# 29ТА ЈУНИОРСКА БАЛКАНСКА МАТЕМАТИЧКА ОЛИМПИЈАДА

*26. јуни, 2025* *Language: Macedonian*

**Задача 1.** За сите позитивни реални броеви $a$, $b$ и $c$, докажи дека важи

$$\frac{(a^2+bc)^2}{b+c}+\frac{(b^2+ca)^2}{c+a}+\frac{(c^2+ab)^2}{a+b}\geq\frac{2abc(a+b+c)^2}{ab+bc+ca}\,.$$

**Задача 2.** Определи ги сите броеви од облик

$$20252025\ldots2025$$

(составени од еден или повеќе последователни блокови 2025) кои се полни квадрати на природни броеви.

**Задача 3.** Нека $ABC$ е правоаголен триаголник со $\angle A=90^\circ$, нека $D$ е подножје на висината повлечена од темето $A$ кон $BC$, и $E$ е средишната точка на отсечката $DC$. Опишаната кружница околу $\triangle ABD$ ја сече $AE$ по втор пат во точка $F$. Нека $X$ е пресечната точка на правите $AB$ и $DF$. Докажи дека $XD=XC$.

**Задача 4.** Нека $n$ е позитивен цел број. Целите броеви од 1 до $n$ се запишани во полињата од $n\times n$ табла (по еден број во поле) така што секој од нив се појавува точно еднаш во секоја редица и точно еднаш во секоја колона. Со $r_i$ го означуваме бројот на парови $(a,b)$ од броеви во $i$-тиот ред $(1\leq i\leq n)$, такви што $a>b$, но $a$ е напишан лево од $b$ (не мора еден до друг). Со $c_j$ го означуваме бројот на парови $(a,b)$ од броеви во $j$-тата колона $(1\leq j\leq n)$, такви што $a>b$, но $a$ е напишан над $b$ (не мора еден до друг). Определи ја најголемата можна вредност на збирот

$$r_1+r_2+\cdots+r_n+c_1+c_2+\cdots+c_n\,.$$

**Забелешка:** Во $n\times n$ таблата редиците 1 до $n$ ги обележуваме од горе кон долу, а колоните од 1 до $n$ ги обележуваме од лево кон десно.

*Време: 4 саати и 30 минути.*
*Секоја задача вреди 10 поени.*

# 29th JUNIOR BALKAN MATHEMATICAL OLYMPIAD

*26th June, 2025* *Language: English*

**Problem 1.** For all positive real numbers $a$, $b$, $c$, prove that

$$\frac{(a^2+bc)^2}{b+c}+\frac{(b^2+ca)^2}{c+a}+\frac{(c^2+ab)^2}{a+b}\geq\frac{2abc(a+b+c)^2}{ab+bc+ca}\,.$$

**Problem 2.** Determine all numbers of the form

$$20252025\ldots2025$$

(consisting of one or more consecutive blocks of 2025) that are perfect squares of positive integers.

**Problem 3.** Let $ABC$ be a right-angled triangle with $\angle A=90^\circ$, let $D$ be the foot of the altitude from $A$ to $BC$, and let $E$ be the midpoint of $DC$. The circumcircle of $\triangle ABD$ intersects $AE$ again at point $F$. Let $X$ be the intersection of the lines $AB$ and $DF$. Prove that $XD=XC$.

**Problem 4.** Let $n$ be a positive integer. The integers from 1 to $n$ are written in the cells of an $n\times n$ table (one integer per cell) so that each of them appears exactly once in each row and exactly once in each column. Denote by $r_i$ the number of pairs $(a,b)$ of numbers in the $i^{\text{th}}$ row $(1\leq i\leq n)$, such that $a>b$, but $a$ is written to the left of $b$ (not necessarily next to it). Denote by $c_j$ the number of pairs $(a,b)$ of numbers in the $j^{th}$ column $(1\leq j\leq n)$, such that $a>b$, but $a$ is written above $b$ (not necessarily next to it). Determine the largest possible value of the sum

$$r_1+r_2+\cdots+r_n+c_1+c_2+\cdots+c_n\,.$$

**Note:** In the $n\times n$ table we label the rows 1 to $n$ from top to bottom, and we label the columns 1 to $n$ from left to right.

*Time: 4 hours and 30 minutes.*
*Every problem is worth 10 points.*

# 29th Junior Balkan Mathematical Olympiad

### Problems with Solutions

## Problem 1

For all positive real numbers $a$, $b$, $c$, prove that

$$\frac{(a^2+bc)^2}{b+c}+\frac{(b^2+ca)^2}{c+a}+\frac{(c^2+ab)^2}{a+b}\geq\frac{2abc(a+b+c)^2}{ab+bc+ca}.$$

## Solutions

**Solution 1.** Apply Cauchy-Schwarz inequality.

$$((b+c)+(c+a)+(a+b))\left(\frac{(a^2+bc)^2}{b+c}+\frac{(b^2+ca)^2}{c+a}+\frac{(c^2+ab)^2}{a+b}\right)\geq$$
$$\left(a^2+b^2+c^2+ab+bc+ca\right)^2$$

**Lemma.** $3(a^2+b^2+c^2+ab+bc+ca)\geq 2(a+b+c)^2$.

**Proof.**

$$3(a^2+b^2+c^2+ab+bc+ca)=$$
$$2(a^2+b^2+c^2+2ab+2bc+2ca)+(a^2+b^2+c^2-ab-bc-ca)=$$
$$2(a+b+c)^2+\frac{1}{2}(a-b)^2+\frac{1}{2}(b-c)^2+\frac{1}{2}(c-a)^2\geq 2(a+b+c)^2,$$

as desired. ◇

Then we have

$$\frac{(a^2+bc)^2}{b+c}+\frac{(b^2+ca)^2}{c+a}+\frac{(c^2+ab)^2}{a+b}\geq\frac{\frac{4}{9}(a+b+c)^4}{2(a+b+c)}=\frac{2}{9}(a+b+c)^3.$$

Again by Cauchy-Schwarz inequality we have

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9.$$

Finally, using these we get

$$\frac{(a^2+bc)^2}{b+c}+\frac{(b^2+ca)^2}{c+a}+\frac{(c^2+ab)^2}{a+b}\ \geq\ \frac{2}{9}(a\,+\,b\,+\,c)^3\ \geq\ \frac{2(a+b+c)^2}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\ =\ \frac{2abc(a+b+c)^2}{ab+bc+ca},$$

as desired. □

**Solution 2.** By using AM-GM we have:

$$\sum_{cyc} \frac{(a^2+bc)^2}{b+c} \geq \sum_{cyc} \frac{\left(2\sqrt{a^2bc}\right)^2}{b+c} = 4abc \sum_{cyc} \frac{a}{b+c}.$$

From here, by using Titu's Lemma, we obtain

$$\begin{aligned} 4abc \sum_{cyc} \frac{a}{b+c} &= 4abc \sum_{cyc} \frac{a^2}{ab+ac} \geq \\ 4abc\frac{(a+b+c)^2}{2(ab+bc+ca)} &= \frac{2abc(a+b+c)^2}{ab+bc+ca}, \end{aligned}$$

as desired. □

**Solution 3.** Using Titu's Lemma and Schur's Inequality, we have

$$\begin{aligned} \sum_{cyc} \frac{(a^2+bc)^2}{b+c} = \sum_{cyc} \frac{(a^3+abc)^2}{a^2(b+c)} \overset{\text{Titu}}{\geq} \frac{\left(\sum a^3+3abc\right)^2}{\sum a^2(b+c)} \geq \\ \overset{\text{Schur}}{\geq} \sum_{cyc} a^2(b+c). \end{aligned}$$

Using the inequality $(ab+bc+ca)^2 \geq 3abc(a+b+c)$, we get:

$$\sum_{cyc} a^2(b+c) = -3abc + (a+b+c)(ab+bc+ca) \geq -3abc + (a+b+c)\frac{3abc(a+b+c)}{ab+bc+ca}.$$

Finally, by $(a+b+c)^2 \geq 3(ab+bc+ca)$:

$$\begin{aligned} -3abc + (a+b+c)\frac{3abc(a+b+c)}{ab+bc+ca} &= \\ abc\left(-3+\frac{(a+b+c)^2}{ab+bc+ca}\right) + \frac{2abc(a+b+c)^2}{ab+bc+ca} &\geq \frac{2abc(a+b+c)^2}{ab+bc+ca} \end{aligned}$$

□

## Problem 2

Determine all numbers of the form

$$20252025\ldots2025$$

(where the block $2025$ is repeated one or more times) that are perfect squares of positive integers.

## Solution

We will prove that the only solution is $2025 = 45^2$.

First, observe that the given number is equal to

$$2025\left(1 + 10^4 + 10^8 + \ldots + 10^{4(n-1)}\right)$$

for some $n \in \mathbb{N}$. Since $2025$ is a perfect square, it must hold that $1 + 10^4 + 10^8 + \ldots + 10^{4(n-1)} = x^2$ **($\star$)** for some $x \in \mathbb{N}$. Multiplying both sides by $10^4 - 1$ yields

$$10^{4n} - 1 = 9999x^2 = 11 \cdot 101 \cdot (3x)^2$$

Applying the difference of squares twice gives

$$(10^n - 1)(10^n + 1)(10^{2n} + 1) = 11 \cdot 101 \cdot (3x)^2 \qquad (\star\star)$$

Using the Euclidean algorithm, we see that $10^n - 1$ and $10^n + 1$ are coprime (their difference is $2$ and they are both odd). Additionaly, $10^{2n} + 1$ is coprime with both $10^n - 1$ and $10^n + 1$ since it is coprime with their product $10^{2n} - 1$ (same reasoning as before). Hence, all three factors on the left-hand side of $(\star\star)$ are pairwise coprime. Since the numbers 11 and 101 are prime, at least one of the factors on the left-hand side of $(\star\star)$ must be a perfect square.

**Lemma 1.** The equation $10^n + 1 = m^2$ has no solutions in the set of positive integers.

*Proof.* Reducing modulo 3 implies $m^2 \equiv 2 \pmod 3$, which cannot hold.

Hence, neither $10^{2n} + 1$ nor $10^n + 1$ can be perfect squares.

**Lemma 2.** The only solution to the equation $10^n - 1 = m^2$ in the set of positive integers is $(n, m) = (1, 3)$.

*Proof.* Consider the equation modulo $4$. For $n \geqslant 2$, $10^n \equiv 0 \pmod 4$, hence the right-hand side is congruent to $3$ modulo $4$. On the other hand, $n = 1$ gives $m = 3$ as a solution.

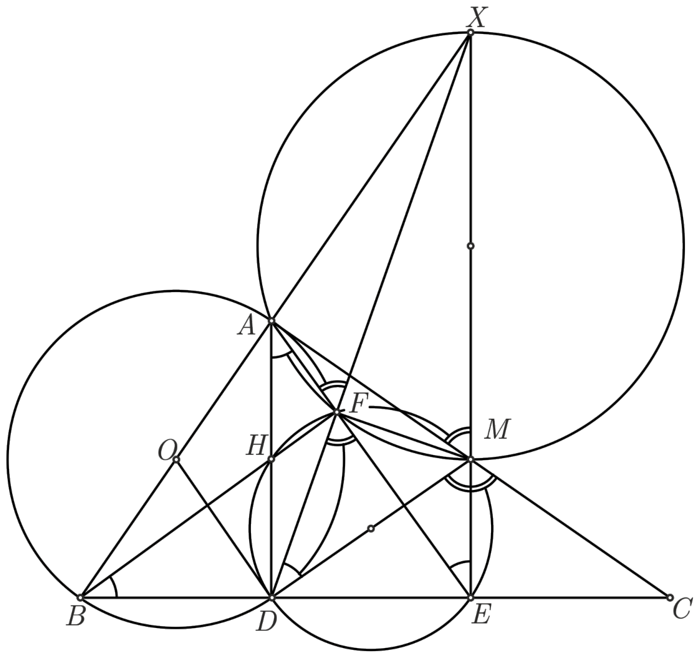
Lemmas $1$ and $2$ imply that the only solution to the given equation is $(n, x) = (1, 1)$. Therefore, the only solution is the number $2025$. $\square$

**Problem 3.** Let $\triangle ABC$ be right-angled at $A$ and let $D$ be the foot of altitude from $A$ to $BC$ and let $E$ be the midpoint of $DC$. The circumcircle of $\triangle ABD$ intersects $AE$ again at point $F$. Let $X$ be the intersection of $AB$ and $DF$. Prove that $XD = XC$.

**Solution 1.** Since $E$ is the midpoint of $DC$, it is sufficient to show that $XE \perp DC$ which is equivalent to proving that $XE \parallel AD$. Let $H$ be the intersection of $BF$ and $AD$. Since $\angle ADB = 90^\circ \Rightarrow \angle AFB = 90^\circ \Rightarrow H$ is the orthocenter of $\triangle ABE \Rightarrow EH \perp AB \Rightarrow EH \parallel AC$. Since $E$ is the midpoint of $DC$ we have that $H$ is also the midpoint of $AD$. Apply (unoriented) Menelaus theorem in $\triangle ABH$ for points $D, F, X$ and in $\triangle BHD$ for points $A, F, E$. We get that

$$\frac{AX}{XB}\frac{BF}{FH}\frac{HD}{DA} = 1 = \frac{DE}{EB}\frac{BF}{FH}\frac{HA}{AD}.$$

Since we have that $AH = HD$ the ratios cancel and we get that $\frac{AX}{XB} = \frac{DE}{EB}$ which implies that $AD \parallel XE$ by Thales. $\square$




**Solution 2.** Let $M$ be the midpoint of $AC$. As midsegment in $\triangle ADC$, $ME \parallel AD$. Also, by Thales' Theorem in the right $\triangle ADC$, we get $MA = MD = MC$.

**Claim 1.** $DFME$ is cyclic. ...(1)

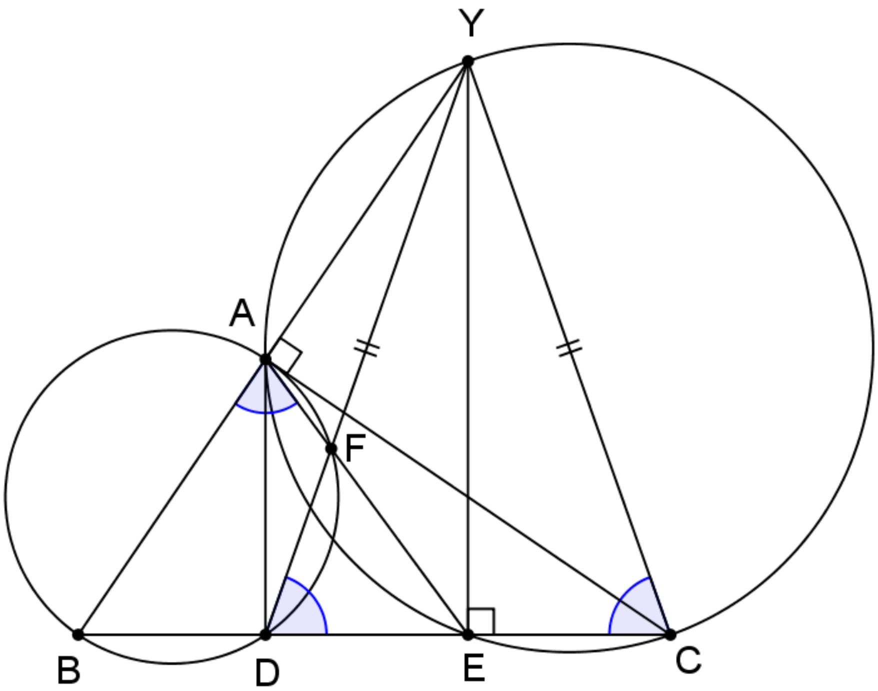
**Proof 1.** $\angle DFE \overset{(ABDF)}{=} \angle DBA = 90^\circ - \angle DCA = \angle DAC \overset{AD \parallel ME}{=} \angle EMC = \angle DME$, so $DFME$ is cyclic. $\diamond$

**Proof 2.** First we prove that $MD$ is tangent to $(ABDF)$. Let $O$ be the midpoint of $AB$ and therefore center of $(ABDF)$. Then, $OA = OD$, so by criterion SSS, we get $\triangle OAM \cong \triangle ODM$, thus $\angle ODM = \angle OAM = 90^\circ$.
Now, $\angle MEF \overset{ME \parallel AD}{=} \angle FAD = \angle MDF$, so $DFME$ is cyclic. $\diamond$

**Claim 2.** $X, M, E$ are collinear.

**Proof 1.** $\angle MFX \overset{(1)}{=} \angle MED = 90^\circ = \angle MAX$, so $MFAX$ is cyclic. ...(2)
$\angle XMA \overset{(2)}{=} \angle XFA = \angle DFE \overset{(1)}{=} \angle DME = \angle CME$ and since $A, M, C$ are collinear, so are $X, M, E$. $\diamond$

**Proof 2.** Same as in Proof 1, $MFAX$ is cyclic.
$\angle XMF \overset{(2)}{=} \angle FAB \overset{(ABDF)}{=} \angle FDE \overset{(1)}{=} 180^\circ - \angle FME$, so $\angle XME = 180^\circ$. $\diamond$




**Proof 3.** $\angle MAB + \angle MEB = 90^\circ + 90^\circ = 180^\circ$, so $MABE$ is cyclic. The three pairwise radical axes of the circles $(MABE)$, $(ABDF)$ and $(DFME)$ are $AB$, $DF$ and $ME$. Then $X = AB \cap DF$ is their radical center, so $X \in ME$. $\diamond$

Now, $X, M, E$ are collinear, i.e. $X$ lies on the side bisector of $DC$, so $XD = XC$. $\square$

**Solution 3.** Let $Y$ be the intersection of the side bisector of $DC$ with line $AB$. Then, $YD = YC$ and therefore $\angle YDC = \angle YCD$. Also, $\angle YEC = 90^\circ = \angle YAC$, so $YAEC$ is cyclic. From there, $\angle BAE = \angle YCE \equiv \angle YCD = \angle YDC$.
On the other hand, from $(ABDF)$, $\angle FDC = \angle BAF \equiv \angle BAE$.
Therefore, $\angle YDC = \angle BAE = \angle FDC$, thus points $D, F, Y$ are collinear, i.e. $Y \in DF$. Since $X = AB \cap DF$, we get $Y \equiv X$, and therefore $XD = XC$. $\square$

## Problem 4.

Let $n$ be a positive integer. The integers from $1$ to $n$ are written in the cells of an $n \times n$ table (one integer per cell) so that each of them appears exactly once in each row and exactly once in each column. Denote by $r_i$ the number of pairs $(a, b)$ of numbers in the $i^{\text{th}}$ row $(1 \leq i \leq n)$, such that $a > b$, but $a$ is written to the left of $b$ (not necessarily next to it). Denote by $c_j$ the number of pairs $(a, b)$ of numbers in the $j^{th}$ column $(1 \leq j \leq n)$, such that $a > b$, but $a$ is written above $b$ (not necessarily next to it). Determine the largest possible value of the sum

$$r_1 + r_2 + \cdots + r_n + c_1 + c_2 + \cdots + c_n\,.$$

**Note:** In the $n \times n$ table we label the rows $1$ to $n$ from top to bottom, and we label the columns $1$ to $n$ from left to right.

## Solution

**Answer:** $\dfrac{n(n-1)(2n-1)}{3}$

Suppose $x$ is in position $i$ in some row/column. Then after it there could be at most $\min(n-i, x-1)$ smaller numbers. Having in mind that this bound is separately for the row of $x$ and for the column of $x$, as well as that $i$ is different for the different appearances of $x$ (as no row/column has a number appearing more than once, by the problem condition), we deduce that the contribution of $x$ to the overall sum is at most

$$\begin{aligned}
&2\sum_{i=1}^{n} \min(n-i, x-1) = 2\sum_{i=n-x+1}^{n} (n-i) + 2\sum_{i=1}^{n-x}(x-1) = 2\sum_{i=0}^{x-1} i + 2(n-x)(x-1) \\
&= x(x-1) + 2(n-x)(x-1) = (2n-x)(x-1).
\end{aligned}$$

Summing through $x = 1, \ldots, n$ now gives the following upper bound for the sum:

$$\begin{aligned}
&\sum_{x=1}^{n}(2n-x)(x-1) = (2n+1)\sum_{x=1}^{n} x - \sum_{x=1}^{n} x^2 - \sum_{x=1}^{n} 2n \\
&= \frac{n(n+1)(2n+1)}{2} - \frac{n(n+1)(2n+1)}{6} - 2n^2 = \frac{n(n-1)(2n-1)}{3}.
\end{aligned}$$

Equality holds e.g. for the table in which the $s$-th row is $n+1-s, n-s, \ldots, 1, n, n-1, \ldots, n+2-s$, since for each $x$ in position $i$ in a row or column there are indeed exactly $\min(n-i, x-1)$ smaller numbers after it. $\square$