Chapter 1

2010 Shortlist JBMO - Problems

1.1 Algebra

A1 The real numbers a, b, c, d satisfy simultaneously the equations

$$abc - d = 1,\ bcd - a = 2,\ cda - b = 3,\ dab - c = -6.$$

Prove that $a + b + c + d \neq 0$.

A2 Determine all four digit numbers $\overline{abcd}$ such that

$$a(a + b + c + d)(a^2 + b^2 + c^2 + d^2)(a^6 + 2b^6 + 3c^6 + 4d^6) = \overline{abcd}.$$

A3 Find all pairs (x, y) of real numbers such that $|x| + |y| = 1340$ and $x^3 + y^3 + 2010xy = 670^3$.

A4 Let a, b, c be positive real numbers such that $abc(a + b + c) = 3$. Prove the inequality

$$(a + b)(b + c)(c + a) \geq 8,$$

and determine all cases when equality holds.

A5 The real positive numbers x, y, z satisfy the relations $x \leq 2$, $y \leq 3$, $x + y + z = 11$. Prove that $\sqrt{xyz} \leq 6$.

1.2 Combinatorics

C1 There are two piles of coins, each containing 2010 pieces. Two players A and B play a game taking turns (A plays first). At each turn, the player on play has to take one or more coins from one pile or exactly one coin from each pile. Whoever takes the last coin is the winner. Which player will win if they both play in the best possible way?

C2 A 9×7 rectangle is tiled with pieces of two types, shown in the picture below.

Find all possible values of the number of the 2×2 pieces which can be used in such a tiling.

1.3 Geometry

G1 Consider a triangle ABC with $\angle ACB = 90^\circ$. Let F be the foot of the altitude from C. Circle ω touches the line segment FB at point P, the altitude CF at point Q and the circumcircle of ABC at point R. Prove that points A, Q, R are collinear and $AP = AC$.

G2 Consider a triangle ABC and let M be the midpoint of the side BC. Suppose $\angle MAC = \angle ABC$ and $\angle BAM = 105^\circ$. Find the measure of $\angle ABC$.

G3 Let ABC be an acute-angled triangle. A circle $\omega_1(O_1, R_1)$ passes through points B and C and meets the sides AB and AC at points D and E, respectively. Let $\omega_2(O_2, R_2)$ be the circumcircle of the triangle ADE. Prove that O_1O_2 is equal to the circumradius of the triangle ABC.

G4 Let AL and BK be angle bisectors in the non-isosceles triangle ABC $(L \in BC, K \in AC)$. The perpendicular bisector of BK intesects the line AL at point M. Point N lies on the line BK such that $LN \parallel MK$. Prove that $LN = NA$.

1.4 Number Theory

NT1 Find all positive integers n such that $n2^{n+1} + 1$ is a perfect square.

NT2 Find all positive integers n such that $36^n - 6$ is a product of two or more consecutive positive integers.

Chapter 2

2010 Shortlist JBMO - Solutions

2.1 Algebra

A1 The real numbers a, b, c, d satisfy simultaneously the equations

$$abc - d = 1,\ bcd - a = 2,\ cda - b = 3,\ dab - c = -6.$$

Prove that $a + b + c + d \neq 0$.
Solution. Suppose that $a + b + c + d = 0$. Then

$$abc + bcd + cda + dab = 0. \tag{1}$$

If $abcd = 0$, then one of numbers, say d, must be 0. In this case $abc = 0$, and so at least two of the numbers a, b, c, d will be equal to 0, making one of the given equations impossible. Hence $abcd \neq 0$ and, from (1),

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 0,$$

implying

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c}.$$

It follows that $(a+b)(b+c)(c+a) = 0$, which is impossible (for instance, if $a+b = 0$, then adding the second and third given equations would lead to $0 = 2+3$, a contradiction). Thus $a + b + c + d \neq 0$.

A2 Determine all four digit numbers $\overline{abcd}$ such that

$$a(a + b + c + d)(a^2 + b^2 + c^2 + d^2)(a^6 + 2b^6 + 3c^6 + 4d^6) = \overline{abcd}.$$

Solution. From $\overline{abcd} < 10000$ and

$$a^{10} \leq a(a+b+c+d)(a^2+b^2+c^2+d^2)(a^6+2b^6+3c^6+4d^6) = \overline{abcd}$$

follows that $a \leq 2$. We thus have two cases:
Case I: $a = 1$.
Obviously $2000 > \overline{1bcd} = (1+b+c+d)(1+b^2+c^2+d^2)(1+2b^6+3c^6+4d^6) \geq (b+1)(b^2+1)(2b^6+1)$, so $b \leq 2$. Similarly one gets $c < 2$ and $d < 2$. By direct check there is no solution in this case.
Case II: $a = 2$.
We have $3000 > \overline{2bcd} = 2(2+b+c+d)(4+b^2+c^2+d^2)(64+2b^6+3c^6+4d^6) \geq 2(b+2)(b^2+4)(2b^6+64)$, imposing $b \leq 1$. In the same way one proves $c < 2$ and $d < 2$. By direct check, we find out that 2010 is the only solution.

A3 Find all pairs (x, y) of real numbers such that $|x|+|y| = 1340$ and $x^3+y^3+2010xy = 670^3$.
Solution. Answer: $(-670; -670)$, $(1005, -335)$, $(-335; 1005)$.
To prove this, let $z = -670$. We have

$$0 = x^3+y^3+z^3-3xyz = \frac{1}{2}(x+y+z)((x-y)^2+(y-z)^2+(z-x)^2).$$

Thus either $x+y+z = 0$, or $x = y = z$. In the latter case we get $x = y = -670$, which satisfies both the equations. In the former case we get $x+y = 670$. Then at least one of x, y is positive, but not both, as from the second equation we would get $x+y = 1340$. If $x > 0 \geq y$, we get $x-y = 1340$, which together with $x+y = 670$ yields $x = 1005$, $y = -335$. If $y > 0 \geq x$ we get similarly $x = -335$, $y = 1005$.

A4 Let a, b, c be positive real numbers such that $abc(a+b+c) = 3$. Prove the inequality

$$(a+b)(b+c)(c+a) \geq 8,$$

and determine all cases when equality holds.
Solution. We have

$$A = (a+b)(b+c)(c+a) = (ab+ac+b^2+bc)(c+a) = (b(a+b+c)+ac)(c+a),$$

so by the given condition

$$A = \left(\frac{3}{ac}+ac\right)(c+a) = \left(\frac{1}{ac}+\frac{1}{ac}+\frac{1}{ac}+ac\right)(c+a).$$

Aplying the AM-GM inequality for four and two terms respectively, we get

$$A \geq 4\sqrt[4]{\frac{ac}{(ac)^3}} \cdot 2\sqrt{ac} = 8.$$

From the last part, it is easy to see that inequality holds when $a = c$ and $\frac{1}{ac} = ac$, i.e. $a = b = c = 1$.

A5 The real positive numbers x, y, z satisfy the relations $x \leq 2$, $y \leq 3$, $x + y + z = 11$. Prove that $\sqrt{xyz} \leq 6$.

Solution. For $x = 2$, $y = 3$ and $z = 6$ the equality holds.

After the substitutions $x = 2 - u$, $y = 3 - v$ with $u \in [0, 2)$, $v \in [0, 3)$, we obtain that $z = 6 + u + v$ and the required inequality becomes

$$(2 - u)(3 - v)(6 + u + v) \leq 36. \tag{1}$$

We shall need the following lemma.

Lemma. If real numbers a and b satisfy the relations $0 < b \leq a$, then for every real number $y \in [0, b)$ the inequality

$$\frac{a}{a + y} \geq \frac{b - y}{b} \tag{2}$$

holds.

Proof of the lemma. The inequality (2) is equivalent to

$$ab \geq ab - ay + by - y^2 \Leftrightarrow y^2 + (a - b)y \geq 0.$$

The last inequality is true, because $a \geq b > 0$ and $y \geq 0$.

The equality in (2) holds if $y = 0$. The lemma is proved. □

By using the lemma we can write the following inequalities:

$$\frac{6}{6 + u} \geq \frac{2 - u}{2}, \tag{3}$$

$$\frac{6}{6 + v} \geq \frac{3 - v}{3}, \tag{4}$$

$$\frac{6 + u}{6 + u + v} \geq \frac{6}{6 + v}. \tag{5}$$

By multiplying the inequalities (3), (4) and (5)[1] we obtain:

$$\frac{6 \cdot 6 \cdot (6 + u)}{(6 + u)(6 + v)(6 + u + v)} \geq \frac{6(2 - u)(3 - v)}{2 \cdot 3(6 + v)} \Leftrightarrow$$

[1] actually (5) does not follow from the lemma (it is not known that $6 + u \geq 6 + v$) but is nevertheless true

$$(2-u)(3-v)(6+u+v) \leq 2\cdot 3\cdot 6 = 36 \Leftrightarrow (1).$$

By virtue of lemma, the equality holds if and only if $u=v=0$.
Alternative solution. With the same substitutions write the inequality as

$$(6-u-v)(6+u+v)+(uv-2u-v)(6+u+v) \leq 36.$$

As the first product on the lefthand side is $36-(u+v)^2 \leq 36$, it is enough to prove that the second product is nonpositive. This comes easily from $|u-1| \leq 1$, $|v-2| \leq 2$ and $uv-2u-v=(u-1)(v-2)-2$, which implies $uv-v-2u \leq 0$.

Alternative solution. From the AM-GM inequality we have
$\frac{x}{2}\cdot\frac{y}{3}\cdot\frac{z}{6} \leq \left(\frac{\frac{x}{2}+\frac{y}{3}+\frac{z}{6}}{3}\right)^3 = \left(\frac{3x+2y+z}{18}\right)^3 = \left(\frac{(x+y+z)+2x+y}{18}\right)^3 \leq$
$\left(\frac{11+2\cdot 2+3}{18}\right)^3 = 1$, and the conclusion follows readily.

2.2 Combinatorics

C1 There are two piles of coins, each containing 2010 pieces. Two players A and B play a game taking turns (A plays first). At each turn, the player on play has to take one or more coins from one pile or exactly one coin from each pile. Whoever takes the last coin is the winner. Which player will win if they both play in the best possible way?
Solution. B wins.
In fact, we will show that A will lose if the total number of coins is a multiple of 3 and the two piles differ by not more than one coin (call this a *balanced* position). To this end, firstly notice that it is not possible to move from one balanced position to another. The winning strategy for B consists in returning A to a balanced position (notice that the initial position is a balanced position).
There are two types of balanced positions; for each of them consider the moves of A and the replies of B.
If the number in each pile is a multiple of 3 and there is at least one coin:
- if A takes $3n$ coins from one pile, then B takes $3n$ coins from the other one.
- if A takes $3n+1$ coins from one pile, then B takes $3n+2$ coins from the other one.
- if A takes $3n+2$ coins from one pile, then B takes $3n+1$ coins from the other one.
- if A takes a coin from each pile, then B takes one coin from one pile.

If the numbers are not multiples of 3, then we have $3m+1$ coins in one pile and $3m+2$ in the other one. Hence:
- if A takes $3n$ coins from one pile, then B takes $3n$ coins from the other one.
- if A takes $3n+1$ coins from the first pile $(n \le m)$, then B takes $3n+2$ coins from the second one.
- if A takes $3n+2$ coins from the second pile $(n \le m)$, then B takes $3n+1$ coins from the first one.
- if A takes $3n+2$ coins from the first pile $(n \le m-1)$, then B takes $3n+4$ coins from the second pile
- if A takes $3n+1$ coins from the second pile $(n \le m)$, then B takes $3n-1$ coins from the first one. This is impossible if A has taken only one coin from the second pile; in this case B takes one coin from each pile.
- if A takes a coin from each pile, then B takes one coin from the second pile.

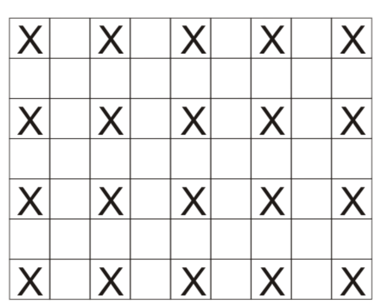
In all these cases, the position after B's move is again a balanced position. Since the number of coins decreases and $(0,0)$ is a balanced position, after a finite number of moves, there will be no coins left after B's move. Thus, B wins.

C2 A 9×7 rectangle is tiled with pieces of two types, shown in the picture below.

Find the possible values of the number of the 2×2 pieces which can be used in such a tiling.

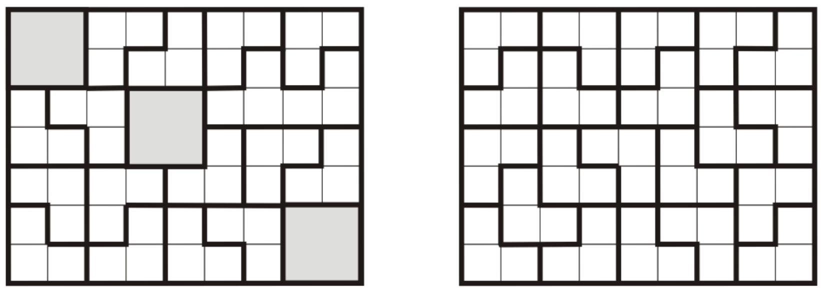
Solution. Answer: 0 or 3.

Denote by *x* the number of the pieces of the type "corner" and by *y* the number of the pieces of the type 2×2. Mark 20 squares of the rectangle as in the figure below.



Obviously, each piece covers at most one marked square.

Thus, $x + y \geq 20, \quad (1)$ and consequently $3x + 3y \geq 60, \quad (2)$.
On the other hand, computing the area of the rectangle, we obtain $3x + 4y = 63, \quad (3)$.
From (2) and (3) it follows that $y \leq 3$ and from (3), $3 \mid y$.
The proof is finished if we produce tilings with 3, respectively 0, 2×2 tiles:



2.3 Geometry

G1[2] Consider a triangle ABC with $\angle ACB = 90^\circ$. Let F be the foot of the altitude from C. Circle ω touches the line segment FB at point P, the altitude CF at point Q and the circumcircle of ABC at point R. Prove that points A, Q, R are collinear and $AP = AC$.
Solution. Let M be the midpoint of AB and let N be the center of ω. Then M is the circumcenter of triangle ABC, so points M, N and R are collinear. From $QN \parallel AM$ we get $\angle AMR = \angle QNR$. Besides that, triangles AMR and QNR are isosceles, therefore $\angle MRA = \angle NRQ$; thus points A, Q, R are collinear.
Right angled triangles AFQ and ARB are similar, which implies $\dfrac{AQ}{AB} = \dfrac{AF}{AR}$, that is $AQ \cdot AR = AF \cdot AB$. The power of point A with respect to ω gives $AQ \cdot AR = AP^2$. Also, from similar triangles ABC and ACF we get $AF \cdot AB = AC^2$. Now, the claim follows from $AC^2 = AF \cdot AB = AQ \cdot AR = AP^2$.

G2 Consider a triangle ABC and let M be the midpoint of the side BC. Suppose $\angle MAC = \angle ABC$ and $\angle BAM = 105^\circ$. Find the measure of $\angle ABC$.
Solution. The angle measure is 30°.
Let O be the circumcenter of the triangle ABM. From $\angle BAM = 105^\circ$ follows $\angle MBO = 15^\circ$. Let M', C' be the projections of points M, C onto the line BO. Since $\angle MBO = 15^\circ$, then $\angle MOM' = 30^\circ$ and consequently $MM' = \dfrac{MO}{2}$. On the other hand, MM' joins the midpoints of two sides of the triangle BCC', which implies $CC' = MO = AO$.

[2] also problem 2, JTST no. 3, Romania, 2012

The relation $\angle MAC = \angle ABC$ implies CA tangent to ω, hence $AO \perp AC$. It follows that $\triangle ACO \equiv \triangle OCC'$, and furthermore $OB \parallel AC$.
Therefore $\angle AOM = \angle AOM' - \angle MOM' = 90^\circ - 30^\circ = 60^\circ$ and $\angle ABM = \dfrac{\angle AOM}{2} = 30^\circ$.

G3 Let ABC be an acute-angled triangle. A circle $\omega_1(O_1, R_1)$ passes through points B and C and meets the sides AB and AC at points D and E, respectively. Let $\omega_2(O_2, R_2)$ be the circumcircle of the triangle ADE. Prove that O_1O_2 is equal to the circumradius of the triangle ABC.
Solution. Recall that, in every triangle, the altitude and the diameter of the circumcircle drawn from the same vertex are isogonal. The proof offers no difficulty, being a simple angle chasing around the circumcircle of the triangle.
Let O be the circumcenter of the triangle ABC. From the above, one has $\angle OAE = 90^\circ - \angle B$. On the other hand $\angle DEA = \angle B$, for $BCED$ is cyclic. Thus $AO \perp DE$, implying that in the triangle ADE cevians AO and AO_2 are isogonal. So, since AO is a radius of the circumcircle of triangle ABC, one obtains that AO_2 is an altitude in this triangle.
Moreover, since OO_1 is the perpendicular bisector of the line segment BC, one has $OO_1 \perp BC$, and furthermore $AO_2 \parallel OO_1$.
Chord DE is common to ω_1 and ω_2, hence $O_1O_2 \perp DE$. It follows that $AO \parallel O_1O_2$, so AOO_1O_2 is a parallelogram. The conclusion is now obvious.

G4 Let AL and BK be angle bisectors in the non-isosceles triangle ABC $(L \in BC, K \in AC)$. The perpendicular bisector of BK intesects the line AL at point M. Point N lies on the line BK such that $LN \parallel MK$. Prove that $LN = NA$.
Solution. The point M lies on the circumcircle of $\triangle ABK$ (since both AL and the perpendicular bisector of BK bisect the arc BK of this circle). Then $\angle ABK = \angle AMK = \angle NLA$. Thus $ABLN$ is cyclic, whence $\angle NAL = \angle NBL = \angle CBK = \angle NLA$. Now it follows that $LN = NA$. (Alternatively, we could finish as follows: $ABLN$ is cyclic and the angle bisector of $\angle ABL$ bisects the arc AL of the circumcircle of $ABLN$. Thus N lies on the perpendicular bisector of AC, which means that $LN = NA$.)

2.4 Number Theory

NT1 Find all positive integers n such that $n2^{n+1} + 1$ is a perfect square.
Solution. Answer: $n = 0$ and $n = 3$.

Clearly $n2^{n+1}+1$ is odd, so, if this number is a perfect square, then $n2^{n+1}+1=(2x+1)^2$, $x\in\mathbb{N}$, whence $n2^{n-1}=x(x+1)$.
The integers x and $x+1$ are coprime, so one of them must divisible by 2^{n-1}, which means that the other must be at most n. This shows that $2^{n-1}\leq n+1$.
An easy induction shows that the above inequality is false for all $n\geq 4$, and a direct inspection confirms that the only convenient values in the case $n\leq 3$ are 0 and 3.

NT2 Find all positive integers n such that 36^n-6 is a product of two or more consecutive positive integers.
Solution. Answer: $n=1$.
Among each four consecutive integers there is a multiple of 4. As 36^n-6 is not a multiple of 4, it must be the product of two or three consecutive positive integers.
Case I. If $36^n-6=x(x+1)$ (all letters here and below denote positive integers), then $4\cdot 36^n-23=(2x+1)^2$, whence $(2\times 6^n+2x+1)(2\times 6^n-2x-1)=23$. As 23 is prime, this leads to $2\times 6^n+2x+1=23$, $2\times 6^n-2x-1=1$. Subtracting these yields $4x+2=22$, $x=5$, $n=1$, which is a solution to the problem.
Case II. If $36^n-6=(y-1)y(y+1)$, then

$$36^n=y^3-y+6=(y^3+8)-(y+2)=(y+2)(y^2-2y+3).$$

Thus each of $y+2$ and y^2-2y+3 can have only 2 and 3 as prime factors, so the same is true for their GCD. This, combined with the identity $y^2-2y+3=(y+2)(y-4)+11$ yields $GCD(y+2;y^2-2y+3)=1$. Now $y+2<y^2-2y+3$ (if $y>2$; $y=1$ and $y=2$ do not work), so $y+2=4^n$, $y^2-2y+3=9^n$. The former identity implies y is even and now by the latter one $9^n\equiv 3 \pmod 4$, while in fact $9^n\equiv 1 \pmod 4$ - a contradiction. So, in this case there is no such n.