Algebra

A1. Let a, b, c be positive real numbers such that $a+b+c+ab+bc+ca+abc=7$. Prove that

$$\sqrt{a^2+b^2+2}+\sqrt{b^2+c^2+2}+\sqrt{c^2+a^2+2} \geq 6.$$

Solution. First we see that $x^2+y^2+1 \geq xy+x+y$. Indeed, this is equivalent to

$$(x-y)^2+(x-1)^2+(y-1)^2 \geq 0.$$

Therefore

$$\begin{aligned} & \sqrt{a^2+b^2+2}+\sqrt{b^2+c^2+2}+\sqrt{c^2+a^2+2} \\ \geq\ & \sqrt{ab+a+b+1}+\sqrt{bc+b+c+1}+\sqrt{ca+c+a+1} \\ =\ & \sqrt{(a+1)(b+1)}+\sqrt{(b+1)(a+1)}+\sqrt{(c+1)(a+1)} \end{aligned}$$

It follows from the AM-GM inequality that

$$\begin{aligned} & \sqrt{(a+1)(b+1)}+\sqrt{(b+1)(a+1)}+\sqrt{(c+1)(a+1)} \\ \geq\ & 3\sqrt[3]{\sqrt{(a+1)(b+1)} \cdot \sqrt{(b+1)(a+1)} \cdot \sqrt{(c+1)(a+1)}} \\ =\ & 3\sqrt[3]{(a+1)(b+1)(c+1)} \end{aligned}$$

On the other hand, the given condition is equivalent to $(a+1)(b+1)(c+1)=8$ and we get the desired inequality.

Obviously, equality is attained if and only if $a=b=c=1$.

Remark. The condition of positivity of a, b, c is superfluous and the equality $\cdots=7$ can be replaced by the inequality $\cdots \geq 7$. Indeed, the above proof and the triangle inequality imply that

$$\begin{aligned} \sqrt{a^2+b^2+2}+\sqrt{b^2+c^2+2}+\sqrt{c^2+a^2+2} &\geq 3\sqrt[3]{(|a|+1)(|b|+1)(|c|+1)} \\ &\geq 3\sqrt[3]{|a+1|.|b+1|.|c+1|} \geq 6. \end{aligned}$$

A2. Let a and b be positive real numbers such that $3a^2+2b^2=3a+2b$. Find the minimum value of

$$A=\sqrt{\frac{a}{b(3a+2)}}+\sqrt{\frac{b}{a(2b+3)}}.$$

Solution. By the Cauchy-Schwarz inequality we have that

$$5(3a^2+2b^2)=5(a^2+a^2+a^2+b^2+b^2)\geq(3a+2b)^2$$

(or use that the last inequality is equivalent to $(a-b)^2\geq 0$).

So, with the help of the given condition we get that $3a+2b\leq 5$. Now, by the AM-GM inequality we have that

$$A\geq 2\sqrt{\sqrt{\frac{a}{b(3a+2)}}\cdot\sqrt{\frac{b}{a(2b+3)}}}=\frac{2}{\sqrt[4]{(3a+2)(2b+3)}}.$$

Finally, using again the AM-GM inequality, we get that

$$(3a+2)(2b+3)\leq\left(\frac{3a+2b+5}{2}\right)^2\leq 25,$$

so $A\geq 2/\sqrt{5}$ and the equality holds if and only if $a=b=1$.

A3. Let a, b, c, d be real numbers such that $0 \leq a \leq b \leq c \leq d$. Prove the inequality

$$ab^3 + bc^3 + cd^3 + da^3 \geq a^2b^2 + b^2c^2 + c^2d^2 + d^2a^2.$$

Solution. The inequality is equivalent to

$$(ab^3 + bc^3 + cd^3 + da^3)^2 \geq (a^2b^2 + b^2c^2 + c^2d^2 + d^2a^2)^2.$$

By the Cauchy-Schwarz inequality,

$$(ab^3 + bc^3 + cd^3 + da^3)(a^3b + b^3c + c^3d + d^3a) \geq (a^2b^2 + b^2c^2 + c^2d^2 + d^2a^2)^2.$$

Hence it is sufficient to prove that

$$(ab^3 + bc^3 + cd^3 + da^3)^2 \geq (ab^3 + bc^3 + cd^3 + da^3)(a^3b + b^3c + c^3d + d^3a),$$

i.e. to prove $ab^3 + bc^3 + cd^3 + da^3 \geq a^3b + b^3c + c^3d + d^3a$.

This inequality can be written successively

$$a(b^3 - d^3) + b(c^3 - a^3) + c(d^3 - b^3) + d(a^3 - c^3) \geq 0,$$

or

$$(a - c)(b^3 - d^3) - (b - d)(a^3 - c^3) \geq 0,$$

which comes down to

$$(a - c)(b - d)(b^2 + bd + d^2 - a^2 - ac - c^2) \geq 0.$$

The last inequality is true because $a - c \leq 0$, $b - d \leq 0$, and $(b^2 - a^2) + (bd - ac) + (d^2 - c^2) \geq 0$ as a sum of three non-negative numbers.

The last inequality is satisfied with equality whence $a = b$ and $c = d$. Combining this with the equality cases in the Cauchy-Schwarz inequality we obtain the equality cases for the initial inequality: $a = b = c = d$.

Remark. Instead of using the Cauchy-Schwarz inequality, once the inequality $ab^3 + bc^3 + cd^3 + da^3 \geq a^3b + b^3c + c^3d + d^3a$ is established, we have $2(ab^3 + bc^3 + cd^3 + da^3) \geq (ab^3 + bc^3 + cd^3 + da^3) + (a^3b + b^3c + c^3d + d^3a) = (ab^3 + a^3b) + (bc^3 + b^3c) + (cd^3 + c^3d) + (da^3 + d^3a) \overset{AM-GM}{\geq} 2a^2b^2 + 2b^2c^2 + 2c^2d^2 + 2d^2a^2$ which gives the conclusion.

A4. Let x, y, z be three distinct positive integers. Prove that

$$(x+y+z)(xy+yz+zx-2) \geq 9xyz.$$

When does the equality hold?

Solution. Since x, y, z are distinct positive integers, the required inequality is symmetric and WLOG we can suppose that $x \geq y+1 \geq z+2$. We consider 2 possible cases:

Case 1. $y \geq z+2$. Since $x \geq y+1 \geq z+3$ it follows that

$$(x-y)^2 \geq 1, \quad (y-z)^2 \geq 4, \quad (x-z)^2 \geq 9$$

which are equivalent to

$$x^2+y^2 \geq 2xy+1, \quad y^2+z^2 \geq 2yz+4, \quad x^2+z^2 \geq 2xz+9$$

or otherwise

$$zx^2+zy^2 \geq 2xyz+z, \quad xy^2+xz^2 \geq 2xyz+4x, \quad yx^2+yz^2 \geq 2xyz+9y.$$

Adding up the last three inequalities we have

$$xy(x+y)+yz(y+z)+zx(z+x) \geq 6xyz+4x+9y+z$$

which implies that $(x+y+z)(xy+yz+zx-2) \geq 9xyz+2x+7y-z$.

Since $x \geq z+3$ it follows that $2x+7y-z \geq 0$ and our inequality follows.

Case 2. $y = z+1$. Since $x \geq y+1 = z+2$ it follows that $x \geq z+2$, and replacing $y = z+1$ in the required inequality we have to prove

$$(x+z+1+z)(x(z+1)+(z+1)z+zx-2) \geq 9x(z+1)z$$

which is equivalent to

$$(x+2z+1)(z^2+2zx+z+x-2)-9x(z+1)z \geq 0$$

Doing easy algebraic manipulations, this is equivalent to prove

$$(x-z-2)(x-z+1)(2z+1) \geq 0$$

which is satisfied since $x \geq z+2$.

The equality is achieved only in the Case 2 for $x = z+2$, so we have equality when $(x, y, z) = (k+2, k+1, k)$ and all the permutations for any positive integer k.

Combinatorics

C1. Consider a regular $2n+1$-gon P in the plane, where n is a positive integer. We say that a point S on one of the sides of P can be seen from a point E that is external to P, if the line segment SE contains no other points that lie on the sides of P except S. We want to color the sides of P in 3 colors, such that every side is colored in exactly one color, and each color must be used at least once. Moreover, from every point in the plane external to P, at most 2 different colors on P can be seen (ignore the vertices of P, we consider them colorless). Find the largest positive integer for which such a coloring is possible.

Solution. Answer: $n = 1$ is clearly a solution, we can just color each side of the equilateral triangle in a different color, and the conditions are satisfied. We prove there is no larger n that fulfills the requirements.

Lemma 1. Given a regular $2n+1$-gon in the plane, and a sequence of $n+1$ consecutive sides $s_1, s_2, \ldots, s_{n+1}$ there is an external point Q in the plane, such that the color of each s_i can be seen from Q, for $i = 1, 2, \ldots, n+1$.

Proof. It is obvious that for a semi-circle S, there is a point R in the plane far enough on the perpendicular bisector of the diameter of S such that almost the entire semi-circle can be seen from R.

Now, it is clear that looking at the circumscribed circle around the $2n+1$-gon, there is a semi-circle S such that each s_i either has both endpoints on it, or has an endpoint that is on the semi-circle, and is not on the semicircle's end. So, take Q to be a point in the plane from which almost all of S can be seen, clearly, the color of each s_i can be seen from Q. $\diamond$

Take $n \geq 2$, denote the sides $a_1, a_2, \ldots, a_{2n+1}$ in that order, and suppose we have a coloring that satisfies the condition of the problem. Let's call the 3 colors red, green and blue. We must have 2 adjacent sides of different colors, say a_1 is red and a_2 is green. Then, by Lemma 1:

(i) We cannot have a blue side among $a_1, a_2, \ldots, a_{n+1}$.

(ii) We cannot have a blue side among $a_2, a_1, a_{2n+1}, \ldots, a_{n+3}$.

We are required to have at least one blue side, and according to 1) and 2), that can only be a_{n+2}, so a_{n+2} is blue. Now, applying Lemma 1 on the sequence of sides $a_2, a_3, \ldots, a_{n+2}$ we get that $a_2, a_3, \ldots, a_{n+1}$ are all green. Applying Lemma 1 on the sequence of sides $a_1, a_{2n+1}, a_{2n}, \ldots, a_{n+2}$ we get that $a_{2n+1}, a_{2n}, \ldots, a_{n+3}$ are all red.

Therefore a_{n+1}, a_{n+2} and a_{n+3} are all of different colors, and for $n \geq 2$ they can all be seen from the same point according to Lemma 1, so we have a contradiction.

C2. Consider a regular $2n$-gon P in the plane, where n is a positive integer. We say that a point S on one of the sides of P can be seen from a point E that is external to P, if the line segment SE contains no other points that lie on the sides of P except S. We want to color the sides of P in 3 colors, such that every side is colored in exactly one color, and each color must be used at least once. Moreover, from every point in the plane external to P, at most 2 different colors on P can be seen (ignore the vertices of P, we consider them colorless). Find the number of distinct such colorings of P(two colorings are considered distinct if at least one side is colored differently).

Solution. Answer: For $n = 2$, the answer is 36; for $n = 3$, the answer is 30 and for $n \geq 4$, the answer is $6n$.

Lemma 1. Given a regular $2n$-gon in the plane and a sequence of n consecutive sides $s_1, s_2, \ldots, s_n$ there is an external point Q in the plane, such that the color of each s_i can be seen from Q, for $i = 1, 2, \ldots, n$.

Proof. It is obvious that for a semi-circle S, there is a point R in the plane far enough on the bisector of its diameter such that almost the entire semi-circle can be seen from R.

Now, it is clear that looking at the circumscribed circle around the $2n$-gon, there is a semi-circle S such that each s_i either has both endpoints on it, or has an endpoint that's on the semi-circle, and is not on the semi-circle's end. So, take Q to be a point in the plane from which almost all of S can be seen, clearly, the color of each s_i can be seen from Q.

Lemma 2. Given a regular $2n$-gon in the plane, and a sequence of $n + 1$ consecutive sides $s_1, s_2, \ldots, s_{n+1}$ there is no external point Q in the plane, such that the color of each s_i can be seen from Q, for $i = 1, 2, \ldots, n + 1$.

Proof. Since s_1 and s_{n+1} are parallel opposite sides of the $2n$-gon, they cannot be seen at the same time from an external point.

For $n = 2$, we have a square, so all we have to do is make sure each color is used. Two sides will be of the same color, and we have to choose which are these 2 sides, and then assign colors according to this choice, so the answer is $\binom{4}{2}.3.2 = 36$.

For $n = 3$, we have a hexagon. Denote the sides as $a_1, a_2, \ldots q_6$, in that order. There must be 2 consecutive sides of different colors, say a_1 is red, a_2 is blue. We must have a green side, and only a_4 and a_5 can be green. We have 3 possibilities:

1) a_4 is green, a_5 is not. So, a_3 must be blue and a_5 must be blue (by elimination) and a_6 must be blue, so we get a valid coloring.

2) Both a_4 and a_5 are green, thus a_6 must be red and a_5 must be blue, and we get the coloring $rbbggr$.

3) a_5 is green, a_4 is not. Then a_6 must be red. Subsequently, a_4 must be red (we assume it is not green). It remains that a_3 must be red, and the coloring is $rbrrgr$.

Thus, we have 2 kinds of configurations:

i) 2 opposite sides have 2 opposite colors and all other sides are of the third color. This can happen in $3.(3.2.1) = 18$ ways (first choosing the pair of opposite sides, then assigning colors),

ii) 3 pairs of consecutive sides, each pair in one of the 3 colors. This can happen in $2.6 = 12$ ways (2 partitioning into pairs of consecutive sides, for each partitioning, 6 ways to assign the colors).

Thus, for $n = 3$, the answer is $18 + 12 = 30$.

Finally, let's address the case $n \geq 4$. The important thing now is that any 4 consecutive sides can be seen from an external point, by Lemma 1.

Denote the sides as $a_1, a_2, \ldots, a_{2n}$. Again, there must be 2 adjacent sides that are of different colors, say a_1 is blue and a_2 is red. We must have a green side, and by Lemma 1, that can only be a_{n+1} or a_{n+2}. So, we have 2 cases:

Case 1: a_{n+1} is green, so a_n must be red (cannot be green due to Lemma 1 applied to $a_1, a_2, \ldots, a_n$, cannot be blue for the sake of $a_2, \ldots, a_{n+1}$. If a_{n+2} is red, so are $a_{n+3}, \ldots, a_{2n}$, and we get a valid coloring: a_1 is blue, a_{n+1} is green, and all the others are red.

If a_{n+2} is green:

a) a_{n+3} cannot be green, because of $a_2, a_1, a_{2n} \ldots, a_{n+3}$.

b) a_{n+3} cannot be blue, because the 4 adjacent sides $a_n, \ldots, a_{n+3}$ can be seen (this is the case that makes the separate treatment of $n \geq 4$ necessary)

c) a_{n+3} cannot be red, because of $a_1, a_{2n}, \ldots, a_{n+2}$.

So, in the case that a_{n+2} is also green, we cannot get a valid coloring.

Case 2: a_{n+2} is green is treated the same way as Case 1.

This means that the only valid configuration for $n \geq 4$ is having 2 opposite sides colored in 2 different colors, and all other sides colored in the third color. This can be done in $n \cdot 3 \cdot 2 = 6n$ ways.

C3. We have two piles with 2000 and 2017 coins respectively. Ann and Bob take alternate turns making the following moves: The player whose turn is to move picks a pile with at least two coins, removes from that pile t coins for some $2 \leqslant t \leqslant 4$, and adds to the other pile 1 coin. The players can choose a different t at each turn, and the player who cannot make a move loses. If Ann plays first determine which player has a winning strategy.

Solution. Denote the number of coins in the two piles by X and Y. We say that the pair (X, Y) is losing if the player who begins the game loses and that the pair (X, Y) is winning otherwise. We shall prove that (X, Y) is loosing if $X - Y \equiv 0, 1, 7 \bmod 8$, and winning if $X - Y \equiv 2, 3, 4, 5, 6 \bmod 8$.

Lemma 1. If we have a winning pair (X, Y) then we can always play in such a way that the other player is then faced with a losing pair.

Proof of Lemma 1. Assume $X \geq Y$ and write $X = Y + 8k + \ell$ for some non-negative integer k and some $\ell \in \{2, 3, 4, 5, 6\}$. If $\ell = 2, 3, 4$ then we remove two coins from the first pile and add one coin to the second pile. If $\ell = 5, 6$ then we remove four coins from the first pile and add one coin to the second pile. In each case we then obtain loosing pair $\square$

Lemma 2. If we are faced with a losing distribution then either we cannot play, or, however we play, the other player is faced with a winning distribution.

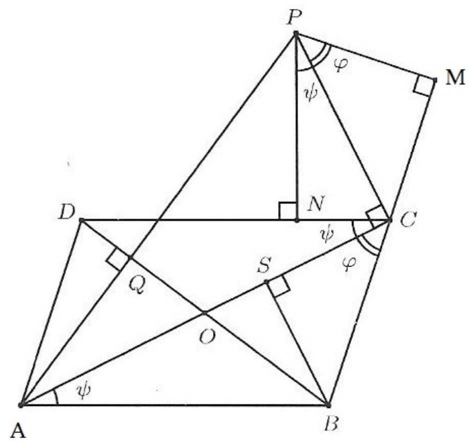
Proof of Lemma 2. Without loss of generality we may assume that we remove k coins from the first pile. The following table show the new difference for all possible values of k and all possible differences $X - Y$. So however we move, the other player will be faced with a winning distribution. $\square$

$k \backslash X - Y$	0	1	7
2	5	6	4
3	4	5	3
4	3	4	2

Since initially the coin difference is 1 mod 8, by Lemmas 1 and 2 Bob has a winning strategy: He can play so that he is always faced with a winning distribution while Ann is always faced with a losing distribution. So Bob cannot lose. On the other hand the game finishes after at most 4017 moves, so Ann has to lose.

Geometry

G1. Given a parallelogram $ABCD$. The line perpendicular to AC passing through C and the line perpendicular to BD passing through A intersect at point P. The circle centered at point P and radius PC intersects the line BC at point X, $(X \neq C)$ and the line DC at point Y, $(Y \neq C)$. Prove that the line AX passes through the point Y.



Solution. Denote the feet of the perpendiculars from P to the lines BC and DC by M and N respectively and let $O = AC \cap BD$. Since the points O, M and N are midpoints of CA, CX and CY respectively it suffices to prove that M, N and O are collinear. According to Menelaus's theorem for $\triangle BCD$ and points M, N and O we have to prove that

$$\frac{BM}{MC} \cdot \frac{CN}{ND} \cdot \frac{DO}{OB} = 1$$

Since $DO = OB$ the above simplifies to $\frac{BM}{CM} = \frac{DN}{CN}$. It follows from $BM = BC + CM$ and $DN = DC - CN = AB - CN$ that the last equality is equivalent to:

$$\frac{BC}{CM} + 2 = \frac{AB}{CN}. \tag{1}$$

Denote by S the foot of the perpendicular from B to AC. Since $\measuredangle BCS = \measuredangle CPM = \varphi$ and $\measuredangle BAC = \measuredangle ACD = \measuredangle CPN = \psi$ we conclude that $\triangle CBS \sim \triangle PCM$ and $\triangle ABS \sim \triangle PCN$. Therefore

$$\frac{CM}{BS} = \frac{CP}{BC} \quad \text{and} \quad \frac{CN}{BS} = \frac{CP}{AB}$$

and thus,

$$CM = \frac{CP.BS}{BC} \quad \text{and} \quad CN = \frac{CP.BS}{AB}.$$

Now equality (1) becomes $AB^2 - BC^2 = 2CP.BS$. It follows from

$$AB^2 - BC^2 = AS^2 - CS^2 = (AS - CS)(AS + CS) = 2OS.AC$$

that

$$DC^2 - BC^2 = 2CP.BS \iff 2OS.AC = 2CP.BS \iff OS.AC = CP.BS.$$

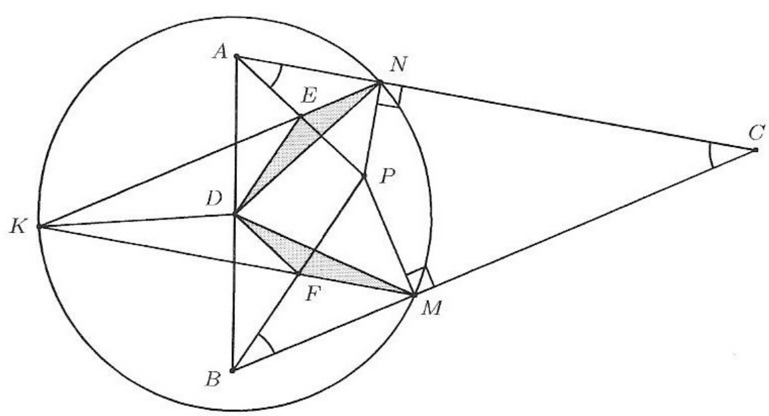
Since $\sphericalangle ACP = \sphericalangle BSO = 90^\circ$ and $\sphericalangle CAP = \sphericalangle SBO$ we conclude that $\triangle ACP \sim \triangle BSO$. This implies $OS.AC = CP.BS$, which completes the proof.

G2. Let ABC be an acute triangle such that AB is the shortest side of the triangle. Let D be the midpoint of the side AB and P be an interior point of the triangle such that

$$\measuredangle CAP = \measuredangle CBP = \measuredangle ACB.$$

Denote by M and N the feet of the perpendiculars from P to BC and AC, respectively. Let p be the line through M parallel to AC and q be the line through N parallel to BC. If p and q intersect at K prove that D is the circumcenter of triangle MNK.

Solution. If $\gamma = \measuredangle ACB$ then $\measuredangle CAP = \measuredangle CBP = \measuredangle ACB = \gamma$. Let $E = KN \cap AP$ and $F = KM \cap BP$. We show that points E and F are midpoints of AP and BP, respectively.



Indeed, consider the triangle AEN. Since $KN \parallel BC$, we have $\measuredangle ENA = \measuredangle BCA = \gamma$. Moreover $\measuredangle EAN = \gamma$ giving that triangle AEN is isosceles, i.e. $AE = EN$. Next, consider the triangle ENP. Since $\measuredangle ENA = \gamma$ we find that

$$\measuredangle PNE = 90^\circ - \measuredangle ENA = 90^\circ - \gamma.$$

Now $\measuredangle EPN = 90^\circ - \gamma$ implies that the triangle ENP is isosceles triangle, i.e. $EN = EP$. Since $AE = EN = EP$ point E is the midpoint of AP and analogously, F is the midpoint of BP. Moreover, D is also midpoint of AB and we conclude that $DFPE$ is parallelogram.

It follows from $DE \parallel AP$ and $KE \parallel BC$ that $\sphericalangle DEK = \sphericalangle CBP = \gamma$ and analogously $\sphericalangle DFK = \gamma$.

We conclude that $\triangle EDN \cong \triangle FMD$ ($ED = FP = FM$, $EN = EP = FD$ and $\sphericalangle DEN = \sphericalangle MFD = 180^\circ - \gamma$) and thus $ND = MD$. Therefore D is a point on the perpendicular bisector of MN. Further,

$$\begin{aligned} \sphericalangle FDE &= \sphericalangle FPE = 360^\circ - \sphericalangle BPM - \sphericalangle MPN - \sphericalangle NPA = \\ &= 360^\circ - (90^\circ - \gamma) - (180^\circ - \gamma) - (90^\circ - \gamma) = 3\gamma. \end{aligned}$$

It follows that

$$\begin{aligned} \sphericalangle MDN &= \sphericalangle FDE - \sphericalangle FDM - \sphericalangle EDN = \sphericalangle FDE - \sphericalangle END - \sphericalangle EDN = \\ &= \sphericalangle FDE - (\sphericalangle END + \sphericalangle EDN) = 3\gamma - \gamma = 2\gamma. \end{aligned}$$

Fianlly, $KMCN$ is parallelogram, i.e. $\sphericalangle MKN = \sphericalangle MCN = \gamma$. Therefore D is a point on the perpendicular bisector of MN and $\sphericalangle MDN = 2\sphericalangle MKN$, so D is the circumcenter of $\triangle MNK$.

Problem G3. Consider triangle ABC such that $AB \leq AC$. Point D on the arc BC of the circumcirle of ABC not containing point A and point E on side BC are such that

$$\sphericalangle BAD = \sphericalangle CAE < \frac{1}{2}\sphericalangle BAC.$$

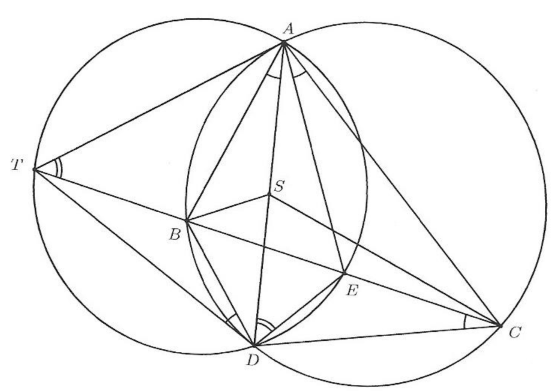
Let S be the midpoint of segment AD. If $\sphericalangle ADE = \sphericalangle ABC - \sphericalangle ACB$ prove that

$$\sphericalangle BSC = 2\sphericalangle BAC.$$

Solution. Let the tangent to the circumcircle of $\triangle ABC$ at point A intersect line BC at T. Since $AB \leq AC$ we get that B lies between T and C. Since $\sphericalangle BAT = \sphericalangle ACB$ and $\sphericalangle ABT = \sphericalangle 180° - \sphericalangle ABC$ we get $\sphericalangle ETA = \sphericalangle BTA = \sphericalangle ABC - \sphericalangle ACB = \sphericalangle ADE$ which gives that A, E, D, T are concyclic. Since

$$\sphericalangle TDB + \sphericalangle BCA = \sphericalangle TDB + \sphericalangle BDA = \sphericalangle TDA = \sphericalangle AET = \sphericalangle ACB + \sphericalangle EAC$$

this means $\sphericalangle TDB = \sphericalangle EAC = \sphericalangle DAB$ which means that TD is tangent to the circumcircle of $\triangle ABC$ at point D.



Using similar triangles TAB and TCA we get

(1) $$\frac{AB}{AC} = \frac{TA}{TC}.$$

Using similar triangles TBD and TDC we get

$$(2) \qquad \frac{BD}{CD} = \frac{TD}{TC}.$$

Using the fact that $TA = TD$ with (1) and (2) we get

$$(3) \qquad \frac{AB}{AC} = \frac{BD}{CD}.$$

Now since $\sphericalangle DAB = \sphericalangle CAE$ and $\sphericalangle BDA = \sphericalangle ECA$ we get that the triangles DAB and CAE are similar. Analogously, we get that triangles CAD and EAB are similar. These similarities give us

$$\frac{DB}{CE} = \frac{AB}{AE} \quad \text{and} \quad \frac{CD}{EB} = \frac{CA}{EA}$$

which, when combined with (3) give us $BE = CE$ giving E is the midpoint of side BC.

Using the fact that triangles DAB and CAE are similar with the fact that E is the midpoint of BC we get:

$$\frac{2DS}{CA} = \frac{DA}{CA} = \frac{DB}{CE} = \frac{DB}{\frac{CB}{2}} = \frac{2DB}{CB}$$

implying that

$$(4) \qquad \frac{DS}{DB} = \frac{CA}{CB}.$$

Since $\sphericalangle SDB = \sphericalangle ADB = \sphericalangle ACB$ we get from (4) that the triangles SDB and ACB are similar, giving us $\sphericalangle BSD = \sphericalangle BAC$. Analogously we get $\triangle SDC$ and $\triangle ABC$ are similar we get $\sphericalangle CSD = \sphericalangle CAB$. Combining the last two equalities we get

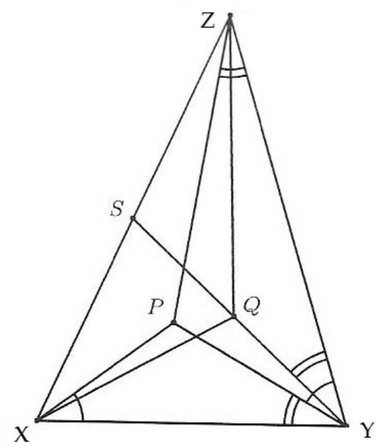
$$2\sphericalangle BAC = \sphericalangle BAC + \sphericalangle CAB = \sphericalangle CSD + \sphericalangle BSD = \sphericalangle CSB.$$

This completes the proof.

Alternative solution (PSC).

Lemma 1. A point P is such that $\sphericalangle PXY = \sphericalangle PYZ$ and $\sphericalangle PZY = \sphericalangle PYX$. If R is the midpoint of XZ then $\sphericalangle XYP = \sphericalangle ZYR$.

Proof. We consider the case when P is inside the triangle XYZ (the other case is treated in similar way). Let Q be the conjugate of P in $\triangle XYZ$ and let YQ intersects XZ at S.



Then $\sphericalangle QXZ = \sphericalangle QYX$ and $\sphericalangle QZX = \sphericalangle QYZ$ and therefore $\triangle SXY \sim \triangle SQX$ and $\triangle SZY \sim \triangle SQZ$. Thus $SX^2 = SQ.SY = SZ^2$ and we conclude that $S \equiv R$. This completes the proof of the Lemma.

For $\triangle DCA$ we have $\sphericalangle CDE = \sphericalangle ECA$ and $\sphericalangle EAC = \sphericalangle ECD$. By the Lemma 1 for $\triangle DCA$ and point E we have that $\sphericalangle SCA = \sphericalangle DCE$. Therefore

$$\sphericalangle DSC = \sphericalangle SAC + \sphericalangle SCA = \sphericalangle SAC + \sphericalangle DCE = \sphericalangle SAC + \sphericalangle BAD = \sphericalangle BAC.$$

By analogy, Lemma 1 applied for $\triangle BDA$ and point E gives $\sphericalangle BSD = \sphericalangle BAC$. Thus, $\sphericalangle BSC = 2\sphericalangle BAC$.

Problem G4. Let ABC be a scalene triangle with circumcircle Γ and circumcenter O. Let M be the midpoint of BC and D be a point on Γ such that $AD \perp BC$. Let T be a point such that $BDCT$ is a parallelogram and Q a point on the same side of BC as A such that

$$\measuredangle BQM = \measuredangle BCA \quad \text{and} \quad \measuredangle CQM = \measuredangle CBA.$$

Let AO intersect Γ again at E and let the circumcircle of ETQ intersect Γ at point $X \neq E$. Prove that the points A, M, and X are collinear.

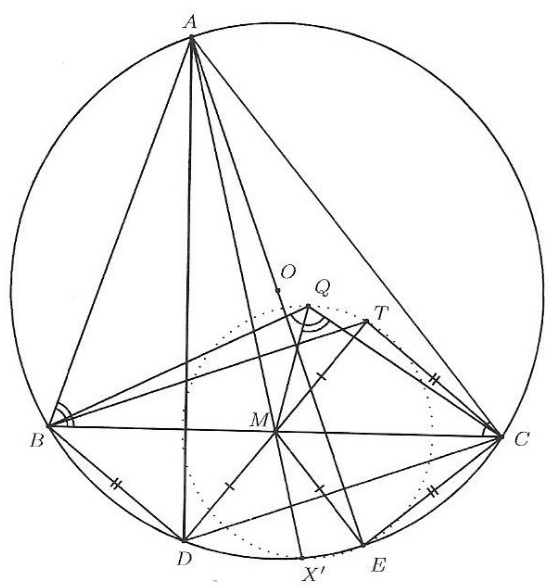
Solution. Let X' be symmetric point to Q in line BC. Now since $\measuredangle CBA = \measuredangle CQM = \measuredangle CX'M$, $\measuredangle BCA = \measuredangle BQM = \measuredangle BX'M$, we have

$$\measuredangle BX'C = \measuredangle BX'M + \measuredangle CX'M = \measuredangle CBA + \measuredangle BCA = 180^\circ - \measuredangle BAC$$

we have that $X' \in \Gamma$. Now since $\measuredangle AX'B = \measuredangle ACB = \measuredangle MX'B$ we have that A, M, X' are collinear. Note that since

$$\measuredangle DCB = \measuredangle DAB = 90^\circ - \measuredangle ABC = \measuredangle OAC = \measuredangle EAC$$

we get that $DBCE$ is an isosceles trapezoid.



Since $BDCT$ is a parallelogram we have $MT = MD$, with M, D, T being collinear, $BD = CT$, and since $BDEC$ is an isosceles trapezoid we have $BD = CE$ and $ME = MD$. Since

$$\sphericalangle BTC = \sphericalangle BDC = \sphericalangle BED, \quad CE = BD = CT \quad \text{and} \quad ME = MT$$

we have that E and T are symmetric with respect to the line BC. Now since Q and X' are symmetric with respect to the line BC as well, this means that $QX'ET$ is an isosceles trapezoid which means that Q, X', E, T are concyclic. Since $X' \in \Gamma$ this means that $X \equiv X'$ and therefore A, M, X are collinear.

Alternative solution (PSC). Denote by H the orthocenter of $\triangle ABC$. We use the following well known properties:

(i) Point D is the symmetric point of H with respect to BC. Indeed, if H_1 is the symmetric point of H with respect to BC then $\sphericalangle BH_1C + \sphericalangle BAC = 180°$ and therefore $H_1 \equiv D$.

(ii) The symmetric point of H with respect to M is the point E. Indeed, if H_2 is the symmetric point of H with respect to M then BH_2CH is parallelogram, $\sphericalangle BH_2C + \sphericalangle BAC = 180°$ and since $EB \parallel CH$ we have $\sphericalangle EBA = 90°$.

Since $DETH$ is a parallelogram and $MH = MD$ we have that $DETH$ is a rectangle. Therefore $MT = ME$ and $TE \perp BC$ implying that T and E are symmetric with respect to BC. Denote by Q' the symmetric point of Q with respect to BC. Then $Q'ETQ$ is isosceles trapezoid, so Q' is a point on the circumcircle of $\triangle ETQ$. Moreover $\sphericalangle BQ'C + \sphericalangle BAC = 180°$ and we conclude that $Q' \in \Gamma$. Therefore $Q' \equiv X$.

It remains to observe that $\sphericalangle CXM = \sphericalangle CQM = \sphericalangle CBA$ and $\sphericalangle CXA = \sphericalangle CBA$ and we infer that X, M and A are collinear.

Problem G5. A point P lies in the interior of the triangle ABC. The lines AP, BP, and CP intersect BC, CA, and AB at points D, E, and F, respectively. Prove that if two of the quadrilaterals $ABDE$, $BCEF$, $CAFD$, $AEPF$, $BFPD$, and $CDPE$ are concyclic, then all six are concyclic.

Solution. We first prove the following lemma:

Lemma 1. Let $ABCD$ be a convex quadrilateral and let $AB \cap CD = E$ and $BC \cap DA = F$. Then the circumcircles of triangles ABF, CDF, BCE and DAE all pass through a common point P. This point lies on line EF if and only if $ABCD$ in concyclic.

Proof. Let the circumcircles of ABF and BCF intersect at $P \neq B$. We have

$$\begin{aligned} \sphericalangle FPC &= \sphericalangle FPB + \sphericalangle BPC = \sphericalangle BAD + \sphericalangle BEC = \sphericalangle EAD + \sphericalangle AED = \\ &= 180^\circ - \sphericalangle ADE = 180^\circ - \sphericalangle FDC \end{aligned}$$

which gives us F, P, C and D are concyclic. Similarly we have

$$\begin{aligned} \sphericalangle APE &= \sphericalangle APB + \sphericalangle BPE = \sphericalangle AFB + \sphericalangle BCD = \sphericalangle DFC + \sphericalangle FCD = \\ &= 180^\circ - \sphericalangle FDC = 180^\circ - \sphericalangle ADE \end{aligned}$$

which gives us E, P, A and D are concyclic. Since $\sphericalangle FPE = \sphericalangle FPB + \sphericalangle EPB = \sphericalangle BAD + \sphericalangle BCD$ we get that $\sphericalangle FPE = 180^\circ$ if and only if $\sphericalangle BAD + \sphericalangle BCD = 180^\circ$ which completes the lemma. We now divide the problem into cases:

Case 1: $AEPF$ and $BFEC$ are concyclic. Here we get that

$$180^\circ = \sphericalangle AEP + \sphericalangle AFP = 360^\circ - \sphericalangle CEB - \sphericalangle BFC = 360^\circ - 2\sphericalangle CEB$$

and here we get that $\sphericalangle CEB = \sphericalangle CFB = 90^\circ$, from here it follows that P is the ortocenter of $\triangle ABC$ and that gives us $\sphericalangle ADB = \sphericalangle ADC = 90^\circ$. Now the quadrilaterals $CEPD$ and $BDPF$ are concyclic because

$$\sphericalangle CEP = \sphericalangle CDP = \sphericalangle PDB = \sphericalangle PFB = 90^\circ.$$

Quadrilaterals $ACDF$ and $ABDE$ are concyclic because

$$\measuredangle AEB = \measuredangle ADB = \measuredangle ADC = \measuredangle AFC = 90^\circ.$$

Case 2: $AEPF$ and $CEPD$ are concyclic. Now by lemma 1 applied to the quadrilateral $AEPF$ we get that the circumcircles of CEP, CAF, BPF and BEA intersect at a point on BC. Since $D \in BC$ and $CEPD$ is concyclic we get that D is the desired point and it follows that $BDPF$, $BAED$, $CAFD$ are all concylic and now we can finish same as Case 1 since $AEDB$ and $CEPD$ are concyclic.

Case 3: $AEPF$ and $AEDB$ are concyclic. We apply lemma 1 as in Case 2 on the quadrilateral $AEPF$. From the lemma we get that $BDPF$, $CEPD$ and $CAFD$ are concylic and we finish off the same as in Case 1.

Case 4: $ACDF$ and $ABDE$ are concyclic. We apply lemma 1 on the quadrilateral $AEPF$ and get that the circumcircles of ACF, ECP, PFB and BAE intersect at one point. Since this point is D (because $ACDF$ and $ABDE$ are concyclic) we get that $AEPF$, $CEPD$ and $BFPD$ are concylic. We now finish off as in Case 1. These four cases prove the problem statement.

Remark. A more natural approach is to solve each of the four cases by simple angle chasing.

Number Theory

NT1. Determine all sets of six consecutive positive integers such that the product of two of them, added to the the product of other two of them is equal to the product of the remaining two numbers.

Solution. Exactly two of the six numbers are multiples of 3 and these two need to be multiplied together, otherwise two of the three terms of the equality are multiples of 3 but the third one is not.

Let n and $n+3$ denote these multiples of 3. Two of the four remaining numbers give remainder 1 when divided by 3, while the other two give remainder 2, so the two other products are either $\equiv 1 \cdot 1 = 1 \pmod 3$ and $\equiv 2 \cdot 2 \equiv 1 \pmod 3$, or they are both $\equiv 1 \cdot 2 \equiv 2 \pmod 3$. In conclusion, the term $n(n+3)$ needs to be on the right hand side of the equality.

Looking at parity, three of the numbers are odd, and three are even. One of n and $n+3$ is odd, the other even, so exactly two of the other numbers are odd. As $n(n+3)$ is even, the two remaining odd numbers need to appear in different terms.

We distinguish the following cases:

I. The numbers are $n-2$, $n-1$, n, $n+1$, $n+2$, $n+3$.

The product of the two numbers on the RHS needs to be larger than $n(n+3)$. The only possibility is $(n-2)(n-1)+n(n+3)=(n+1)(n+2)$ which leads to $n=3$. Indeed, $1 \cdot 2 + 3 \cdot 6 = 4 \cdot 5$.

II. The numbers are $n-1$, n, $n+1$, $n+2$, $n+3$, $n+4$.

As $(n+4)(n-1)+n(n+3)=(n+1)(n+2)$ has no solutions, $n+4$ needs to be on the RHS, multiplied with a number having a different parity, so $n-1$ or $n+1$.

$(n+2)(n-1)+n(n+3)=(n+1)(n+4)$ leads to $n=3$. Indeed, $2 \cdot 5 + 3 \cdot 6 = 4 \cdot 7$.

$(n+2)(n+1)+n(n+3)=(n-1)(n+4)$ has no solution.

III. The numbers are n, $n+1$, $n+2$, $n+3$, $n+4$, $n+5$.

We need to consider the following situations:

$(n+1)(n+2) + n(n+3) = (n+4)(n+5)$ which leads to $n = 6$; indeed $7 \cdot 8 + 6 \cdot 9 = 10 \cdot 11$;

$(n+2)(n+5) + n(n+3) = (n+1)(n+4)$ obviously without solutions, and

$(n+1)(n+4) + n(n+3) = (n+2)(n+5)$ which leads to $n = 2$ (not a multiple of 3).

In conclusion, the problem has three solutions:

$$1 \cdot 2 + 3 \cdot 6 = 4 \cdot 5, \quad 2 \cdot 5 + 3 \cdot 6 = 4 \cdot 7, \quad \text{and} \quad 7 \cdot 8 + 6 \cdot 9 = 10 \cdot 11.$$

NT2. Determine all positive integers n such that $n^2|(n-1)!$.

First solution. This is true for all positive integers n unless $n = 8, 9, p, 2p$ for some prime p. It is easy to check that $8^2 \nmid (8-1)!$ and $9^2 \nmid (9-1)!$ by determining the largest powers of 2 and 3 which divide the right hand sides. It is also immediate that $p^2 \nmid (p-1)!$ and $(2p)^2 \nmid (2p-1)!$ as $(p-1)!$ is not divisible by p, while the largest power of p dividing $(2p-1)!$ is 1.

The case $n = 1$ is also clearly true. So it remains to show that $n^2|(n-1)!$ in all other cases. It is enough to show that in those cases, for every prime p which divides n, the largest power of p dividing n^2 is less than or equal to the largest power of p dividing $(n-1)!$. So let us write $n = mp^r$ where $(m, p) = 1$. The largest power of p dividing $(n-1)!$ is

$$\left\lfloor \frac{n-1}{p} \right\rfloor + \left\lfloor \frac{n-1}{p^2} \right\rfloor + \cdots \geqslant (mp^{r-1} - 1) + \cdots + (m-1) = m\frac{p^r - 1}{p-1} - r$$

So it is enough to prove that

$$m\frac{p^r - 1}{p-1} \geqslant 3r.$$

We will distinguish between the cases $p = 2, p = 3$ and $p \geqslant 5$.

Case 1: Suppose $p = 2$. We will further distinguish the cases $r \geqslant 4$ and $r \leqslant 3$

Case 1A: Suppose $r \geqslant 4$. Then

$$m\frac{p^r - 1}{p-1} \geqslant 2^r - 1 = 8(1+1)^{r-3} - 1 \geqslant 8(1 + r - 3) - 1 = 3r + (5r - 17) \geqslant 3r.$$

Here, we have used Bernoulli's inequality.

Case 1B: Suppose $r \leqslant 3$. Because $n \neq 2, 4, 8$, then n has another prime divisor and so $m \geqslant 3$. Then

$$m\frac{p^r - 1}{p-1} \geqslant 3(2^r - 1) \geqslant 3r$$

where the last inequality is easily verifiable for $r \leqslant 3$. (It also follows by applying Bernoulli's inequality.)

Case 2: Suppose $p = 3$. We will further distinguish three cases. The case $r \geqslant 3$ alone, and the cases $r = 2$ and $r = 1$ separately.

Case 2A: Suppose $r \geqslant 3$. Then

$$m\frac{p^r-1}{p-1} \geqslant \frac{3^r-1}{2} = \frac{9(1+2)^{r-2}-1}{2} \geqslant \frac{9(1+2(r-2))-1}{2} = 3r + (6r-14) \geqslant 3r.$$

Case 2B: Suppose $r = 2$. Because $n \neq 9$, then n has another prime divisor and so $m \geqslant 2$. Then

$$m\frac{p^r-1}{p-1} \geqslant 8 \geqslant 6 = 3r.$$

Case 2C: Suppose $r = 1$. Because $n \neq 3, 6$, then n has another divisor which is bigger than 2. So $m \geqslant 4$. Then

$$m\frac{p^r-1}{p-1} \geqslant 4 \geqslant 3 = 3r.$$

Case 3: Suppose $p \geqslant 5$. We will further distinguish the cases $r \geqslant 2$ and $r = 1$.

Case 3A: Suppose $r \geqslant 2$. Then

$$m\frac{p^r-1}{p-1} \geqslant \frac{5^r-1}{4} = \frac{5(1+4)^{r-1}-1}{4} \geqslant \frac{5(1+4(r-1))-1}{4} = 3r + 2(r-2) \geqslant 3r.$$

Case 3B: Suppose $r = 1$. Because $n \neq p, 2p$, then n has another divisor which is bigger than 2. So $m \geqslant 3$. Then

$$m\frac{p^r-1}{p-1} \geqslant 3 = 3r.$$

Second solution. (PSC) Let $n \neq 8, 9, p, 2p$, where p is prime.

Let n be odd and p be the smallest prime divisor of n. If $n = p^2$, then $p \geq 5$, $p < 2p < 3p < 4p$ participate in $(n-1)!$ and so $p^4 = n^2 | (n-1)!$. If $p < \frac{n}{p}$, then $p < 2p$ and $\frac{n}{p} < \frac{2n}{p}$ all are less that n and therefore participate in $(n-1)!$. So $n^2 | 4n^2 = p.2p.\frac{n}{p} \cdot \frac{2n}{p} | (n-1)!$.

Let n be even and $n = 2^k m$, where k is positive integer and m is odd. If $m = 1$, then $k \geq 4$ and $2 < 2^2 < 2^{k-2} < 2^{k-1}$ shows that $n^2 = 2^{2k} | (n-1)!$ for $k \geq 5$ and the case $k = 4$ is seen directly. Let now $m > 1$. If $k \geq 2$, then the divisors $2 < m < 2^{k-1}m$ and 2^k of n work. If $k = 1$, then m is not prime, and let p is the smallest prime divisor of m. Now 4, $p < 2p$ and $\frac{m}{p} < \frac{2m}{p}$ work when $m \neq p^2$, and 4, $p < 2p < 3p < 4p$ work when $m = p^2$.

NT3. Find all pairs of positive integers (x, y) such that $2^x + 3^y$ is a perfect square.

Solution. In order for the expression $2^x + 3^y$ to be a perfect square, a positive integer t such that $2^x + 3^y = t^2$ should exist.

Case 1. If x is even, then there exists a positive integer z such that $x = 2z$. Then

$$(t - 2^z)(t + 2^z) = 3^y$$

Since $t + 2^z - (t - 2^z) = 2^{z+1}$, which implies $gcd(t - 2^z, t + 2^z)|2^{z+1}$, it follows that $gcd(t - 2^z, t + 2^z) = 1$, hence $t - 2^z = 1$ and $t + 2^z = 3^y$, so we have $2^{z+1} + 1 = 3^y$.

For $z = 1$ we have $5 = 3^y$ which clearly have no solution. For $z \geq 2$ we have (modulo 4) that y is even. Let $y = 2k$. Then $2^{z+1} = (3^k - 1)(3^k + 1)$ which is possible only when $3^k - 1 = 2$, i.e. $k = 1$, $y = 2$, which implies that $t = 5$. So the pair $(4, 2)$ is a solution to our problem.

Case 2. If y is even, then there exists a positive integer w such that $y = 2w$, and

$$(t - 3^w)(t + 3^w) = 2^x$$

Since $t + 3^w - (t - 3^w) = 2 \cdot 3^w$, we have $gcd(t - 2^z, t + 2^z)|2 \cdot 3^w$, which means that $gcd(t - 3^w, t + 3^w) = 2$. Hence $t - 3^w = 2$ and $t + 3^w = 2^{x-1}$. So we have

$$2 \cdot 3^w + 2 = 2^{x-1} \Rightarrow 3^w + 1 = 2^{x-2}.$$

Here we see modulo 3 that $x - 2$ is even. Let $x - 2 = 2m$, then $3^w = (2^m - 1)(2^m + 1)$, whence $m = 1$ since $gcd(2^m - 1, 2^m + 1) = 1$. So we arrive again to the solution $(4, 2)$.

Case 3. Let x and y be odd. For $x \geq 3$ we have $2^x + 3^y \equiv 3 \pmod 4$ while $t^2 \equiv 0, 1 \pmod 4$, a contradiction. For $x = 1$ we have $2 + 3^y = t^2$. For $y \geq 2$ we have $2 + 3^y \equiv 2 \pmod 9$ while $t^2 \equiv 0, 1, 4, 7 \pmod 9$. For $y = 1$ we have $5 = 2 + 3 = t^2$ clearly this doesn't have solution.

Note. The proposer's solution used Zsigmondy's theorem in the final steps of cases 1 and 2.

NT4. Solve in nonnegative integers the equation $5^t + 3^x 4^y = z^2$.

Solution. If $x = 0$ we have

$$z^2 - 2^{2y} = 5^t \iff (z + 2^y)(z - 2^y) = 5^t.$$

Putting $z + 2^y = 5^a$ and $z - 2^y = 5^b$ with $a + b = t$ we get $5^a - 5^b = 2^{y+1}$. This gives us $b = 0$ and now we have $5^t - 1 = 2^{y+1}$. If $y \geq 2$ then consideration by modulo 8 gives $2|t$. Putting $t = 2s$ we get $(5^s - 1)(5^s + 1) = 2^{y+1}$. This means $5^s - 1 = 2^c$ and $5^s + 1 = 2^d$ with $c + d = y + 1$. Subtracting we get $2 = 2^d - 2^c$. Then we have $c = 1, d = 2$, but the equation $5^s - 1 = 2$ has no solutions over nonnegative integers. Therefore so $y \geq 2$ in this case gives us no solutions. If $y = 0$ we get again $5^t - 1 = 2$ which again has no solutions in nonnegative integers. If $y = 1$ we get $t = 1$ and $z = 3$ which gives us the solution $(t, x, y, z) = (1, 0, 1, 3)$.
Now if $x \geq 1$ then by modulo 3 we have $2|t$. Putting $t = 2s$ we get

$$3^x 4^y = z^2 - 5^{2s} \iff 3^x 4^y = (z + 5^s)(z - 5^s).$$

Now we have $z + 5^s = 3^m 2^k$ and $z - 5^s = 3^n 2^l$, with $k + l = 2y$ and $m + n = x \geq 1$. Subtracting we get

$$2.5^s = 3^m 2^k - 3^n 2^l.$$

Here we get that $\min\{m, n\} = 0$. We now have a couple of cases.
Case 1. $k = l = 0$. Now we have $n = 0$ and we get the equation $2.5^s = 3^m - 1$. From modulo 4 we get that m is odd. If $s \geq 1$ we get modulo 5 that $4|m$, a contradiction. So $s = 0$ and we get $m = 1$. This gives us $t = 0, x = 1, y = 0, z = 2$.
Case 2. $\min\{k, l\} = 1$. Now we deal with two subcases:
Case 2a. $l > k = 1$. We get $5^s = 3^m - 3^n 2^{l-1}$. Since $\min\{m, n\} = 0$, we get that $n = 0$. Now the equation becomes $5^s = 3^m - 2^{l-1}$. Note that $l - 1 = 2y - 2$ is even. By modulo 3 we get that s is odd and this means $s \geq 1$. Now by modulo 5 we get $3^m \equiv 2^{2y-2} \equiv 1, -1 \pmod 5$. Here we get that m is even as well, so we write $m = 2q$. Now we get $5^s = (3^q - 2^{y-1})(3^q + 2^{y-1})$.

Therefore $3^q - 2^{y-1} = 5^v$ and $3^q + 2^{y-1} = 5^u$ with $u + v = s$. Then $2^y = 5^u - 5^v$, whence $v = 0$ and we have $3^q - 2^{y-1} = 1$. Plugging in $y = 1, 2$ we get the solution $y = 2, q = 1$. This gives us $m = 2, s = 1, n = 0, x = 2, t = 2$ and therefore $z = 13$. Thus we have the solution $(t, x, y, z) = (2, 2, 2, 13)$. If $y \geq 3$ we get modulo 4 that q, $q = 2r$. Then $(3^r - 1)(3^r + 1) = 2^{y-1}$. Putting $3^r - 1 = 2^e$ and $3^r + 1 = 2^f$ with $e + f = y - 1$ and subtracting these two and dividing by 2 we get $2^{f-1} - 2^{e-1} = 1$, whence $e = 1, f = 2$. Therefore $r = 1, q = 2, y = 4$. Now since $2^4 = 5^u - 1$ does not have a solution, it follows that there are no more solutions in this case.

Case 2b. $k > l = 1$. We now get $5^s = 3^m 2^{k-1} - 3^n$. By modulo 4 (which we can use since $0 < k - 1 = 2y - 2$) we get $3^n \equiv -1 \pmod 4$ and therefore n is odd. Now since $\min\{m, n\} = 0$ we get that $m = 0$, $0 + n = m + n = x \geq 1$. The equation becomes $5^s = 2^{2y-2} - 3^x$. By modulo 3 we see that s is even. We now put $s = 2g$ and obtain $(2^{y-1} - 5^g)(5^g + 2^{y-1}) = 3^x$. Putting $2^{y-1} - 5^g = 3^h$, $2^{y-1} + 5^g = 3^i$, where $i + h = x$, and subtracting the equations we get $3^i - 3^h = 2^y$. This gives us $h = 0$ and now we are solving the equation $3^x + 1 = 2^y$.

The solution $x = 0, y = 1$ gives $1 - 5^g = 1$ without solution. If $x \geq 1$ then by modulo 3 we get that y is even. Putting $y = 2y_1$ we obtain $3^x = (2^{y_1} - 1)(2^{y_1} + 1)$. Putting $2^{y_1} - 1 = 3^{x_1}$ and $2^{y_1} + 1 = 3^{x_2}$ and subtracting we get $3^{x_2} - 3^{x_1} = 2$. This equation gives us $x_1 = 0, x_2 = 1$. Then $y_1 = 1, x = 1, y = 2$ is the only solution to $3^x + 1 = 2^y$ with $x \geq 1$. Now from $2 - 5^g = 1$ we get $g = 0$. This gives us $t = 0$. Now this gives us the solution $1 + 3.16 = 49$ and $(t, x, y, z) = (0, 1, 2, 7)$.

This completes all the cases and thus the solutions are $(t, x, y, z) = (1, 0, 1, 3)$, $(0, 1, 0, 2)$, $(2, 2, 2, 13)$, and $(0, 1, 2, 7)$.

Note. The problem can be simplified by asking for solutions in positive integers (without significant loss in ideas).

NT5. Find all positive integers n such that there exists a prime number p, such that

$$p^n - (p-1)^n$$

is a power of 3.

Note. A power of 3 is a number of the form 3^a where a is a positive integer.

Solution. Suppose that the positive integer n is such that

$$p^n - (p-1)^n = 3^a \tag{1}$$

for some prime p and positive integer a.

If $p = 2$, then $2^n - 1 = 3^a$ by (1), whence $(-1)^n - 1 \equiv 0 \pmod 3$, so n should be even. Setting $n = 2s$ we obtain $(2^s - 1)(2^s + 1) = 3^a$. It follows that $2^s - 1$ and $2^s + 1$ are both powers of 3, but since they are both odd, they are co-prime, and we have $2^s - 1 = 1$, i.e. $s = 1$ and $n = 2$.

If $p = 3$, then (1) gives $3|2^n$, which is impossible.

Let $p \geq 5$. Then it follows from (1) that we can not have $3|p-1$. This means that $2^n - 1 \equiv 0 \pmod 3$, so n should be even, and let $n = 2k$. Then

$$p^{2k} - (p-1)^{2k} = 3^a \iff (p^k - (p-1)^k)(p^k + (p-1)^k) = 3^a.$$

If $d = (p^k - (p-1)^k, p^k + (p-1)^k)$, then $d|2p^k$. However, both numbers are powers of 3, so $d = 1$ and $p^k - (p-1)^k = 1$, $p^k + (p-1)^k = 3^a$.

If $k = 1$, then $n = 2$ and we can take $p = 5$. For $k \geq 2$ we have $1 = p^k - (p-1)^k \geq p^2 - (p-1)^2$ (this inequality is equivalent to $p^2(p^{k-2} - 1) \geq (p-1)^2((p-1)^{k-2} - 1)$, which is obviously true). Then $1 \geq p^2 - (p-1)^2 = 2p - 1 \geq 9$, which is absurd.

It follows that the only solution is $n = 2$.