes $(k, \ell) \to (k + a, \ell - 2)$, where $a \in \{-2, -1, 0, 1, 2, 3, 4\}$.

We have translated the starting game into another game with changing (the starting pair of non-negative integers) (k, ℓ) by moves described above and the player who cannot make the move looses, i.e. the player who manages to play the move $(k, \ell) \to (0, 0)$ wins. We claim that the second player wins if and only if $3 \mid k$ and $3 \mid \ell$.

We notice that all moves have their inverse modulo 3, namely after the move $(k, \ell) \to (k + a, \ell + b)$, the other player plays $(k + a, \ell + b) \to (k + a + c, \ell + b + d)$, where

$$(c, d) \in \{(0, -1), (0, -2), (-1, 0), (-1, -1), (-1, -2), (-2, 0), (-2, -1), (-2, -2)\}$$

is chosen such that $3 \mid a + c$ and $3 \mid b + d$. Such (c, d) can be chosen as all possible residues different from $(0, 0)$ modulo 3 are contained in the set above and there is no move that keeps k and ℓ the same modulo 3. If the starting numbers (k, ℓ) are divisible by 3, then after the move of the first player at least one of k and ℓ will not be divisible by 3, and then the second player will play the move so that k and ℓ become divisible by 3 again. In this way, the first player can never finish the game, so the second player wins. In all other cases, the first player will make such a move to make k and ℓ divisible by 3 and then he becomes the second player in the game, and by previous reasoning, wins.

The remaining part of the problem is to compute the number of positive integers $n \leqslant 2019$ which are winning for the second player. Those are the n which are divisible by exactly $2^{3k}5^{3\ell}$, $k, \ell \in \mathbb{N}_0$. Here, exact divisibility by $2^{3k}5^{3\ell}$ in this context means that $2^{3k} \mid\mid n$ and $5^{3\ell} \mid\mid n$, even for $\ell = 0$, or $k = 0$. For example, if we say that n is exactly divisible by 8, it means that $8 \mid n$, $16 \nmid n$ and $5 \nmid n$. We start by noting that for each ten consecutive numbers, exactly four of them coprime to 10. Then we find the desired amount by dividing 2019 by numbers $2^{3k}5^{3\ell}$ which are less than 2019, and then computing the number of numbers no bigger than $\left\lfloor \dfrac{2019}{2^{3k}5^{3\ell}} \right\rfloor$ which are coprime to 10.

First, there are $4 \cdot 201 + 4 = 808$ numbers (out of positive integers $n \leqslant 2019$) coprime to 10. Then, there are $\left\lfloor \frac{2019}{8} \right\rfloor = 252$ numbers divisible by 8, and $25 \cdot 4 + 1 = 101$ among them are exactly divisible by 8. There are $\left\lfloor \frac{2019}{64} \right\rfloor = 31$ numbers divisible by 64, giving $3 \cdot 4 + 1 = 13$ divisible exactly by 64. And there are two numbers, 512 and $3 \cdot 512$, which are divisible by exactly 512. Similarly, there are $\left\lfloor \frac{2019}{125} \right\rfloor = 16$ numbers divisible by 125, implying that $4 + 2 = 6$ of them are exactly divisible by 125. Finally, there is only one number divisible by exactly 1000, and this is 1000 itself. All other numbers that are divisible by exactly $2^{3k}5^{3\ell}$ are greater than 2019. So, we obtain that $808 + 101 + 13 + 2 + 6 + 1 = 931$ numbers not bigger that 2019 are winning for the statistician.

Alternative Solution by PSC. Let us call a positive integer n *losing* if $n = 2^r5^sk$ where $r \equiv s \equiv 0 \bmod 3$ and $(k, 10) = 1$. We call all other positive integers *winning*.

Lemma 1. If n is losing, them $\frac{mn}{100}$ is winning for all $m \in \{1, 2, \ldots, 99\}$ such that $100|mn$.

Proof of Lemma 1. Let $m = 2^t5^uk'$. For $\frac{mn}{100}$ to be losing, we would need $t \equiv u \equiv 2 \bmod 3$. But then $m \geqslant 100$, a contradiction. $\square$

Lemma 2. If n is winning, then there is an $m \in \{1, 2, \ldots, 99\}$ such that $100|mn$ and $\frac{mn}{100}$ is losing.

Proof of Lemma 2. Let $n = 2^r5^sk$ where $(k, 10) = 1$. Pick $t, u \in \{0, 1, 2\}$ such that $t \equiv (2 - r) \bmod 3$ and $u \equiv (2 - s) \bmod 3$ and let $m = 2^t5^s$. Then $100|mn$ and $\frac{mn}{100}$ is winning. Furthermore $m < 100$ as otherwise $m = 100, t = u = 2$ giving $r \equiv s \equiv 0 \bmod 3$ contradicting the fact that n was winning. $\square$

Combining Lemmas 1 and 2 we obtain that the second player wins if and only if the game starts from a losing number.

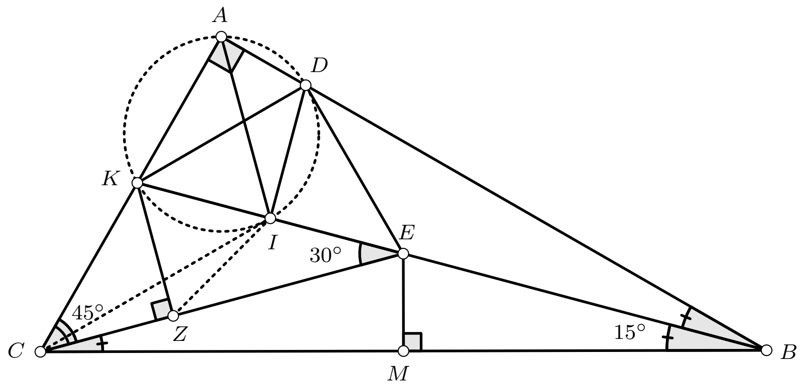
GEOMETRY

G1. Let ABC be a right-angled triangle with $\hat{A} = 90^\circ$ and $\hat{B} = 30^\circ$. The perpendicular at the midpoint M of BC meets the bisector BK of the angle $\hat{B}$ at the point E. The perpendicular bisector of EK meets AB at D. Prove that KD is perpendicular to DE.

Solution. Let I be the incenter of ABC and let Z be the foot of the perpendicular from K on EC. Since KB is the bisector of $\hat{B}$, then $\angle EBC = 15^\circ$ and since EM is the perpendicular bisector of BC, then $\angle ECB = \angle EBC = 15^\circ$. Therefore $\angle KEC = 30^\circ$. Moreover, $\angle ECK = 60^\circ - 15^\circ = 45^\circ$. This means that KZC is isosceles and thus Z is on the perpendicular bisector of KC.

Since $\angle KIC$ is the external angle of triangle IBC, and I is the incenter of triangle ABC, then $\angle KIC = 15^\circ + 30^\circ = 45^\circ$. Thus, $\angle KIC = \frac{\angle KZC}{2}$. Since also Z is on the perpendicular bisector of KC, then Z is the circumcenter of IKC. This means that $ZK = ZI = ZC$. Since also $\angle EKZ = 60^\circ$, then the triangle ZKI is equilateral. Moreover, since $\angle KEZ = 30^\circ$, we have that $ZK = \frac{EK}{2}$, so $ZK = IK = IE$.

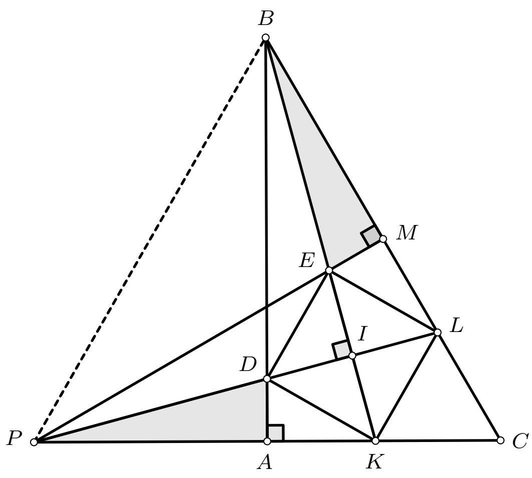
Therefore DI is perpendicular to EK and this means that $DIKA$ is cyclic. So $\angle KDI = \angle IAK = 45^\circ$ and $\angle IKD = \angle IAD = 45^\circ$. Thus $ID = IK = IE$ and so KD is perpendicular to DE as required.



Alternative Question by Proposers. We can instead ask to prove that $ED = 2AD$. (After proving $KD \perp DE$ we have that the triangle EDK is right angled and isosceles, therefore $ED = DK = 2AD$.) This alternative is probably more difficult because the perpendicular relation is hidden.

Alternative Solution by PSC. Let P be the point of intersection of EM with AC. The triangles ABC and MPC are equal since they have equal angles and $MC = \frac{BC}{2} = AC$. They also share the angle $\hat{C}$, so they must have identical incenter.

Let I be the midpoint of EK. We have $\angle PEI = \angle BEM = 75^\circ = \angle EKP$. So the triangle PEK is isosceles and therefore PI is a bisector of $\angle CPM$. So the incenter of MPC belongs on PI. Since it shares the same incentre with ABC, then I is the common incenter. We can now finish the proof as in the first solution.



Alternative Solution by PSC. Let P be the point of intersection of EM with AC and let I be the midpoint of EK. Then the triangle PBC is equilateral. We also have $\angle PEI = \angle BEM = 75^\circ$ and $\angle PKE = 75^\circ$, so PEK is isosceles. We also have $PI \perp EK$ and $DI \perp EK$, so the points P, D, I are collinear.

Furthermore, $\angle PBI = \angle BPI = 45^\circ$, and therefore $BI = PI$.

We have $\angle DPA = \angle EBM = 15^\circ$ and also $BM = \frac{AB}{2} = AC = PA$. So the right-angled triangles PDA and BEM are equal. Thus $PD = BE$.

So

$$EI = BI - BE = PI - PD = DI\,.$$

Therefore $\angle DEI = \angle IDE = 45^\circ$. Since $DE = DK$, we also have $\angle DEI = \angle DKI = \angle KDI = 45^\circ$. So finally, $\angle EDK = 90^\circ$.

Coordinate Geometry Solution by PSC. We may assume that $A = (0, 0), B = (0, \sqrt{3})$ and $C = (1, 0)$. Since $m_{BC} = -\sqrt{3}$, then $m_{EM} = \frac{\sqrt{3}}{3}$. Since also $M = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, then the equation of EM is $y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}$. The slope of BK is

$$m_{BK} = \tan(105^\circ) = \frac{\tan(60^\circ) + \tan(45^\circ)}{1 - \tan(60^\circ)\tan(45^\circ)} = -(2 + \sqrt{3})\,.$$

So the equation of BK is $y = -(2 + \sqrt{3})x + \sqrt{3}$ which gives $K = (2\sqrt{3} - 3, 0)$ and $E = (2 - \sqrt{3}, \sqrt{3} - 1)$. Letting I be the midpoint of EK we get $I = (\frac{\sqrt{3}-1}{2}, \frac{\sqrt{3}-1}{2})$. Thus I is equidistant from the sides AB, AC, so AI is the bisector of $\hat{A}$, and thus I is the incenter of triangle ABC. We can now finish the proof as in the first solution.

Metric Solution by PSC. We can assume that $AC = 1$. Then $AB = \sqrt{3}$ and $BC = 2$. So $BM = MC = 1$. From triangle BEM we get $BE = EC = \sec(15^\circ)$ and $EM = \tan(15^\circ)$. From triangle BAK we get $BK = \sqrt{3}\sec(15^\circ)$. So $EK = BK - BE = (\sqrt{3} - 1)\sec(15^\circ)$. Thus, if N is the midpoint of EK, then $EN = NK = \frac{\sqrt{3}-1}{2}\sec(15^\circ)$ and $BN = BE + EN = \frac{\sqrt{3}+1}{2}\sec(15^\circ)$. From triangle BDN we get $DN = BN\tan(15^\circ) = \frac{\sqrt{3}+1}{2}\tan(15^\circ)\sec(15^\circ)$. It is easy to check that $\tan(15^\circ) = 2 - \sqrt{3}$. Thus $DN = \frac{\sqrt{3}-1}{2}\sec(15^\circ) = EN$. So $DN = EN = EK$ and therefore $\angle EDN = \angle KDN = 45^\circ$ and $\angle KDE = 90^\circ$ as required.

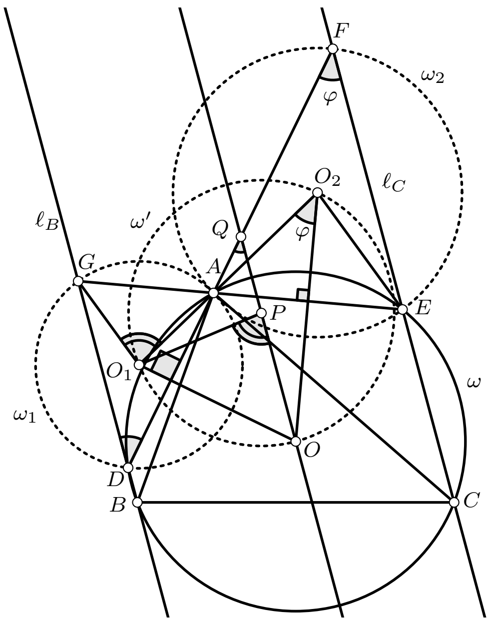
G2. Let ABC be a triangle and let ω be its circumcircle. Let ℓ_B and ℓ_C be two parallel lines passing through B and C respectively. The lines ℓ_B and ℓ_C intersect with ω for the second time at the points D and E respectively, with D belonging on the arc AB, and E on the arc AC. Suppose that DA intersects ℓ_C at F, and EA intersects ℓ_B at G. If O, O_1 and O_2 are the circumcenters of the triangles ABC, ADG and AEF respectively, and P is the center of the circumcircle of the triangle OO_1O_2, prove that OP is parallel to ℓ_B and ℓ_C.

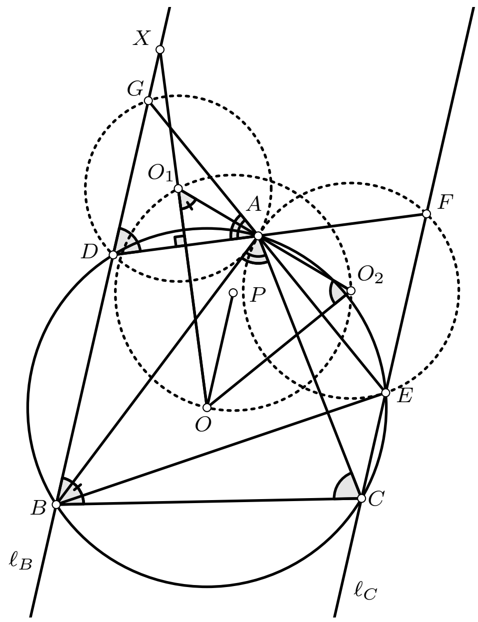
Solution. We write ω_1, ω_2 and ω' for the circumcircles of AGD, AEF and OO_1O_2 respectively. Since O_1 and O_2 are the centers of ω_1 and ω_2, and because DG and EF are parallel, we get that

$$\angle GAO_1 = 90^\circ - \frac{\angle GO_1A}{2} = 90^\circ - \angle GDA = 90^\circ - \angle EFA = 90^\circ - \frac{\angle EO_2A}{2} = \angle EAO_2\,.$$

So, because G, A and E are collinear, we come to the conclusion that O_1, A and O_2 are also collinear.

Let $\angle DFE = \varphi$. Then, as a central angle $\angle AO_2E = 2\varphi$. Because AE is a common chord of both ω and ω_2, the line OO_2 that passes through their centers bisects $\angle AO_2E$, thus $\angle AO_2O = \varphi$. By the collinearity of O_1, A, O_2, we get that $\angle O_1O_2O = \angle AO_2O = \varphi$. As a central angle in ω', we have $\angle O_1PO = 2\varphi$, so $\angle POO_1 = 90^\circ - \varphi$. Let Q be the point of intersection of DF and OP. Because AD is a common chord of ω and ω_1, we have that OO_1 is perpendicular to DA and so $\angle DQP = 90^\circ - \angle POO_1 = \varphi$. Thus, OP is parallel to ℓ_C and so to ℓ_B as well.





Alternative Solution by PSC. Let us write α, β, γ for the angles of ABC. Since $ADBC$ is cyclic, we have $\angle GDA = 180^\circ - \angle BDA = \gamma$. Similarly, we have

$$\angle GAD = 180^\circ - \angle DAE = \angle EBD = \angle BEC = \angle BAC = \alpha\,,$$

where we have also used the fact that ℓ_B and ℓ_C are parallel.

Thus, the triangles ABC and AGD are similar. Analogously, AEF is also similar to them.

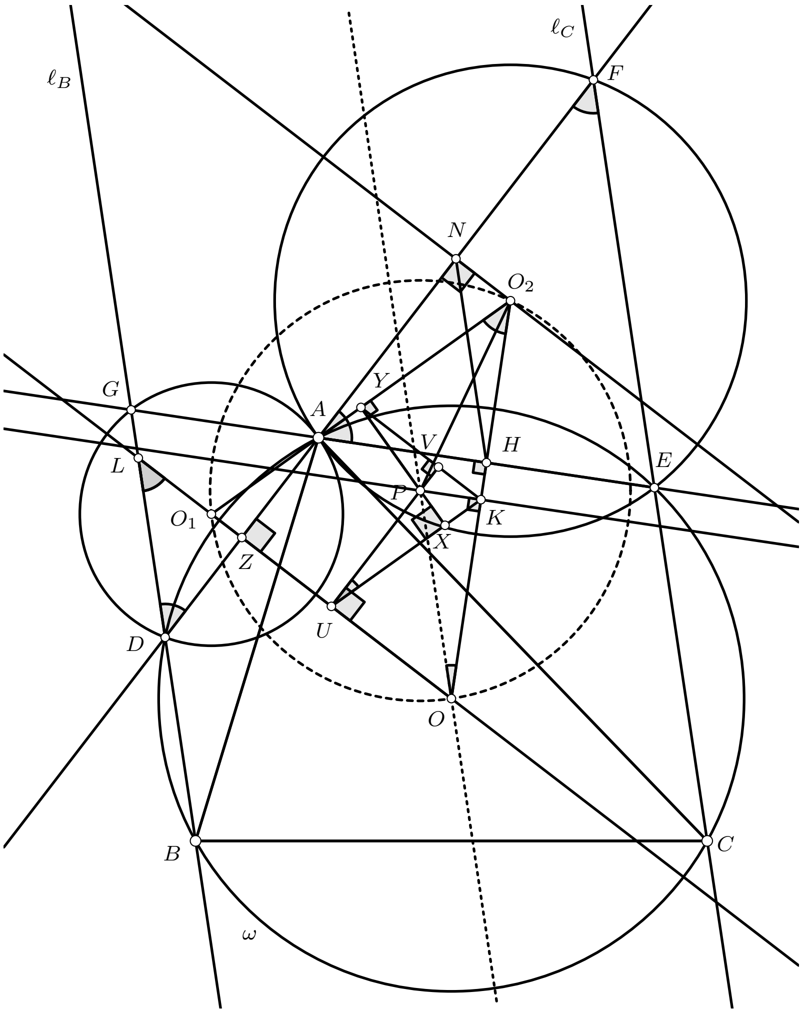
Since AD is a common chord of ω and ω_1 then AD is perpendicular to OO_1. Thus,

$$\angle OO_1A = \frac{1}{2}\angle DO_1A = \angle DGA = \beta\,.$$

Similarly, we have $\angle OO_2A = \gamma$. Since O_1, A, O_2 are collinear (as in the first solution) we get that OO_1O_2 is also similar to ABC. Their circumcentres are P and O respectively, thus $\angle POO_1 = \angle OAB = 90^\circ - \gamma$.

Since OO_1 is perpendicular to AD, letting X be the point of intersection of OO_1 with GD, we get that $\angle DXO_1 = 90^\circ - \gamma$. Thus OP is parallel to ℓ_B and therefore to ℓ_C as well.

Alternative Solution by PSC.



Let L and Z be the points of intesecrion of OO_1 with ℓ_b and DA respectively. Since LZ is perpendicular on DA, and since ℓ_b is parallel to ℓ_c, then

$$\angle DLO = 90^\circ - \angle LDZ = 90^\circ - \angle DFE = 90^\circ - \angle AFE\,.$$

Since AE is a common chord of ω and ω_2, then it is perpendicular to OO_2. So letting H be their point of intersection, we get

$$\angle DLO = 90^\circ - \angle AFE = 90^\circ - \angle AO_2H = \angle O_2AH\,. \tag{1}$$

Let K, Y, U be the projections of P onto OO_2, O_1O_2 and OO_1 respectively. Then $YKUO_1$ is a parallelogram and so the extensions of PY and PU meet the segments UK and KY at points X, V such that $YX \perp KU$ and $UV \perp KY$.

Since the points O_1, A, O_2 are collinear, we have

$$\angle FAO_2 = O_1AZ = 90^\circ - \angle AO_1Z = 90^\circ - \angle YKU = \angle PUK = \angle POK = \angle POK\,, \quad (2)$$

where the last equality follows since $PUOK$ is cyclic.

Since $AZOH$ is also cyclic, we have $\angle FAH = \angle O_1OO_2$. From this, together with (1) and (2) we get

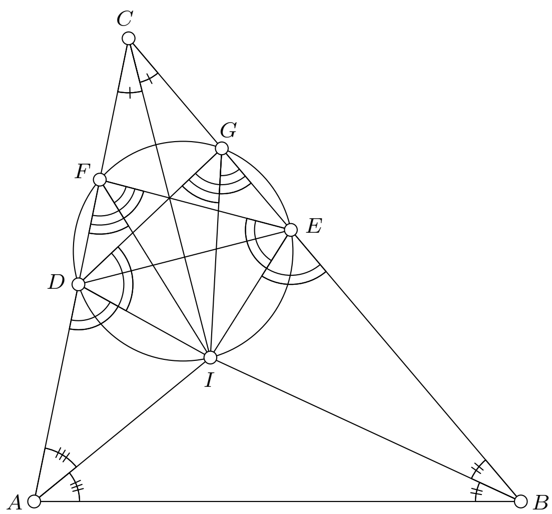
$$\angle DLO = \angle O_2AH = \angle FAH - \angle FAO_2 = \angle O_1OO_2 - \angle POK = \angle UOP = \angle LOP\,.$$

Therefore OP is parallel to ℓ_B and ℓ_C.

G3. Let ABC be a triangle with incenter I. The points D and E lie on the segments CA and BC respectively, such that $CD = CE$. Let F be a point on the segment CD. Prove that the quadrilateral $ABEF$ is circumscribable if and only if the quadrilateral $DIEF$ is cyclic.

Solution. Since $CD = CE$ it means that E is the reflection of D on the bisector of $\angle ACB$, i.e. the line CI. Let G be the reflection of F on CI. Then G lies on the segment CE, the segment EG is the reflection of the segment DF on the line CI. Also, the quadraliteral $DEGF$ is cyclic since $\angle DFE = \angle EGD$.

Suppose that the quadrilateral $ABEF$ is circumscribable. Since $\angle FAI = \angle BAI$ and $\angle EBI = \angle ABI$, then I is the centre of its inscribed circle. Then $\angle DFI = \angle EFI$ and since segment EG is the reflection of segment DF on the line CI, we have $\angle EFI = \angle DGI$. So $\angle DFI = \angle DGI$ which means that quadrilateral $DIGF$ is cyclic. Since the quadrilateral $DEGF$ is also cyclic, we have that the quadrilateral $DIEF$ is cyclic.



Suppose that the quadrilateral $DIEF$ is cyclic. Since quadrilateral $DEGF$ is also cyclic, we have that the pentagon $DIEGF$ is cyclic. So $\angle IEB = 180° - \angle IEG = \angle IDG$ and since segment EG is the reflection of segment DF on the line CI, we have $\angle IDG = \angle IEF$. Hence $\angle IEB = \angle IEF$, which means that EI is the angle bisector of $\angle BEF$. Since $\angle IFA = \angle IFD = \angle IGD$ and since the segment EG is the reflection of segment DF on the line CI, we have $\angle IGD = \angle IFE$, hence $\angle IFA = \angle IFE$, which means that FI is the angle bisector of $\angle EFA$. We also know that AI and BI are the angle bisectors of $\angle FAB$ and $\angle ABE$. So all angle bisectors of the quadrilateral $ABEF$ intersect at I, which means that it is circumscribable.

Comment by PSC. There is no need for introducing the point G. One can show that triangles CID and CIE are equal and also that the triangles CDM and CEM are equal, where M is the midpoint of DE. From these, one can deduce that $\angle CDI = \angle CEI$ and $\angle IDE = \angle IED$ and proceed with similar reasoning as in the solution.

G4. Let ABC be a triangle such that $AB \neq AC$, and let the perpendicular bisector of the side BC intersect lines AB and AC at points P and Q, respectively. If H is the orthocenter of the triangle ABC, and M and N are the midpoints of the segments BC and PQ respectively, prove that HM and AN meet on the circumcircle of ABC.

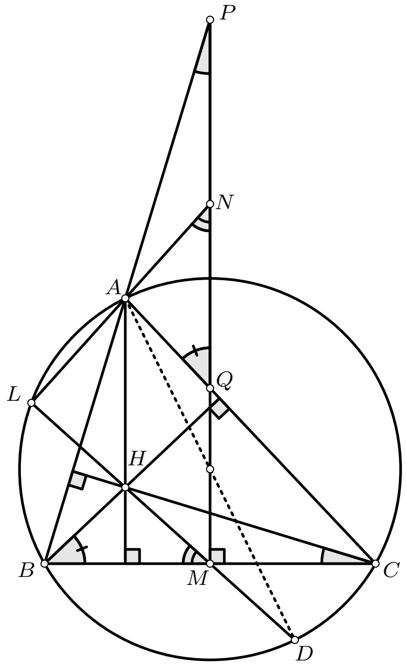
Solution. We have

$$\angle APQ = \angle BPM = 90^\circ - \angle MBP = 90^\circ - \angle CBA = \angle HCB\,,$$

and

$$\angle AQP = \angle MQC = 90^\circ - \angle QCM = 90^\circ - \angle ACB = \angle CBH\,.$$

From these two equalities, we see that the triangles APQ and HCB are similar. Moreover, since M and N are the midpoints of the segments BC and PQ respectively, then the triangles AQN and HBM are also similar. Therefore, we have $\angle ANQ = \angle HMB$.



Let L be the intersection of AN and HM. We have

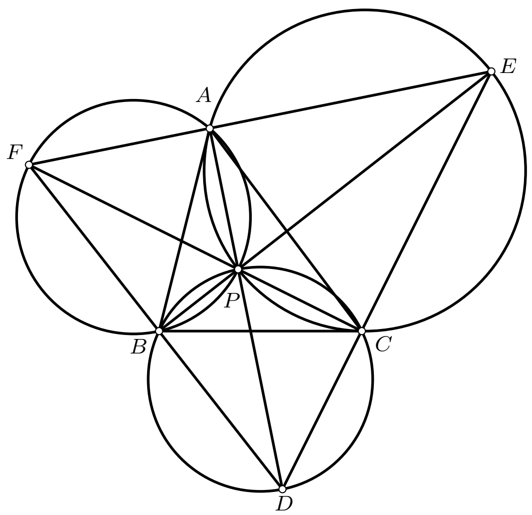
$$\angle MLN = 180^\circ - \angle LNM - \angle NML = 180^\circ - \angle LMB - \angle NML = 180^\circ - \angle NMB = 90^\circ\,.$$

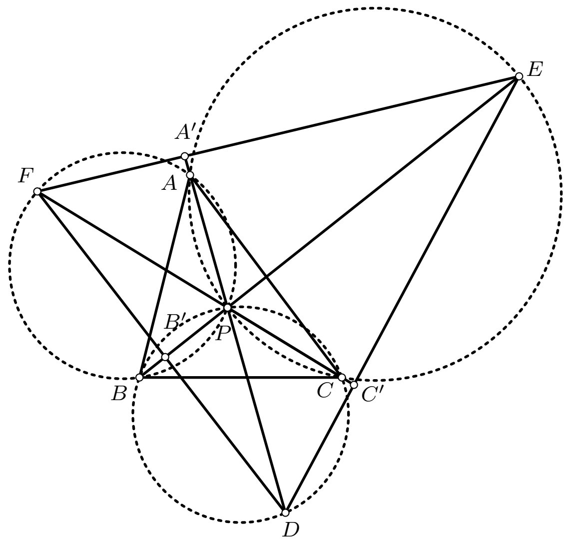
Now let D be the point on the circumcircle of ABC diametrically oposite to A. It is known that D is also the relfection of point H over the point M. Therefore, we have that D belongs on MH and that $\angle DLA = \angle MLA = \angle MLN = 90^\circ$. But, as DA is the diameter of the circumcirle of ABC, the condition that $\angle DLA = 90^\circ$ is enough to conclude that L belongs on the circumcircle of ABC.

Remark by PSC. There is a spiral similarity mapping AQP to HBC. Since the similarity maps AN to HM, it also maps AH to NM, and since these two lines are parallel, the centre of the similarity is $L = AN \cap HM$. Since the similarity maps BC to QP, its centre belongs on the circumcircle of BCX, where $X = BQ \cap PC$. But X is the reflection of A on QM and so it must belong on the circumcircle of ABC. Hence so must L.

G5. Let P be a point in the interior of a triangle ABC. The lines AP, BP and CP intersect again the circumcircles of the triangles PBC, PCA, and PAB at D, E and F respectively. Prove that P is the orthocenter of the triangle DEF if and only if P is the incenter of the triangle ABC.

Solution. If P is the incenter of ABC, then $\angle BPD = \angle ABP + \angle BAP = \frac{\hat{A}+\hat{B}}{2}$, and $\angle BDP = \angle BCP = \frac{\hat{C}}{2}$. From triangle BDP, it follows that $\angle PBD = 90^\circ$, i.e. that EB is one of the altitudes of the triangle DEF. Similarly, AD and CF are altitudes, which means that P is the orhocenter of DEF.





Notice that AP separates B from C, B from E and C from F. Therefore AP separates E from F, which means that P belongs to the interior of $\angle EDF$. It follows that $P \in \operatorname{Int}(\Delta\, DEF)$.

If P is the orthocenter of DEF, then clearly DEF must be acute. Let $A' \in EF$, $B' \in DF$ and $C' \in DE$ be the feet of the altitudes. Then the quadrilaterals $B'PA'F$, $C'PB'D$, and $A'PC'E$ are cyclic, which means that $\angle B'FA' = 180^\circ - \angle B'PA' = 180^\circ - \angle BPA = \angle BFA$. Similarly, one obtains that $\angle C'DB' = \angle CDB$, and $\angle A'EC' = \angle AEC$.

- If $B \in \operatorname{Ext}(\Delta\, FPD)$, then $A \in \operatorname{Int}(\Delta\, EPF)$, $C \in \operatorname{Ext}(\Delta\, DPE)$, and thus $B \in \operatorname{Int}(\Delta\, FPD)$, contradiction.
- If $B \in \operatorname{Int}(\Delta\, FPD)$, then $A \in \operatorname{Ext}(\Delta\, EPF)$, $C \in \operatorname{Int}(\Delta\, DPE)$, and thus $B \in \operatorname{Ext}(\Delta\, FPD)$, contradiction.

This leaves us with $B \in FD$. Then we must have $A \in EF$, $C \in DE$, which means that $A = A'$, $B = B'$, $C = C'$. Thus ABC is the orthic triangle of triangle DEF and it is well known that the orthocenter of an acute triangle DEF is the incenter of its orthic triangle.

G6. Let ABC be a non-isosceles triangle with incenter I. Let D be a point on the segment BC such that the circumcircle of BID intersects the segment AB at $E \neq B$, and the circumcircle of CID intersects the segment AC at $F \neq C$. The circumcircle of DEF intersects AB and AC at the second points M and N respectively. Let P be the point of intersection of IB and DE, and let Q be the point of intersection of IC and DF. Prove that the three lines EN, FM and PQ are parallel.

Solution. Since $BDIE$ is cyclic, and BI is the bisector of $\angle DBE$, then $ID = IE$. Similarly, $ID = IF$, so I is the circumcenter of the triangle DEF. We also have

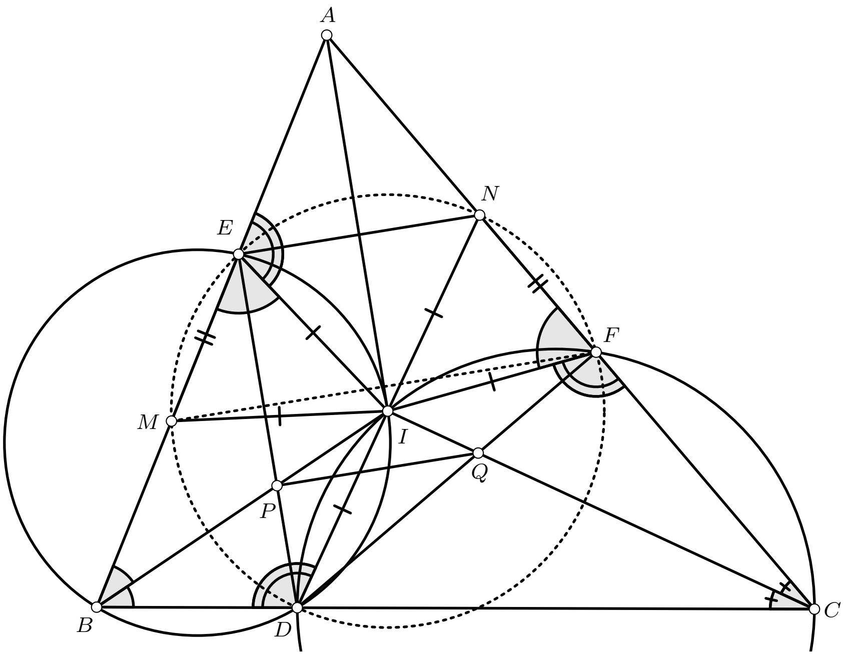
$$\angle IEA = \angle IDB = \angle IFC\,,$$

which implies that $AEIF$ is cyclic. We can assume that A, E, M and A, N, F are collinear in that order. Then $\angle IEM = \angle IFN$. Since also $IM = IE = IN = IF$, the two isosceles triangles IEM and INF are congruent, thus $EM = FN$ and therefore EN is parallel to FM. From that, we can also see that the two triangles IEA and INA are congruent, which implies that AI is the perpendicular bisector of EN and MF.

Note that $\angle IDP = \angle IDE = \angle IBE = \angle IBD$, so the triangles IPD and IDB are similar, which implies that $\frac{ID}{IB} = \frac{IP}{ID}$ and $IP \cdot IB = ID^2$. Similarly, we have $IQ \cdot IC = ID^2$, thus $IP \cdot IB = IQ \cdot IC$. This implies that $BPQC$ is cyclic, which leads to

$$\angle IPQ = \angle ICB = \frac{\hat{C}}{2}\,.$$

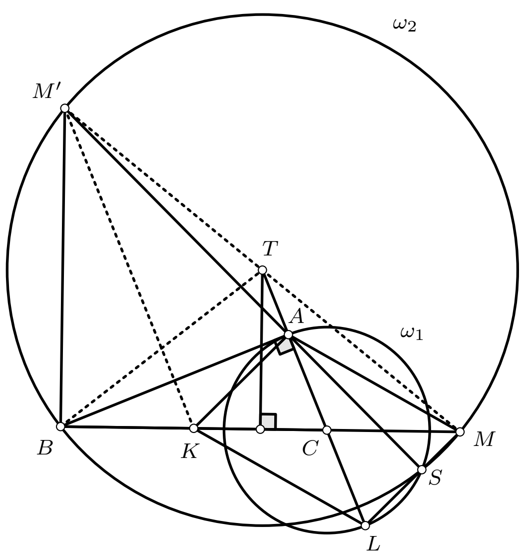
But $\angle AIB = 90^\circ + \frac{\hat{C}}{2}$, so AI is perpendicular to PQ. Hence, PQ is parallel to EN and FM.

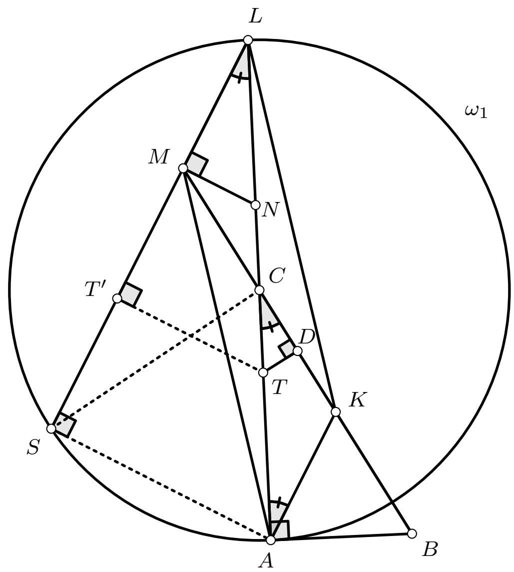


G7. Let ABC be a right-angled triangle with $\hat{A} = 90^\circ$. Let K be the midpoint of BC, and let $AKLM$ be a parallelogram with centre C. Let T be the intersection of the line AC and the perpendicular bisector of BM. Let ω_1 be the circle with centre C and radius CA and let ω_2 be the circle with centre T and radius TB. Prove that one of the points of intersection of ω_1 and ω_2 is on the line LM.

Solution. Let M' be the symmetric point of M with respect to T. Observe that T is equidistant from B and M, therefore M belongs on ω_2 and $M'M$ is a diameter of ω_2. It suffices to prove that $M'A$ is perpendicular to LM, or equivalently, to AK. To see this, let S be the point of intersection of $M'A$ with LM. We will then have $\angle M'SM = 90^\circ$ which shows that S belongs on ω_2 as $M'M$ is a diameter of ω_2. We also have that S belongs on ω_1 as AL is diameter of ω_1.

Since T and C are the midpoints of $M'M$ and KM respectively, then TC is parallel to $M'K$ and so $M'K$ is perpendicular to AB. Since $KA = KB$, then KM' is the perpendicular bisector of AB. But then the triangles KBM' and KAM' are equal, showing that $\angle M'AK = \angle M'BK = \angle M'BM = 90^\circ$ as required.





Alternative Solution by Proposers. Since $CA = CL$, then L belongs on ω_1. Let S be the other point of intersection of ω_1 with the line LM. We need to show that S belongs on ω_2. Since $TB = TM$ (T is on the perpendicular bisector of BM) it is enough to show that $TS = TM$.

Let N, T' be points on the lines AL and LM respectively, such that $MN \perp LM$ and $TT' \perp LM$. It is enough to prove that T' is the midpoint of SM. Since AL is diameter of ω_1 we have that $AS \perp LS$. Thus, it is enough to show that T is the midpoint of AN. We have

$$AT = \frac{AN}{2} \Leftrightarrow AC - CT = \frac{AL - LN}{2} \Leftrightarrow 2AC - 2CT = AL - LN \Leftrightarrow LN = 2CT$$

as $AL = 2AC$. So it suffices to prove that $LN = 2CT$.

Let D be the midpoint of BM. Since $BK = KC = CM$, then D is also the midpont of KC. The triangles LMN and CTD are similar since they are right-angled with

$\angle TCD = \angle CAK = \angle MLN$. ($AK = KC$ and AK is parallel to LM.) So we have

$$\frac{LN}{CT} = \frac{LM}{CD} = \frac{AK}{CD} = \frac{CK}{CD} = 2\,,$$

as required.

NUMBER THEORY

N1. Find all prime numbers p for which there are non-negative integers x, y and z such that the number

$$A = x^p + y^p + z^p - x - y - z$$

is a product of exactly three distinct prime numbers.

Solution. For $p = 2$, we take $x = y = 4$ and $z = 3$. Then $A = 30 = 2 \cdot 3 \cdot 5$. For $p = 3$ we can take $x = 3$ and $y = 2$ and $z = 1$. Then again $A = 30 = 2 \cdot 3 \cdot 5$. For $p = 5$ we can take $x = 2$ and $y = 1$ and $z = 1$. Again $A = 30 = 2 \cdot 3 \cdot 5$.

Assume now that $p \geqslant 7$. Working modulo 2 and modulo 3 we see that A is divisible by both 2 and 3. Moreover, by Fermat's Little Theorem, we have

$$x^p + y^p + z^p - x - y - z \equiv x + y + z - x - y - z = 0 \bmod p\,.$$

Therefore, by the given condition, we have to solve the equation

$$x^p + y^p + z^p - x - y - z = 6p\,.$$

If one of the numbers x, y and z is bigger than or equal to 2, let's say $x \geqslant 2$, then

$$6p \geqslant x^p - x = x(x^{p-1} - 1) \geqslant 2(2^{p-1} - 1) = 2^p - 2\,.$$

It is easy to check by induction that $2^p - 2 > 6p$ for all primes $p \geqslant 7$. This contradiction shows that there are no more values of p which satisfy the required property.

Remark by PSC. There are a couple of other ways to prove that $2^p - 2 > 6p$ for $p \geqslant 7$. For example, we can use the Binomial Theorem as follows:

$$2^p - 2 \geqslant 1 + p + \frac{p(p-1)}{2} + \frac{p(p-1)(p-2)}{6} - 2 \geqslant 1 + p + 3p + 5p - 2 > 6p\,.$$

We can also use Bernoulli's Inequality as follows:

$$2^p - 2 = 8(1+1)^{p-3} - 2 \geqslant 8(1 + (p-3)) - 2 = 8p - 18 > 6p$$

The last inequality is true for $p \geqslant 11$. For $p = 7$ we can see directly that $2^p - 2 > 6p$.

N2. Find all triples (p, q, r) of prime numbers such that all of the following numbers are integers

$$\frac{p^2+2q}{q+r}, \qquad \frac{q^2+9r}{r+p}, \qquad \frac{r^2+3p}{p+q}\,.$$

Solution. We consider the following cases:

1st Case: If $r = 2$, then $\frac{r^2+3p}{p+q} = \frac{4+3p}{p+q}$. If p is odd, then $4+3p$ is odd and therefore $p+q$ must be odd. From here, $q = 2$ and $\frac{r^2+3p}{p+q} = \frac{4+3p}{p+2} = 3 - \frac{2}{p+2}$ which is not an integer. Thus $p = 2$ and $\frac{r^2+3p}{p+q} = \frac{10}{q+2}$ which gives $q = 3$. But then $\frac{q^2+9r}{r+p} = \frac{27}{4}$ which is not an integer. Therefore r is an odd prime.

2nd Case: If $q = 2$, then $\frac{q^2+9r}{r+p} = \frac{4+9r}{r+p}$. Since r is odd, then $4+9r$ is odd and therefore $r+p$ must be odd. From here $p = 2$, but then $\frac{r^2+3p}{p+q} = \frac{r^2+6}{4}$ which is not integer. Therefore q is an odd prime.

Since q and r are odd primes, then $q+r$ is even. From the number $\frac{p^2+2q}{q+r}$ we get that $p = 2$. Since $\frac{p^2+2q}{q+r} = \frac{4+2q}{q+r} < 2$, then $4+2q = q+r$ or $r = q+4$. Since

$$\frac{r^2+3p}{p+q} = \frac{(q+4)^2+6}{2+q} = q+6+\frac{10}{2+q}\,,$$

is an integer, then $q = 3$ and $r = 7$. It is easy to check that this triple works. So the only answer is $(p, q, r) = (2, 3, 7)$.

N3. Find all prime numbers p and nonnegative integers $x \neq y$ such that $x^4 - y^4 = p(x^3 - y^3)$.

Solution. If $x = 0$ then $y = p$ and if $y = 0$ then $x = p$. We will show that there are no other solutions.

Suppose $x, y > 0$. Since $x \neq y$, we have

$$p(x^2 + xy + y^2) = (x + y)(x^2 + y^2)\,. \tag{$*$}$$

If p divides $x + y$, then $x^2 + y^2$ must divide $x^2 + xy + y^2$ and so it must also divide xy. This is a contradiction as $x^2 + y^2 \geqslant 2xy > xy$.

Thus p divides $x^2 + y^2$, so $x + y$ divides $x^2 + xy + y^2$. As $x + y$ divides $x^2 + xy$ and $y^2 + xy$, it also divides x^2, xy and y^2. Suppose $x^2 = a(x + y)$, $y^2 = b(x + y)$ and $xy = c(x + y)$. Then $x^2 + xy + y^2 = (a + b + c)(x + y)$, $x^2 + y^2 = (a + b)(x + y)$, while $(x + y)^2 = x^2 + y^2 + 2xy = (a + b + 2c)(x + y)$ yields $x + y = a + b + 2c$.

Substituting into $(*)$ gives

$$p(a + b + c) = (a + b + 2c)(a + b)\,.$$

Now let $a + b = dm$ and $c = dc_1$, where $\gcd(m, c_1) = 1$. Then

$$p(m + c_1) = (m + 2c_1)dm\,.$$

If $m + c_1$ and m had a common divisor, it would divide c_1, a contradiction. So $\gcd(m, m + c_1) = 1$. and similarly, $\gcd(m + c_1, m + 2c_1) = 1$. Thus $m + 2c_1$ and m divide p, so $m + 2c_1 = p$ and $m = 1$. Then $m + c_1 = d$ so $c \geqslant d = a + b$. Now

$$xy = c(x + y) \geqslant (a + b)(x + y) = x^2 + y^2\,,$$

again a contradiction.

Alternative Solution by PSC. Let $d = \gcd(x, y)$. Then $x = da$ and $y = db$ for some a, b such that $\gcd(a, b) = 1$. Then

$$d^4(a^4 - b^4) = pd^3(a^3 - b^3)\,,$$

which gives

$$d(a + b)(a^2 + b^2) = p(a^2 + ab + b^2)\,. \tag{$*$}$$

If a prime q divides both $a+b$ and a^2+ab+b^2, then it also divides $(a+b)^2-(a^2+ab+b^2) = ab$. So q divides a or q divides b. Since q also divides $a + b$, it must divide both a and b. This is impossible as $\gcd(a, b) = 1$. So $\gcd(a + b, a^2 + ab + b^2) = 1$ and similarly $\gcd(a^2 + b^2, a^2 + ab + b^2) = 1$. Then $(a + b)(a^2 + b^2)$ divides p and since $a + b \leqslant a^2 + b^2$, then $a + b = 1$.

If $a = 0, b = 1$ then $(*)$ gives $d = p$ and so $x = 0, y = p$ which is obviously a solution. If $a = 1, b = 0$ we similarly get the solution $x = p, y = 0$. These are the only solutions.

N4. Find all integers x, y such that

$$x^3(y+1) + y^3(x+1) = 19\,.$$

Solution. Substituting $s = x + y$ and $p = xy$ we get

$$2p^2 - (s^2 - 3s)p + 19 - s^3 = 0\,. \tag{1}$$

This is a quadratic equation in p with discriminant $D = s^4 + 2s^3 + 9s^2 - 152$.

For each s we have $D < (s^2 + s + 5)^2$ as this is equivalent to $(2s + 5)^2 + 329 > 0$.

For $s \geqslant 11$ and $s \leqslant -8$ we have $D > (s^2+s+3)^2$ as this is equivalent to $2s^2-6s-161 > 0$, and thus also to $2(s + 8)(s - 11) > -15$.

We have the following cases:

- If $s \geqslant 11$ or $s \leqslant -8$, then D is a perfect square only when $D = (s^2 + s + 4)^2$, or equivalently, when $s = -21$. From (1) we get $p = 232$ (which yields no solution) or $p = 20$, giving the solutions $(-1, -20)$ and $(-20, -1)$.
- If $-7 \leqslant s \leqslant 10$, then D is directly checked to be perfect square only for $s = 3$. Then $p = \pm 2$ and only $p = 2$ gives solutions, namely $(2, 1)$ and $(1, 2)$.

Remark by PSC. In the second bullet point, one actually needs to check 18 possible values of s which is actually quite time consuming. We did not see many possible shortcuts. For example, D is always a perfect square modulo 2 and modulo 3, while modulo 5 we can only get rid the four cases of the form $s \equiv 0 \bmod 5$.

N5. Find all positive integers x, y, z such that

$$45^x - 6^y = 2019^z .$$

Solution. We define $v_3(n)$ to be the non-negative integer k such that $3^k | n$ but $3^{k+1} \nmid n$. The equation is equivalent to

$$3^{2x} \cdot 5^x - 3^y \cdot 2^y = 3^z \cdot 673^z .$$

We will consider the cases $y \neq 2x$ and $y = 2x$ separately.

Case 1. Suppose $y \neq 2x$. Since $45^x > 45^x - 6^y = 2019^z > 45^z$, then $x > z$ and so $2x > z$. We have

$$z = v_3\left(3^z \cdot 673^z\right) = v_3\left(3^{2x} \cdot 5^x - 3^y \cdot 2^y\right) = \min\{2x, y\} ,$$

as $y \neq 2x$. Since $2x > z$, we get $z = y$. Hence the equation becomes $3^{2x} \cdot 5^x - 3^y \cdot 2^y = 3^y \cdot 673^y$, or equivalently,

$$3^{2x-y} \cdot 5^x = 2^y + 673^y .$$

Case 1.1. Suppose $y = 1$. Doing easy manipulations we have

$$3^{2x-1} \cdot 5^x = 2 + 673 = 675 = 3^3 \cdot 5^2 \Longrightarrow 45^{x-2} = 1 \Longrightarrow x = 2 .$$

Hence one solution which satisfies the condition is $(x, y, z) = (2, 1, 1)$.

Case 1.2. Suppose $y \geqslant 2$. Using properties of congruences we have

$$1 \equiv 2^y + 673^y \equiv 3^{2x-y} \cdot 5^y \equiv (-1)^{2x-y} \bmod 4 .$$

Hence $2x - y$ is even, which implies that y is even. Using this fact we have

$$0 \equiv 3^{2x-y} \cdot 5^y \equiv 2^y + 673^y \equiv 1 + 1 \equiv 2 \bmod 3 ,$$

which is a contradiction.

Case 2. Suppose $y = 2x$. The equation becomes $3^{2x} \cdot 5^x - 3^{2x} \cdot 2^{2x} = 3^z \cdot 673^z$, or equivalently,

$$5^x - 4^x = 3^{z-2x} \cdot 673^z .$$

Working modulo 3 we have

$$(-1)^x - 1 \equiv 5^x - 4^x \equiv 3^{z-2x} \cdot 673^z \equiv 0 \bmod 3 ,$$

hence x is even, say $x = 2t$ for some positive integer t. The equation is now equivalent to

$$\left(5^t - 4^t\right)\left(5^t + 4^t\right) = 3^{z-4t} \cdot 673^z .$$

It can be checked by hand that $t = 1$ is not possible. For $t \geqslant 2$, since 3 and 673 are the only prime factors of the right hand side, and since, as it is easily checked $\gcd(5^t - 4^t, 5^t + 4^t) = 1$ and $5^t - 4^t > 1$, the only way for this to happen is when $5^t - 4^t = 3^{z-4t}$ and $5^t + 4^t = 673^z$ or $5^t - 4^t = 673^z$ and $5^t + 4^t = 3^{z-4t}$. Adding together we have

$$2 \cdot 5^t = 3^{z-4t} + 673^z .$$

Working modulo 5 we have

$$0 \equiv 2 \cdot 5^t \equiv 3^{z-4t} + 673^z \equiv 3^{4t} \cdot 3^{z-4t} + 3^z \equiv 2 \cdot 3^z \bmod 5\,,$$

which is a contradiction. Hence the only solution which satisfies the equation is $(x, y, z) = (2, 1, 1)$.

Alternative Solution by PSC. Working modulo 5 we see that $-1 \equiv 4^z \bmod 5$ and therefore z is odd. Now working modulo 4 and using the fact that z is odd we get that $1 - 2^y \equiv 3^z \equiv 3 \bmod 4$. This gives $y = 1$. Now working modulo 9 we have $-6 \equiv 3^z \bmod 9$ which gives $z = 1$. Now since $y = z = 1$ we get $x = 2$ and so $(2, 1, 1)$ is the unique solution.

N6. Find all triples (a,b,c) of nonnegative integers that satisfy

$$a! + 5^b = 7^c\,.$$

Solution. We cannot have $c = 0$ as $a! + 5^b \geqslant 2 > 1 = 7^0$.

Assume first that $b = 0$. So we are solving $a! + 1 = 7^c$. If $a \geqslant 7$, then $7|a!$ and so $7 \nmid a! + 1$. So $7 \nmid 7^c$ which is impossible as $c \neq 0$. Checking $a < 7$ by hand, we find the solution $(a,b,c) = (3,0,1)$.

We now assume that $b > 0$. In this case, if $a \geqslant 5$, we have $5|a!$, and since $5|5^b$, we have $5|7^c$, which obviously cannot be true. So we have $a \leqslant 4$. Now we consider the following cases:

Case 1. Suppose $a = 0$ or $a = 1$. In this case, we are solving the equation $5^b + 1 = 7^c$. However the Left Hand Side of the equation is always even, and the Right Hand Side is always odd, implying that this case has no solutions.

Case 2. Suppose $a = 2$. Now we are solving the equation $5^b + 2 = 7^c$. If $b = 1$, we have the solution $(a,b,c) = (2,1,1)$. Now assume $b \geqslant 2$. We have $5^b + 2 \equiv 2 \bmod 25$ which implies that $7^c \equiv 2 \bmod 25$. However, by observing that $7^4 \equiv 1 \bmod 25$, we see that the only residues that 7^c can have when divided with 25 are $7, 24, 18, 1$. So this case has no more solutions.

Case 3. Suppose $a = 3$. Now we are solving the equation $5^b + 6 = 7^c$. We have $5^b + 6 \equiv 1 \bmod 5$ which implies that $7^c \equiv 1 \bmod 5$. As the residues of 7^c modulo 5 are $2, 4, 3, 1$, in that order, we obtain $4|c$.

Viewing the equation modulo 4, we have $7^c \equiv 5^b + 6 \equiv 1 + 2 \equiv 3 \bmod 4$. But as $4|c$, we know that 7^c is a square, and the only residues that a square can have when divided by 4 are $0, 1$. This means that this case has no solutions either.

Case 4. Suppose $a = 4$. Now we are solving the equation $5^b + 24 = 7^c$. We have $5^b \equiv 7^c - 24 \equiv 1 - 24 \equiv 1 \bmod 3$. Since $5 \equiv 2 \bmod 3$, we obtain $2|b$. We also have $7^c \equiv 5^b + 24 \equiv 4 \bmod 5$, and so we obtain $c \equiv 2 \bmod 4$. Let $b = 2m$ and $c = 2n$. Observe that

$$24 = 7^c - 5^b = (7^n - 5^m)(7^n + 5^m)\,.$$

Since $7^n + 5^m > 0$, we have $7^n - 5^m > 0$. There are only a few ways to express $24 = 24 \cdot 1 = 12 \cdot 2 = 8 \cdot 3 = 6 \cdot 4$ as a product of two positive integers. By checking these cases we find one by one, the only solution in this case is $(a,b,c) = (4,2,2)$.

Having exhausted all cases, we find that the required set of triples is

$$(a,b,c) \in \{(3,0,1), (1,2,1), (4,2,2)\}\,.$$

N7. Find all perfect squares n such that if the positive integer $a \geqslant 15$ is some divisor of n then $a + 15$ is a prime power.

Solution. We call positive a integer a "nice" if $a + 15$ is a prime power.

From the definition, the numbers $n = 1, 4, 9$ satisfy the required property. Suppose that for some $t \in \mathbb{Z}^+$, the number $n = t^2 \geqslant 15$ also satisfies the required property. We have two cases:

1. If n is a power of 2, then $n \in \{16, 64\}$ since

$$2^4 + 15 = 31, \quad 2^5 + 15 = 47, \quad \text{and} \quad 2^6 + 15 = 79$$

are prime, and $2^7 + 15 = 143 = 11 \cdot 13$ is not a prime power. (Thus 2^7 does not divide n and therefore no higher power of 2 satisfies the required property.)

2. Suppose n has some odd prime divisor p. If $p > 3$ then $p^2|n$ and $p^2 > 15$ which imply that p^2 must be a nice number. Hence

$$p^2 + 15 = q^m$$

for some prime q and some $m \in \mathbb{Z}^+$. Since p is odd, then $p^2 + 15$ is even, thus we can conclude that $q = 2$. I.e.

$$p^2 + 15 = 2^m .$$

Considering the above modulo 3, we can see that $p^2 + 15 \equiv 0, 1 \bmod 3$, so $2^m \equiv 1 \bmod 3$, and so m is even. Suppose $m = 2k$ for some $k \in \mathbb{Z}^+$. So we have $(2^k - p)(2^k + p) = 15$ and $(2^k + p) - (2^k - p) = 2p \geqslant 10$. Thus

$$2^k - p = 1 \quad \text{and} \quad 2^k + p = 15 ,$$

giving $p = 7$ and $k = 3$. Thus we can write $n = 4^x \cdot 9^y \cdot 49^z$ for some non-negative integers x, y, z.

Note that 27 is not nice, so $27 \nmid n$ and therefore $y \leqslant 1$. The numbers 18 and 21 are also not nice, so similarly, x, y and y, z cannot both positive. Hence, we just need to consider $n = 4^x \cdot 49^z$ with $z \geqslant 1$.

Note that 7^3 is not nice, so $z = 1$. By checking directly, we can see that $7^2 + 15 = 2^6, 2 \cdot 7^2 + 15 = 113, 4 \cdot 7^2 + 15 = 211$ are nice, but $8 \cdot 7^2$ is not nice, so only $n = 49, 196$ satisfy the required property.

Therefore, the numbers n which satisfy the required property are $1, 4, 9, 16, 49, 64$ and 196.

Remark by PSC. One can get rid of the case $3|n$ by noting that in that case, we have $9|n$. But then $n^2 + 15$ is a multiple of 3 but not a multiple of 9 which is impossible. This simplifies a little bit the second case.