25. ТУРНИР ГРАДОВА

Јесење коло.

Припремна варијанта, 19. октобар 2003. год.

8-9. разред (млађи узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

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1. (3 поена) Свака страна кутије облика паралелопипеда са ивицама 3, 4, 5 подељена је на јединична квадратиће. Може ли се у сваки квадратић уписати број тако да збир бројева у сваком прстену ширине 1, описаном око те кутије, буде 120?

2. (4 поена) У седмоуглу $A_1A_2A_3A_4A_5A_6A_7$ дијагонале A_1A_3, A_2A_4,A_3A_5, A_4A_6, A_5A_7,-A_6A_1 и A_7A_2 једнаке су међу собом. Дијагонале A_1A_4, A_2A_5,A_3A_6,A_4A_7,A_5A_1,A_6A_2 и A_7A_3 такође су међусобно једнаке. Мора ли тај седмоугао бити правилан?

3. (4 поена) За сваки цео број од $n+1$ до $2n$ укључујући и $2n$ (где је n природан број) је одабран његов највећи непаран делилац и сви ти делиоци су сабрани. Доказати да је тај збир једнак n^2.

4. (4 поена) N тачака у равни, од којих никоје три не леже на једној правој, су међусобно спојене дужима (свака са сваком). Неке дужи су обојене црвеном, а остале плавом бојом. Црвене дужи чине затворену изломљену линију без самопресецања, као и плаве. Наћи све N за које је то могуће.

5. (5 поена) На траци 1•N на првих 25 поља (с лева) стоји 25 жетона. Жетон се може померити на суседно слободно десно поље или прескочити суседни жетон са десне стране на следеће поље (уколико је оно празно), померање у лево није дозволено. За које најмање N је могуће све жетоне поређати без размака у обрнутом поретку?

25. ТУРНИР ГРАДОВА

Јесење коло.

Припремна варијанта, 19. октобар 2003. год.

10-11 разред (старији узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

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1. (3 поена) За сваки цео број од $n+1$ до $2n$ укључујући и $2n$ (где је n природан број) је одабран његов највећи непаран делилац и сви ти делиоци су сабрани. Доказати да је тај збир једнак n^2.

2. (4 поена) Који је најмањи број квадрата 1×1 које треба нацртати да би се добила слика квадрата величине 25×25 подељеног на 625 квадрата 1×1?

3. (5 поена) Продавац и купац имају укупно 1999 рубаља у апоенима од 1, 5, 10, 50, 100, 500 и 1000 рубаља. Мачка у џаку кошта целобројан број рубаља и купац има довољно новца да је купи. Доказати да купац може купити мачку и добити тачан износ кусура.

4. Четири правоугла троугла су конструисана ван јединичног квадрата тако да су нихове хипотенузе четири странице квадрата. Нека су A, B, C, D темена тих троуглова наспрам хипотенуза и нека су O_1, O_2, O_3, O_4 центри уписаних кругова у те троуглове. Доказати да:

(3 поена) а) површина четвороугла ABCD није већа од 2.

(3 поена) б) површина четвороугла $O_1O_2O_3O_4$ није већа од 1.

5. (6 поена) Папирни тетраедар је расечен дуж неких својих ивица тако да се може развити у раван. Да ли се може десити да се тај развој у раван не може остварити без преклапања (то јест, у једном слоју)?

25. ТУРНИР ГРАДОВА

Јесење коло.

Основна варијанта. 26. октобар 2003. год.

8-9. разред (млађи узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

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1. (4 поена) Сто целих позитивних бројева образује растућу аритметичку прогресију. Могу ли било која два од тих бројева бити узајамно прости?

2. (5 поена) У групи младића и девојака неки се међусобно познају, а неки не. Две проводаџике знају ко кога познаје. Једна проводаџика је изјавила:"Ја могу истовремено оженити све плавокосе момке тако да се сваки од њих ожени девојком коју познаје." Друга проводаџика каже:" А ја могу да удесим судбину свих плавуша: свака ће се удати за познатог јој момка!" Тај дијалог је слушао један љубитељ математике, па је казао:
"У том случају могуће је учинити и једно и друго!". Да ли је он у праву?

3. (5 поена) Нађите све целе позитивне бројеве k, за које се могу наћи такви цели позитивни бројеви m и n да је $m(m+k)=n(n+1)$.

4. (6 поена) Који је најмањи број поља које треба означити на табли 15×15, тако да ловац (шаховски), постављен на било које поље, туче не мање од два означена поља? (Ловац туче по две дијагонале на чијем пресеку стоји; ловац, постављен на једно означено поље, туче и то поље.)

5. (7 поена) Дат је квадрат ABCD. Унутар квадрата лежи тачка O. Докажите да се збир углова OAB, OBC, OCD и ODA разликује од 180^0 за не више од 45^0.

6. (7 поена) По површини (спољашњости) једне кутије (облика правоуглог пралелопипеда) мили мрав. У почетку он стоји на једном врху (темену) паралелопипеда. Да ли је истина да се међу свим тачкама на површини кутије које су на највећем растојању од мрава, налази теме наспрамног угла? (Растојање међу тачкама, према схватању мрава, представља дужину најкраћег пута међу тим тачкама, идући по површини паралелопипеда.)

7. (8 поена) Играју двоје. Први има 1000 парних картица (2,4, ..., 2000), а други 1001 непарну (1,3, ..., 2001). Потезе вуку наизменично. Почиње први играч. Потез се састоји у следећем: играч који је на потезу извлачи и показује једну од својих картица, а други играч, пошто погледа ту картицу, извлачи и показује једну од својих картица; онај играч чији је број на картици већи записује себи један поен, а обе показане картице се одлажу на страну. Игра се 1000 потеза (и једна картица другог играча остаје неискоришћена). Који највећи број поена може гарантовати себи сваки играч (ма како играо његов супарник)?

25. ТУРНИР ГРАДОВА

Јесење коло.

Основна варијанта. 26. октобар 2003. год.

10-11. разред (старији узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

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1. (4 поена) У групи младића и девојака неки се међусобно познају, а неки не. Две проводаџике знају ко кога познаје. Једна проводаџика је изјавила:"Ја могу истовремено оженити све плавокосе момке тако да се сваки од њих ожени девојком коју познаје." Друга проводаџика каже:" А ја могу да удесим судбину свих плавуша: свака ће се удати за познатог јој момка!" Тај дијалог је слушао један љубитељ математике, па је казао:
"У том случају могуће је учинити и једно и друго!". Да ли је он у праву?

2. (4 поена) Докажите да се сваки позитиван број може представити у облику:

$$3^{u_1} \cdot 2^{v_1} + 3^{u_2} \cdot 2^{v_2} + \ldots + 3^{u_k} \cdot 2^{v_k},$$

где су $u_1>u_2>\ldots u_k\geq 0$ и $0\leq v_1<v_2<\ldots<v_k$ цели бројеви.

3. (6 поена) По површини (спољашњости) једне кутије (облика правоуглог пралелопипеда) мили мрав. У почетку он стоји на једном врху (темену) паралелопипеда. Да ли је истина да се међу свим тачкама на површини кутије које су на највећем растојању од мрава, налази теме наспрамног угла? (Растојање међу тачкама, према схватању мрава, представља дужину најкраћег пута међу тим тачкама, идући по површини паралелопипеда.)

4. (7 поена) Дат је троугао ABC. У њему је тачка H пресек висина, I центар уписане кружнице, O центар описане кружнице, K тачка пресека тетиве уписане кружнице са страницом BC. Зна се да су одсечци IO и BC паралелни. Докажи да су и одсечци AO и HK паралелни.

5. (7 поена) Играју двоје. Први има 1000 парних картица (2, 4, ..., 2000), а други 1001 непарну (1, 3, ..., 2001). Потезе вуку наизменично. Почиње први играч. Потез се састоји у следећем: играч који је на потезу извлачи и показује једну од својих картица, а други играч, пошто погледа ту картицу, извлачи и показује једну од својих картица; онај играч чији је број на картици већи записује себи један поен, а обе показане картице се одлажу на страну. Игра се 1000 потеза (и једна картица другог играча остаје неискоришћена). Који највећи број поена може гарантовати себи сваки играч (ма како играо његов супарник)?

25. ТУРНИР ГРАДОВА

Пролећно коло.

Припремна варијанта, 22. фебруар 2004. год.

8–9. разред (млађи узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

1. (3 поена). У троуглу ABC симетрала угла A, симетраала страницe AB и висина из темена B секу се у једној тачки. Докажите да се симетрала угла A, симетрала странице AC и висина из тачке C такође секу у једној тачки.

2. (3 поена). Нађи све природне бројеве n за које се могу наћи n узастопних природних бројева чији је збир прост број.

3. а) (3 поена). Имамо три једнаке велике посуде. У једној се налази $3l$ сирупа, у другом $20\ l$ воде, а трећа је празна. Дозвољено је сипати из једне посуде сву течност у другу посуду или, пак, просути сву течност (нпр. у лавабо). Такође је дозвољено одабрати две посуде и доливати у једну од њих течност из треће све док се нивои течности у одабраним посудама не изједначе. Како се може добити 10 литара 30-процентног сирупа?

б) (2 поена). Исто питање, али ако сада имамо N литара воде. За које целобројно N је могуће добити 10 литара 30-процентног сирупа?

4. (5 поена). Природном броју $a>1$ дописан (приписан) је исти тај број и добијен је број b који је дељив са a^2. Нађите $\frac{b}{a^2}$. (Пронађите све одговоре и докажите да других нема).

5. (6 поена). Два десетоцифрена броја зовемо "суседним" ако се они разликују само у једној цифри на некој декадној позицији. На пример, бројеви 1234567890 и 1234507890 су "суседни". Колико је највише могуће написати десетоцифрених бројева тако да међу њима нема ниједног пара "суседних" бројева?

25. ТУРНИР ГРАДОВА

Пролећно коло.

Припремна варијанта, 22. фабруар 2004. год.

10–11 разред (старији узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

1. (4 поена). *Дужи AB, BC и CD изломљене линије $ABCD$ су једнаке по дужини и додирују неку кружницу са центром O.* Докажите да тачка додира те кружнице са дужи BC, тачка O и тачка пресека правих AC и BD леже на једној правој.

2. (4 поена). Природном броју $a>1$ дописан (приписан) је исти тај број и добијен је број b који је дељив са a^2. Нађите $\frac{b}{a^2}$. (Наведите све одговоре и докажите да других нема).

3. (4 поена). Обим конвексног четвороугла је 2004, а једна од дијагонала је 1001. Може ли друга дијагонала бити једнака 1? Једнака 2? Једнака 1001?

4. (5 поена). Познато је да се међу члановима неке аритметичке прогресије a_1, a_2, a_3, a_4, ... налазе бројеви a_1^2, a_2^2, a_3^2. Докажите да су чланови те прогресије цели бројеви.

5. (5 поена). Два десетоцифрена броја зовемо "суседним" ако се они разликују у само једној цифри на некој декадној позицији. На пример, бројеви 1234567890 и 1234507890 су "суседни". Колико је највише могуће написати десетоцифрених бројева тако да међу њима нема ниједног пара "суседних" бројева?

25. ТУРНИР ГРАДОВА

Пролећно коло.

Основна варијанта, 29. фебруар 2004. год.

8–9. разред (млађи узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

1. (4 поена). Коначан аритметички низ (прогресија) састоји се из целих бројева, а његова сума је степен двојке. Докажите да је број чланова тога низа такође степен двојке.

2. (5 поена). Колики максималан број жетона можемо поређати на табли 8×8 тако да сваки буде на удару? (Ако поља шаховске табле x, y, z стоје једно до другог на дијагонали, жетон a стоји на пољу x, жетон b на пољу y и поље z је слободно, тада је жетон b под ударом.)

3. (5 поена). Курс акција компаније "Рогови и копита" повећава се или пада сваки пут за n процената, где је n фиксирани цео број, $0<n<100$ (курс се рачуна са неограниченом тачношћу). Постоји ли такво n за које курс акција може два пута узети (имати) исту вредност?

4. (6 поена). Две кружнице секу се у тачкама A и B. Њихова заједничка тангента (она која је ближа тачки B) додирује кружнице у тачкама E и F. Права AB сече праву EF у тачки M. На продужетку AM иза тачке M изабрана је тачка K тако да је $KM=MA$. Права KE по други пут сече кружницу, којој припада тачка E, у тачки C. Права KF по други пут сече кружницу, којој припада тачка F, у тачки D. Докажите да тачке C, D и A леже на истој правој.

5. (6 поена). Имамо билијарски сто у облику многоугла (не мора бити конвексан) код кога су ма које две суседне странице нормалне једна на другу. У теменима се налазе тачкасте рупе у које лоптица, кад ту дође, пропада. Из темена A с унутрашњим углом 90^0 полази тачкаста лоптица и креће се унутар многоугла, одбијајући се од његових страница по закону "упадни угао једнак је углу одбијања". Доказати да се она никада неће вратити у тачку A.

6. (7 поена). У почетку је на табли написан број 2004! (2004 факторијел, то јест број $1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2004$). Два играча играју наизменично. Играч који је на реду ("на потезу") од написаног броја одузима неки природан број који је дељив са не више од 20 различитих простих бројева (али тако да разлика буде ненегативна), записује на табли ту разлику, а стари број брише. Побеђује онај играч који добије 0. Који од играча, онај који почиње или његов противник, може себи обезбедити победу и како он треба да игра?

25. ТУРНИР ГРАДОВА

Пролећно коло.

Основна варијанта, 29. фебруар 2004. године.

10–11 разред (старији узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

1. (4 поена). Курс акција компаније "Рогови и копита" повећава се или пада сваки пут за n процената, где је n фиксираани цео број, $0<n<100$ (курс се рачуна са неограниченом тачношћу). Постоји ли такво n за које курс акција може два пута узети (имати) исту вредност?

2. (6 поена). Имамо билијарски сто у облику многоугла (не мора бити конвексан) код кога сваки угао има цео број степени, а угао A има тачно 1 степен. У теменима се налазе тачкасте рупе у које лоптица, кад ту дође, пропада. Из темена A полази тачкаста лоптица и креће се унутар многоугла, одбијајући се од његових страница по закону "упадни угао једнак је углу одбијања". Доказати да се она никада неће вратити у тачку A.

3. (6 поена). Правоугаона пројекција тростране пирамиде на неку раван има максимално могућу површину.Докажите да је та раван паралелна или једној од страна пирамиде, или двема мимоилазним ивицама пирамиде.

4. (6 поена). У почетку је на табли написан број 2004! (2004 факторијел, то јест број $1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2004$). Два играча играју наизменично. Играч који је на реду ("на потезу") од написаног броја одузима неки природан број који је дељив са не више од 20 различитих простих бројева (али тако да разлика буде ненегативна), записује на табли ту разлику, а стари број брише. Побеђује онај играч који добије 0. Који од играча, онај који почиње или његов противник, може себи обезбедити победу и како он треба да игра?

5. (7 поена). У равни су дати парабола $y = x^2$ и кружница, тако да имају тачно две зааједничке тачке: A и B. Показало се да се тангенте на кружницу и параболу у тачки A поклапају. Да ли ће се обавезно такође поклопити и тангенте на кружницу и параболу у тачки B?

6. Пред мађионичара се ставља шпил од 36 карата тако да им је полеђина горе. Он каже боју карте на врху (једну од четири: пик, каро, херц или треф), после чега се карта отвара, показује и ставља на страну. После тога мађионичар каже боју следеће карте, итд. Задатак мађионичара је да погоди боју што је могуће више пута. Стварно полеђине карата нису симетричне и играч види у ком од два положаја лежи карта на врху. Шпил је припремио подмићени (подплаћени) службеник. Службеник зна поредак карата у шпилу и, иако га не може променити, ипак може дошапнути (помоћи) мађионичару, намештајући полеђине карата овако или онако, сагласно договору. Може ли мађионичар, заахваљујући таквој помоћи, гарантовано обезбедити погађање боје:

а) (3 поена) код више од половине карата?

б) (5 поена) код не мање од 20 карата?

25-й Международный математический Турнир городов

2003/04 учебный год

Решения задач

Весенний тур

Тренировочный вариант, 8-9 классы

1.1. [3] В треугольнике *ABC* биссектриса угла *A*, серединный перпендикуляр к стороне *AB* и высота, опущенная из вершины *B*, пересекаются в одной точке. Доказать, что биссектриса угла *A*, серединный перпендикуляр к *AC* и высота, опущенная из *C*, также пересекаются в одной точке. (*П.Кожевников*)

Решение: Пусть *BK* – высота, *M* – середина *AB*. Из условия $AK = AM = {}^1/_2AB \Rightarrow \angle A = 60°$. Осталось обратить рассуждение со сменой обозначений.

1.2. [3] Найти все натуральные *n*, для которых найдутся *n* идущих подряд натуральных чисел, сумма которых простое число. (*Г.Гальперин*)

Решение: $n = 1, 2$. Например, $2 + 3 = 5$. Если $n > 2$, то при нечетном *n* указанная сумма делится на *n*, а при четном – на $a_1 + a_n$, причем в обоих случаяк частное больше 1.

1.3. В сосуде *A* – 3 л сиропа, в сосуде *B* – *n* л воды, сосуд *C* пуст. Разрешаются следующие операции (в любом количестве и порядке):

вылить **всю** жидкость из любого сосуда в раковину или другой сосуд;

перелить часть жидкости из одного сосуда во второй так, чтобы объемы жидкости во втором и третьем (не участвующем в операции) сосудах сравнялись.

Требуется получить 10 л разбавленного 30-процентного сиропа.

а) [3] Доказать, что это можно сделать при $n = 20$.

б) [2] Найти все целые *n*, при которых это возможно. (*А.Шаповалов*)

Решение:

а) Перельем из *B* в *C* 3 л воды и выльем ее (из *C*) в раковину; повторим эту операцию несколько раз, пока в *B* не останется 5 л воды, причем в последний раз перельем воду из *C* не в раковину , а в *A*. Теперь в *A* – 6 л 50% сиропа, а *C* снова пуст. Перельем 5 л из *A* в *C*. Теперь можно перелить 4 л воды из *B* в *A*, а оставшийся в *B* литр вылить в раковину. Осталось вернуть 5 л смеси из *C* в *A*, где образуется 10 л 30% сиропа.

б) Все некратные 3 числа, большие 6. Заметим, что указанный в а) алгоритм годится для всех $n \equiv 2 \pmod 3$. Если $n \equiv 1 \pmod 3$, то алгоритм еще проще: с помощью *C* выливаем из *B* по 3 л в раковину, пока в *B* не останется 7 л; после чего переливаем их в *A*.

Если же *n* кратно 3, то в начале объем жидкости в каждом сосуде кратен 3. Разрешенные операции не могут испортить это свойство, поэтому получить 10 л жидкости (независимо от ее состава) в одном сосуде невозможно.

1.4. [5] К натуральному числу $a > 1$ приписали это же число и получили число *b*, кратное a^2. Найти все возможные значения числа $\frac{b}{a^2}$. (*И.Богданов*)

Решение: Если число a n-значно, то $\frac{b}{a^2} = \frac{(10^n+1)a}{a^2} = \frac{10^n+1}{a}$. Ясно, что $a \neq 10^{n-1}$, значит, $1 < \frac{10^n+1}{a} < 10$. Число $10^n + 1$ (а тем более, частное) не делится ни на 2, ни на 3 (сумма цифр равна 2), ни на 5, поэтому единственное возможное частное – 7. Такое частное можно получить, например, при $a = 143 = \frac{1001}{7}$.

1.5. [6] Два 10-значных числа назовем соседними, если они различаются только одной цифрой в каком-то из разрядов. Какое наибольшее количество десятизначных чисел можно выписать так, чтобы никакие два из них не были соседними? (*Д.Калинин*)

Решение: $9{\cdot}10^8$. Рассмотрим на множестве 10-значных чисел отображение f, которое увеличивает последнюю цифру на 1, а если она равна 9, заменяет ее нулем. Заметим, что если множество M *хорошее* (т.е. состоит из попарно не соседних чисел), то и множества M, $f(M)$, $f^{(2)}(M) = f(f(M))$, …, $f^{(9)}(M)$ хорошие. Более того, эти множества попарно не пересекаются, поскольку $f^{(k)}$ переводит каждое число в соседнее.

Поэтому хорошее множество не может содержать более $^1/_{10}$ количества всех 10-значных чисел (а их $9{\cdot}10^9$). С другой стороны, если в качестве взять M множество всех 10-значных чисел с суммой цифр, кратной 10 (оно, очевидно, хорошее), то множества M, $f(M)$, …, $f^{(9)}(M)$ содержат все 10-значные числа. Следовательно, каждое из них состоит из $9{\cdot}10^8$ элементов.

Тренировочный вариант, 10-11 классы

2.1. [4] Звенья AB, BC и CD ломаной $ABCD$ равны по длине и касаются некоторой окружности с центром O. Д/ч точка K касания этой окружности со звеном BC, точка O и точка пересечения прямых AC и BD лежат на одной прямой. (*М.Макаров*)

Решение: BO – биссектриса равнобедренного тр-ка $ABC \Rightarrow BO \perp AC$. Аналогично $CO \perp BD$. Таким образом, прямые CA, BD и OK – высоты тр-ка BOC.

2.2. [4] См. 1.4.

2.3. [4] Периметр выпуклого 4-угольника равен 2004, одна из диагоналей равна 1001. Может ли 2-я диагональ быть равна **а**) 1; **б**) 2; **в**) 1001? (*А.Толпыго*)

Решение:

а) Нет. Из неравенства тр-ка легко следует, что сумма диагоналей больше полупериметра.

б) Да. Рассмотрим дельтоид с диагоналями длины 1001 и 2. При движении меньшей диагонали вдоль большей от ее конца к середине пееримeтр меняется от $2 + 2\sqrt{1001^2+1} > 2004$ до $2\sqrt{1001^2+4} < 2004$. Поэтому при некотором положении диагонали периметр равен 2004.

в) Да. Рассмотрим прямоугольник с диагоналями длины 1001. При изменении угла между диагоналями от 0° до 90° периметр изменяется от 2002 до $2002\sqrt{2} > 2004$. Поэтому найдется угол, при котором периметр равен 2004.

2.4. [5] Известно, что среди членов некоторой арифметической прогрессии a_1, a_2, a_3, a_4, ... есть числа a_1^2, a_2^2 и a_3^2. Д/ч эта прогрессия состоит из целых чисел. (*А.Быстриков*)

Решение: $a_3 - a_2 = a_2 - a_1 = d$ – разность прогрессии. $a_1 + a_2 = \dfrac{a_2^2 - a_1^2}{d}$ и $a_2 + a_3 = \dfrac{a_3^2 - a_2^2}{d}$ – целые числа $\Rightarrow$ $a_3 - a_1 = 2d$ – целое, а d – целое или полуцелое. Таким же будет число $2a_1 = (a_1 + a_2) - d$. Итак, возможны 3 случая: a_1 и d (а значит, и все члены прогрессии) – целые; d – целое, a_1 – полуцелое; d – полуцелое, a_1 – дробь со знаменателем 4. Но в последних двух случаях $a_1^2 - a_1$ – не “кратно”d.

2.5. [5] См. 1.5.

Основной вариант, 9-10 классы

3.1. [4] Конечная арифметическая прогрессия состоит из целых чисел, и ее сумма – степень двойки. Докажите, что количество членов прогрессии – тоже степень двойки.
Указание: Удвоенная сумма прогрессии делится на количество членов.

3.2. [5] Какое максимальное число шашек можно расставить на доске 8×8 так, чтобы каждая была под боем? (Если поля X, Y, Z стоят одно за другим по диагонали, шашка a стоит на X, b – на Y, и поле Z свободно, то шашка b под боем).

Решение: Ответ 32. Шашки, очевидно, нельзя ставить на граничные поля (а их 28). Разобьем оставшийся квадрат 6×6 на 4 квадрата 3×3. В каждом из этих квадратов должно быть хотя бы одно свободное поле (иначе шашка, стоящая в его центре, не атакована). Итого, должны быть свободны не менее 32 полей.

32 шашки расставить можно: например, оставив свободными все граничные и 4 центральных поля.

3.3. [5] Курс акций компании “Рога и копыта” повышается или понижается каждый раз на n процентов, где n – фиксированное целое число, $0 < n < 100$ (курс вычисляется с неограниченной точностью). Существует ли n, для которого курс акций может дважды принять одно и то же значение?

Решение: ответ: «нет». Если это произошло после k повышений и l понижений, то $(100 + n)^k(100 - n)^l = 100^{k+l}$. Один из множителей в левой части четен $\Rightarrow$ n четно. По той же причине n кратно 5 $\Rightarrow$ $n = 10m$. Подставляя, получим $(10 + m)^k(10 - m)^l = 10^{k+l}$. Аналогично доказываем, что m кратно 10 $\Rightarrow$ n делится на 100. Противоречие.

3.4. [6] 2 окружности пересекаются в точках A и B. Их общая касательная (та, которая ближе к точке B) касается окружностей в точках E и F. Прямая AB пересекает прямую EF в точке M. На продолжении AM за точку M выбрана точка K так, что $KM = MA$. Прямая KE вторично пересекает окружность, содержащую точку E, в точке C. Прямая KF вторично пересекает окружность, содержащую точку F, в точке D. Доказать что точки C, D и A лежат на одной прямой.

Решение: $ME^2 = MB{\cdot}MA = MF^2$ $\Rightarrow$ $ME = MF$ $\Rightarrow$ $ME{\cdot}MF = MB{\cdot}MA = MB{\cdot}MK$ $\Rightarrow$ 4-угольник $BEKF$ вписанный.

$\angle BAC + \angle BAD = \angle BEK + \angle BFK = 180°$ $\Rightarrow$ точки C, D и A лежат на одной прямой.

3.5. [6] Имеется биллиардный стол в виде многоугольника (не обязательно выпуклого), у которого каждые соседние стороны перпендикулярны друг другу. В вершинах находятся точечные лузы, попав в которые шарик проваливается. Из вершины A с внутренним углом 90° вылетает точечный шарик и движется внутри многоугольника, отражаясь от сторон по закону "угол падения равен углу отражения". Доказать, что он никогда не вернется в вершину A.

Решение: Если шар вылетает "вдоль борта", то он сваливается в ближайшую лузу. В противном случае угол между фиксированной стороной и отрезками пути шара постоянен (он не меняется при отражениях). Существует лишь один луч "внутри" стола с таким направлением и вершиной в $A \quad \Rightarrow$ шар мог вернуться в A только по той же прямой, по которой вылетел, т.е. в этом случае 1-й и последний отрезки пути шара совпадают. Но тогда 2-й и предпоследний отрезки также совпадают… Итак, путь шара состоит из **четного** числа отрезков. Но в нужном для возврата направлении шар двигается на **нечетных** отрезках. Противоречие.

3.6. [7] Первоначально на доске написано число 2004!. 2 игрока ходят по очереди. Игрок в свой ход вычитает из написанного числа какое-нибудь натуральное число, которое имеет не более 20 простых делителей (так, чтобы разность была неотрицательна), записывает на доске эту разность, а старое число стирает. Выигрывает тот, кто получит 0. Кто из играющих – начинающий, или его соперник, – может гарантировать себе победу, и как ему следует играть?

Решение: 2-й. Пусть P – произведение 21 наименьших простых чисел. 2004! делится на P. 1-й не может вычесть число, кратное P, поэтому после его хода останется число, дающее при делении на P ненулевой остаток r. 2-й вычитает этот остаток (любое число, имеющее более 20 простых делителей, $\geq P$, поэтому r не таково) и, тем самым, снова оставляет на доске число, кратное P. Далее он повторяет эту же стратегию. 1-й не может выиграть: после каждого его хода на доске остается число, не кратное P, тем более, не равное нулю.

Основной вариант, 11-12 классы

4.1. [4] См. 3.3.

4.2. [6] Имеется биллиардный стол в виде многоугольника (не обязательно выпуклого), у которого все углы составляют целое число градусов, а угол A – в точности 1°. В вершинах находятся точечные лузы, попав в которые шарик проваливается. Из вершины A вылетает точечный шарик и движется внутри многоугольника, отражаясь от сторон по закону "угол падения равен углу отражения". Докажите, что он никогда не вернется в вершину A.

Решение: Если шар вылетает "вдоль борта", то он сваливается в ближайшую лузу.

В противном случае угол α между 1-м отрезком пути шара и стороной AB – не "целый" (меньше 1°). Если некоторый отрезок пути составляет угол φ со стороной AB, то следующий отрезок (после отражения от стороны XY) составляет со стороной AB угол $2m° - \varphi$, где m – некоторое целое число (в этом проще всего убедиться, отразив оба отрезка пути, а также сторону AB относительно прямой XY). Следовательно, *каждый* из отрезков пути составляет с AB угол вида $2n° \pm \varphi$, причем выбор знака зависит от четности. Поэтому вернуться в вершину шар может только по той же прямой, по которой вылетел, причем после **четного** числа отражений. Далее см. решение 3.5.

4.3. [6] Прямоугольная проекция треугольной пирамиды на некоторую плоскость имеет максимально возможную площадь. Доказать, что эта плоскость параллельна либо одной из граней, либо двум скрещивающимся ребрам пирамиды.

Решение: Пусть A_1, B_1, C_1, D_1 – проекции вершин A, B, C, D пирамиды. Эта проекция может быть выпуклым 4-угольником с вершинами A_1, B_1, C_1, D_1 или тр-ком (если, например, точка D_1 находится внутри тр-ка $A_1B_1C_1$). Таким образом, "максимально возможная площадь проекции" равна наибольшему из 7 чисел: наибольших значений площадей тр-ков $A_1B_1C_1$, $A_1B_1D_1$, $A_1C_1D_1$, $B_1C_1D_1$ и наибольших значений площадей 4-угольников $A_1B_1C_1D_1$, $A_1B_1D_1C_1$, $A_1C_1B_1D_1$ (возможно, невыпуклых).

Наибольшее значение $S_{A_1B_1C_1}$ достигается, когда плоскость проекции параллельна грани ABC (ибо $S_{A_1B_1C_1} = S_{ABC}\cos\alpha$, где α – угол между плоскостями $A_1B_1C_1$ и ABC).

Как известно, $S_{A_1B_1C_1D_1} = 2\,S_{K_1L_1M_1N_1}$, где K_1, L_1, M_1, N_1 – середины сторон A_1B_1, B_1C_1, C_1D_1, D_1A_1, т.е. проекции середин K, L, M, N ребер AB, BC, CD, DA. $KLMN$ – пар-мм, стороны которого параллельны ребрам AC и BD (см. задачу 4.6; в частности, точки K, L, M, N лежат в одной плоскости). Следовательно, $S_{A_1B_1C_1D_1} = 2S_{KLMN}\cos\alpha$ и максимальна, когда плоскость проекции параллельна плоскости $KLMN$, т.е. ребрам AC и BD.

4.4. [6] См. 3.6

4.5. [7] На плоскости даны парабола $y = x^2$ и окружность, имеющие ровно 2 общие точки: A и B. Оказалось, что касательные к окружности и параболе в точке A совпадают. Обязательно ли тогда касательные к окружности и параболе в точке B также совпадают?

Решение: Нет. Приведем контрпример. Рассмотрим окружность $(x-a)^2 + (y-b)^2 - r^2 = 0$, соответствующую условию для точек $A(1,\ 1)$ и $B(-3,\ 9)$ (такая окружность существует: ее центр находится на пересечении нормали к параболе в точке A и серединного перпендикуляра к AB).

При подстановке $y = x^2$ в уравнение окружности мы получим уравнение 4-й степени, не содержащее члена 3-й степени $\Rightarrow$ сумма корней равна нулю. Мы знаем 3 корня этого уравнения $x_{1,2} = 1$ (в силу касания этот корень кратный), $x_3 = -3$. Отсюда $x_4 = 1$, т.е. в B касания нет (корень x_3 не кратный), и других точек пересечения также нет.

Замечание для сомневающихся. Доказать, что если касательные к окружности $f(x, y) = 0$ и параболе $y = x^2$ в точке $M(x_0, x_0^2)$ совпадают, то x_0 – кратный корень уравнения $f(x, x^2) = 0$.

Заметим, что $f(x, y) - f(x, z)$ делится на $y - z$ (это видно, например, из формулы разности квадратов). Пусть $y = g(x)$ – уравнение общей касательной (g – линейная функция). Эта касательная имеет как с окружностью, так и с параболой единственную общую точку M, поэтому x_0 – кратный корень квадратных уравнений $x^2 - g(x) = 0$ и $f(x, g(x)) = 0$.

$f(x, x^2) = f(x, g(x)) + (f(x, x^2) - f(x, g(x)))$. Первое слагаемое делится на $(x - x_0)^2$, второе – на $x^2 - g(x)$, что в свою очередь делится на $(x - x_0)^2$ $\Rightarrow$ $f(x, x^2)$ делится на $(x - x_0)^2$.

4.6. Перед экстрасенсом кладут колоду из 36 карт рубашкой вверх. Он называет масть верхней карты, после чего карту открывают, показывают ему и откладывают в сторону. После этого экстрасенс называют масть следующей карты, и т.д. Задача экстрасенса – угадать масть как можно большее число раз. На деле рубашки карт несимметричны, и экстрасенс видит, в каком из 2 положений лежит верхняя карта. Колода подготовлена подкупленным служащим. Служащий знает порядок карт в колоде, и хотя изменить его не может, зато может подсказать, располагая рубашки карт так или иначе согласно договоренности.

Может ли экстрасенс с помощью такой подсказки гарантированно обеспечить угадывание масти

а) [3] более чем у половины карт?

б) [5] не менее чем у 20 карт?

Решение:

а) Да. 1-ми 2 рубашками можно “закодировать” масть 2-й карты, следующими 2-мя – масть 4-й и т.д. Когда в колоде остались 2 карты достаточно закодировать лишь их порядок, что можно сделать при помощи рубашки 35-й карты.

б) Да. Рассмотрим 17 карт: все нечётные карты, кроме 1-й и 35-й, и 2-ю карту. Среди них найдутся 5 карт одной масти. Назовём эту масть *основной*. 1-ми 2 рубашками помощник кодирует основную масть. Экстрасенс называет основную масть на каждую из указанных 17 карт, тем самым угадывая не менее 5 из них. Кроме того, по методу а) он угадает масти всех четных карт, начиная с 4-й и масть 35-й карты. Итого, 17 + 5 = **23**.

2-й способ. 1-ми 2 рубашками кодируется масть **2-й** карты, если 24-я карта **красная**, и **3-й**, если 24-я карта – **черная**. Экстрасенс называет эту масть и на 2-ю и на 3-ю карту. Если они одной масти, то он угадывает обе, если же разной – то одну, но при этом узнает цвет 24-й карты. В дальнейшем рубашкой 24-й карты кодируется недостающая информация о ней. Так из 3 карт – 2-й, 3-й и 24-й – угадываются, как минимум, две.

Аналогично, угадываются по две из 4-й, 5-й и 25-й карт (используются 3-я и 4-я рубашки); …, из 22-й, 23-й и 34-й карт. Итого 22 карты + (см. а)) 2 последние – **24** карты.

Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Fall 2003.**

1 [**3**] There is $3 \times 4 \times 5$ - box with its faces divided into 1×1 - squares. Is it possible to place numbers in these squares so that the sum of numbers in every stripe of squares (one square wide) circling the box, equals 120?

2 [4] In 7-gon $A_1A_2A_3A_4A_5A_6A_7$ diagonals A_1A_3, A_2A_4, A_3A_5, A_4A_6, A_5A_7, A_6A_1 and A_7A_2 are congruent to each other and diagonals A_1A_4, A_2A_5, A_3A_6, A_4A_7, A_5A_1, A_6A_2 and A_7A_3 are also congruent to each other. Is the polygon necessarily regular?

3 [4] For any integer $n+1, \ldots, 2n$ (n is a natural number) consider its greatest odd divisor. Prove that the sum of all these divisors equals n^2.

4 [4] There are N points on the plane; no three of them belong to the same straight line. Every pair of points is connected by a segment. Some of these segments are colored in red and the rest of them in blue. The red segments form a closed broken line without self-intersections (each red segment having only common endpoints with its two neighbors and no other common points with the other segments), and so do the blue segments. Find all possible values of N for which such a disposition of N points and such a choice of red and blue segments are possible.

5 [**5**] 25 checkers are placed on 25 leftmost squares of $1 \times N$ board. Checker can either move to the empty adjacent square to its right or jump over adjacent right checker to the next square if it is empty. Moves to the left are not allowed. Find minimal N such that all the checkers could be placed in the row of 25 successive squares but in the reverse order.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2003.**
Solutions

1. Answer: yes.

   Example: Placing 5 into each square of two 4×3-faces, 8 into each square of two 5×3-faces and 9 into each square of two 4×5-faces , we satisfy all the given requirements.

2. Assume that a 7-gon is convex. In order for a polygon to be regular, one needs to prove equalities of all its sides and angles. Let us color the sides of the polygon with white, the first set of equal diagonals with blue, and the second set of equal diagonals with red. All triangles created by two blue and one red sides are equal (S-S-S). Therefore the next two sets of angles are equal:

   (a) Angles at the bases of isosceles triangles mentionned (let us denote them by α).

   (b) Angles at the vertices of isosceles triangles mentionned (let us denote them by β).

   Consider the triangles created by three different colors. All the angles opposed to white sides are equal (each equals $\alpha - \beta$). Then all sides of the polygon are equal (S-A-S).

   Now, all the triangles, created by two white sides and one blue side are equal (S-S-S). Thus all angles of the polygon are equal.

   Therefore, the polygon is regular.

3. Let us denote the greatest odd divisor of number K as $\text{god}(K)$. Obviously, $\text{god}(K) \leq K$. Assume, that $\text{god}(l) = \text{god}(m)$, $l, m \in [n+1; 2n]$, $l < m$. Then $m \geq 2l$, which is impossible.

   Therefore, for all (n) numbers from $[n+1, 2n]$ their corresponding greatest odd divisors are distinct. It means, that the set $\{\text{god}(m), n+1 \leq m \leq 2n\}$ coincides with $\{1, 3, \dots, 2n-1\}$. Then, the sum in question is $1 + 3 + \cdots + 2n + 1 = n^2$.

4. Let us note that

   (a) from each point emanates exactly $(n-1)$ segments (each pair of points is connected).

   (b) from each point emanates either 2 blue/red segments or 2 blue and 2 red segments (each broken line is closed, without intersection and there are no isolated points).

   Therefore, we have 2 possible cases to consider:

   (i) n=3.

   (ii) n=5.

   Case (i) is trivial, corresponding to a triangle with all sides of the same color.

   Case (ii) is also possible:

   Example. Consider points $A(0;4); B(-4;-4); C(4;-4); D(-1;0); E(1;0)$. Connect them by red segments in the order A, B, C, D, E, A and by blue segments in the order A, D, B, D, C, A.

5. ANSWER: 50.

Let us prove first that the arrangement in question is impossible, if $N < 50$. Case $N = 49$ means that checker "25" stays on its initial place, so the only possible move is that checker "24" jumps through checker "25" on the 26-th place (its final position). After that any movement to the right is impossible.

Example, that $N = 50$ works.

(a) Checker "25" moves on the 26-th place (its final position).

(b) Checker "23" jumps through "24" and "25", then moves on the 28-th place (its final position).

(c) Checker "21" jumps through "22", "24", "25" and "23", then moves on the 30-th place (its final position)

and so on...

Finally, checker "1" jumps through "2", "4", ... "22", "24", "25", "23",..., "3", then moves on the 50-th place (its final position).

Now,

(a) Checker "2" moves one space to the right and then jumps through "4", ..., "22", "24", "25", "23",...,"3" occupying the 49-th place (its final position).

(b) Checker "4" moves one space to the right and then jumps through "6", ..., "22", "24", "25", "23",...,"5" occupying the 47-th place (its final position)

and so on ...

Seniors
(Grades 11,12)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Fall 2003.**

1 [**3**] For any integer $n+1, \ldots, 2n$ (n is a natural number) consider its greatest odd divisor. Prove that the sum of all these divisors equals n^2.

2 [**4**] What least possible number of unit squares (1×1) must be drawn in order to get a picture of 25×25-square divided into 625 of unit squares?

3 [**5**] A salesman and a customer altogether have 1999 rubles in coins and bills of 1, 5, 10, 50, 100, 500 , 1000 rubles. The customer has enough money to buy a Cat in the Bag which costs the integer number of rubles. Prove that the customer can buy the Cat and get the correct change.

4 Each side of 1×1 square is a hypothenuse of an exterior right triangle. Let A, B, C, D be the vertices of the right angles and O_1, O_2, O_3, O_4 be the centers of the incircles of these triangles. Prove that

a) [3] The area of quadrilateral $ABCD$ does not exceed 2;

b) [3] The area of quadrilateral $O_1O_2O_3O_4$ does not exceed 1.

6 [**5**] A paper tetrahedron is cut along some of so that it can be developed onto the plane. Could it happen that this development cannot be placed on the plane in one layer?

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2003.**
Solutions

1. Let us denote the greatest odd divisor of number K as god(K). Obviously, god$(K) \le K$. Assume, that god$(l) =$ god(m), $l, m \in [n+1; 2n]$, $l < m$. Then $m \ge 2l$, which is impossible.

   Therefore, for all (n) numbers from $[n+1, 2n]$ their corresponding greatest odd divisors are distinct. It means, that the set $\{\text{god}(m), n+1 \le m \le 2n\}$ coincides with $\{1, 3, \dots, 2n-1\}$. Then, the sum in question is $1 + 3 + \dots + 2n + 1 = n^2$.

2. Let us solve the problem for a $(2n+1) \times (2n+1)$-square. Note that all unit boundary squares $(8n)$ should be drawn (otherwise there would be "holes" in a frame). Now we have a $(2n-1) \times (2n-1)$ square with the frame. Let it be colored as a chess board, with black squares at the angles. Drawing only white squares would give us the whole picture in question. The number of white squares is equal to $\big((2n-1)^2 - 1\big)/2$. Then the total number of the unit squares used equals $8n + ((2n-1)^2 - 1)/2 = 360$, if $n = 25$.

   Let us show that it is indeed the minimal number. Let us tile a $(2n-1) \times (2n-1)$ square with dominos (2×1 rectangles). Then we use $\big((2n-1)^2 - 1\big)/2$ dominos with 1 unit square left. In each domino we have to draw at least one square, so the least number of squares drawn is $\big((2n-1)^2 - 1\big)/2$.

3. Let Customer give all his money to Salesman. The value of change could vary from 0 (if Cat is a "gift") to 1999 rubles and we show that any integer value could be created within a set of bills. It is enough to solve the problem in the case of the following (minimal) set of sixteen bills $\{1000, 500, 100, 100, 100, 100, 50, 10, 10, 10, 10, 5, 1, 1, 1, 1\}$. Actually, representing 1999 as 1000+900+90+9 one can see that in the "minimal" case we must have exactly one 1000 ruble bill, one 500 ruble bill, four 100 ruble bills, one 50 ruble bill, four 10 ruble bills, one 5 ruble bill and four 1 ruble bills. Now, notice that if a transaction could be made with the minimal set of bills, then it could be also made with any other set of bills. Actually, each smaller nomination divides every larger one. Therefore, if in an arbitrary set we have more than, say, four ruble bills, we wrap five rubles by a rubber band and consider it as a 5 ruble bill and so on. So, any set of bills could be reduced to the minimal set.

   It is easy to see that any value of change in the form $ABCD$, where $A = 0, 1$ and B, C, D are any digits from 0 to 9, could be paid with the minimal set of bills.

4. Lemma. The area of a quadrilateral, placed into a circle with radius R does not exceed $2R^2$.

   Proof. Let $ABCD$ be the quadrilateral in mention, and O the center of the circle. Then, Area$(\triangle ABO) = \frac{1}{2} AO \cdot BO \cdot \sin(\angle AOB) \le \frac{1}{2} R^2$ and

$$\text{Area}(ABCD) = \text{Area}(\triangle ABO) + \text{Area}(\triangle BCO) + \text{Area}(\triangle CDO) + \text{Area}(\triangle DAO) \le 2R^2.$$

   Let O be the center of the given square $KLMN$.

(a) Notice, that points A, B, C, D lie on semicircles with diameter 1. Let S be the midpoint of side KL of $\triangle AKL$. Then $AO \leq OS + AS = \frac{1}{2} + \frac{1}{2} = 1$. In a similar way, we have $BO \leq 1$, $CO \leq 1$, $DO \leq 1$, meaning that points A, B, C, D lie in a circle with the center O and radius 1. Then according to Lemma, Area$(ABCD) \leq 2$.

(b) Let O_1 be the center of an incircle of $\triangle AKL$. Then, for any position of A $\angle KO_1L = 135°$ (O_1 is the intersection of bisectors of $\triangle KAL$); therefore O_1 lies on the circumference with center O and radius $\sqrt{2}/2$. Then $OO_1 = \sqrt{2}$. In similar way we have $OO_2 = OO_3 = OO_4 = \sqrt{2}$. Therefore, according to the Lemma, the Area$(O_1O_2O_3O_4) \leq 1$.

5. Answer: yes. Under "development" we understand that the figure could be flattened on a plane (folded, but not teared or crumpled).

Example. Consider a Cartesian coordinate system and an "envelop" $PABQD$ (consisting of two layers with PQ open; the first layer is a rectangle $PABQ$ and the second layer consists of triangles PAD, ADB and BDQ), where $D(0,0), A(-1/2, \sqrt{3}/2), B(1/2; \sqrt{3}/2), Q(1/2; 0), P(-1/2, 0), C(0, 0)$.

Now, let us consider this figure in a space. Pull point D up while pushing point C down until points P and Q coincide (in the mid point of segment CD). We get tetrahedron $ABCD$, with edge CD cut. Reversing the process, we could develop a (regular) tetrahedron $ABCD$ into a planar figure. It is easy to see that

(a) this procedure could be implemented in a space.

(b) a tetrahedron is a closed 3-D figure, so by cutting only one edge it is impossible to develop it in one layer.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper[1] **Fall 2003.**

1. An increasing arithmetic progression consists of one hundred positive integers. Is it possible that every two of them are relatively prime?

2. Smallville is populated by unmarried men and women, some of them are mutual acquaintants. The City's two Official Matchmakers are aware of all the mutual acquaintances. One of them claimed: "I can arrange it so that every brown haired man will marry a woman with whom he is mutually acquainted." The other claimed, "I can arrange it so that every blond haired woman will marry a man with whom she is mutually acquainted." An amateur mathematician overheard their conversation and said, "Then both arrangements can be made at the same time!" Is he right?

3. Determine all positive integers k such that there exist positive integers m and n satisfying $m(m+k) = n(n+1)$.

4. In chess, a bishop attacks any square on the two diagonals that contain the square on which it stands, including that square itself. Several squares on a 15×15 chessboard are to be marked so that a bishop placed on any square of the board attacks at least two of the marked squares. Determine the minimal number of such marked squares.

5. Prove that $135^\circ \le \angle OAB + \angle OBC + \angle OCD + \angle ODA \le 225^\circ$ for any point O inside a square $ABCD$.

6. An ant crawls on the outer surface of a rectangular box. The distance between two points on a surface is defined as the length of the shortest path the ant needs to crawl to reach one point from the other. Is it true that if the ant is at a vertex, then the opposite vertex is the point on the surface which is at the greatest distance away?

7. In a game, Boris has 1000 cards numbered $2, 4, \ldots, 2000$ while Anna has 1001 cards numbered $1, 3, \ldots, 2001$. The game lasts 1000 rounds. In an odd-numbered round, Boris plays any card of his. Anna sees it and plays a card of hers. The player whose card has the larger number wins the round, and both cards are discarded. An even-numbered round is played in the same manner except that Anna plays first. At the end of the game, Anna discards her unused card. What is the maximal number of rounds each player can guarantee to win, regardless of how the opponent plays?

Note: The problems are worth 4, 5, 5, 6, 7, 7 and 8 points respectively.

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[1]Courtesy of Andy Liu.

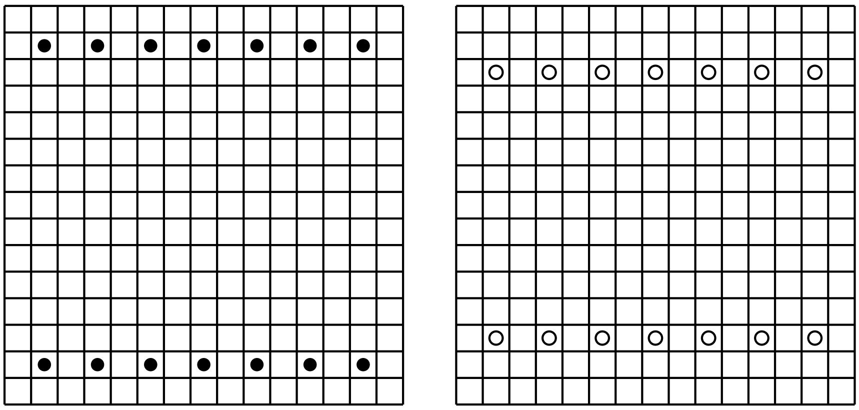
Solution to Senior A-Level Fall 2003

1. Let the arithmetic progression be $\{a_k\}$ where $a_k = k(101!) + 1$ for $k = 1, 2, \ldots, 100$. Let d be the greatest common divisor of a_i and a_j where $i < j$. Then d divided $a_j - a_i = (j - i)(101!)$. If d has a prime divisor p, then $p \leq 101$. On the other hand, since p divides $i(100!) + 1$, we must have $p > 101$. This contradiction shows that d has no prime divisors. It follows that $d = 1$.

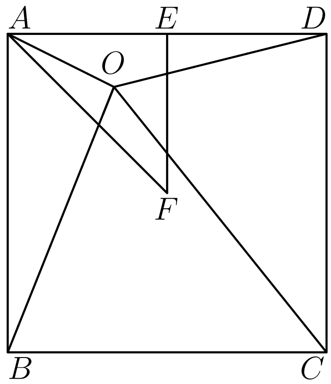
2. Construct a graph as follows. One group of vertices $X_1, X_2, \ldots, X_n$ represent the brown haired men. Another group of vertices $Y_1, Y_2, \ldots, Y_m$ represent the blond haired women. From each of these vertices, draw an arc pointing to the vertex representing the mate promised by the appropriate matchmaker. If a man represented by X_k is promised a woman who is not blond haired, a new vertex W_k is introduced to represent the mate. Similarly, if a woman represented by Y_k is promised a man who is not brown haired, a new vertex Z_k is introduced to represent the mate. Now some of these arcs may form a cycle. All vertices on such a cycle are X-vertices or Y-vertices, as W-vertices and Z-vertices have no out-going arcs. Moreover, since the arcs point alternately at the two groups, the cycle must be of even length. Hence the amateur mathematician can prescribe marriages, according to alternate arcs along the cycle, between brown haired men and blond haired women. There are also arcs which form a path. Such a path must consist of X-vertices and Y-vertices, ecept that it terminates at a W-vertex or an Z-vertex. The amateur mathematician can prescribe marriages according to alternate arcs along the path, starting with the initial arc. If the path is of even length, all marriages here are between brown haired men and blond haired women. If the path is of odd length, this is still the case except for the marriage corresponding to the terminating arc. In any cases, all brown haired men and all blond haired women are married off with mutual acquaintants.

3. Multiply the given equation by 4 and rearranging terms, we have $(2m+k)^2-(2n+1)^2 = k^2-1$. Suppose k is odd. We may set $2m + k + 2n + 1 = \frac{k^2-1}{2}$ and $2m + k - 2n - 1 = 2$, yielding $m = \frac{(k-2)^2-1}{8}$ and $n = \frac{k^2-1}{8} - 1$. Both are integers since $x^2 \equiv 1 \pmod 8$ for odd x, and both are positive if $k \geq 5$. For $k = 3$, we have $(2m + 3)^2 - (2n + 1)^2 = 8$, but the only two squares differing by 8 are 9 and 1. Hence we must have $m = n = 0$. Suppose k is even. e may set $2m+k+2n+1 = k^2-1$ and $2m+k-2n-1 = 1$, yielding $m = \frac{k^2}{4} - \frac{k}{2}$ and $n = \frac{k^2}{4} - 1$. Both are clearly integers, but are only positive if $k \geq 4$. For $k = 2$, we have $(2m + 2)^2 - (2n + 1)^2 = 3$, but the only two squares differing by 3 are 4 and 1. Again we must have $m = n = 0$. In summary, positive integers m and n exist if and only if $k \geq 4$.

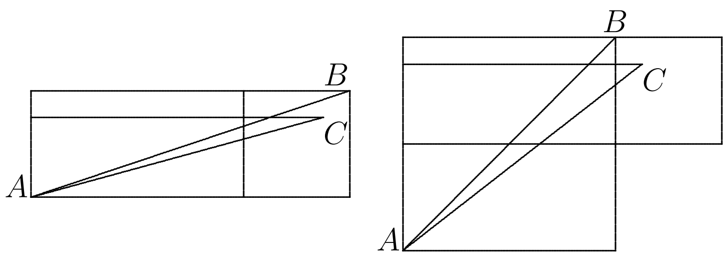
4. Colour the squares in the 15×15 board in the usual chessboard patterm, with black squares at the corners. In the diagram on the left, 14 black squares are marked with black circles. It is easy to verify that any bishop placed on a black square will attack at least 2 of the marked black squares. To show that 14 is optimal, consider the 28 black squares along the edge of the board. Each must attack at least 2 of the marked black squares. Yet any black square can be attacked by at most 2 of the black squares along the edge. Similarly, 14 white squares can be marked as shown in the diagram on the right, with white circles. As with black sqaures, this is both necessary and sufficient.



5. Let E be the midpoint of AD and let F be the centre of $ABCD$. We may assume by symmetry that O is in triangle AEF. Hence $OA \leq OD$ and $OC \geq OB$, so that $\angle ODA \leq \angle OAD$ and $\angle OBC \geq \angle OCB$. Now $90^\circ = \angle OAB + \angle OAD \geq \angle OAB + \angle ODA \geq \angle OAB \geq 45^\circ$ while $90^\circ + 45^\circ \geq \angle OBC + \angle OCD \geq \angle OCB + \angle OCD = 90^\circ$. Addition yields the desired result.



6. Let A and B be opposite vertices of a $4 \times 4 \times 8$ box and let C be a point on the 4×4 face with B as a vertex, such that the distance from C to each side containing B is 1. To go from A to B or C efficiently, we unfold the box in one of two ways, both shown in the diagram below, and travel in a straight line. In the diagram on the left, $AB = \sqrt{12^2 + 4^2} = \sqrt{160}$ while $AC = \sqrt{11^2 + 3^2} = \sqrt{130}$. In the diagram on the right, $AB = \sqrt{8^2 + 8^2} = \sqrt{128}$ while $AC = \sqrt{9^2 + 7^2} = \sqrt{130}$. All other ways of unfolding the box yield longer distances for AB and AC. Hence the minimum distance from A to B is $\sqrt{128}$ and that from A to C is $\sqrt{130}$, showing that C is further from A than B.



7. Let us first play the game with 5 cards, with Boris holding 2 and 4 and Anna holding 1, 3 and 5. If Boris leads 2, Anna wins both rounds by playing 3 and then leading 5. If Boris leads 4, Anna wins both rounds by playing 5 and then leading 3. We shall prove by induction on n that if there are $4n+1$ cards, than Boris can guarantee winning $n-1$ rounds while Anna can guarantee winning $n+1$ rounds. The best strategy for Boris is to lead 2 in the first round. If Anna is going to win this round, she should definitely play 3. Moreover, there is no reason why she should lose this round by playing 1, because 1 and 3 are equivalent after 2 has been discarded. Hence we may assume that Anna wins the first round with 3. If in the second round Anna does not lead $4n+1$, Boris can win that round by playing the lowest card above Anna's. At this point, although there are now some gaps among the $4n-3$ cards still in play, the holdings of Boris and Anna are in the alternating pattern. Hence we may play the balance of the game as though the numbers on the cards have been adjusted to run from 1 to $4n-3$, and reach the desired conclusion by induction. Suppose Anna leads $4n+1$ in the second round. Clearly, Boris should concede by playing 4, but he can win the next round by leading $4n$, forcing Anna to concded with 1 or 5 (now equivalent) and restoring the alternating pattern. If in any subsequent round, Anna does not lead her highest card, Boris can win that round and at the same time restore the alternating pattern. If Anna continues to lead her highest card, Boris can do likewise. He will then lose the first and the last rounds, and wins every second round in between, winning altogether $n-1$ rounds. Anna's simplest strategy is to win the first round by playing the lowest card above Boris's, and leading 1 in the second round. The desired conclusion then follows from the induction hypothesis.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper [1] **Fall 2003.**

1. Smallville is populated by unmarried men and women, some of them are mutual acquaintants. The City's two Official Matchmakers are aware of all the mutual acquaintances. One of them claimed: "I can arrange it so that every brown haired man will marry a woman with whom he is mutually acquainted." The other claimed, "I can arrange it so that every blond haired woman will marry a man with whom she is mutually acquainted." An amateur mathematician overheard their conversation and said, "Then both arrangements can be made at the same time!" Is he right?

2. Prove that one can represent every positive integer in the form $3^{u_1} \cdot 2^{v_1} + 3^{u_2} \cdot 2^{v_2} + \ldots + 3^{u_k} \cdot 2^{v_k}$ where $u_1 > u_2 > \cdots > u_k \geq 0$ and $0 \leq v_1 < v_2 < \cdots < v_k$ are integers.

3. An ant crawls on the outer surface of a rectangular box. The distance between two points on a surface is defined as the length of the shortest path the ant needs to crawl to reach one point from the other. Is it true that if the ant is at a vertex, then the opposite vertex is the point on the surface which is at the greatest distance away?

4. Triangle ABC has orthocentre H, incentre I and circumcentre O. K is the point where the incircle touches BC. If IO is parallel to BC, prove that AO is parallel to HK.

5. In a game, Boris has 1000 cards numbered $2, 4, \ldots, 2000$ while Anna has 1001 cards numbered $1, 3, \ldots, 2001$. The game lasts 1000 rounds. In an odd-numbered round, Boris plays any card of his. Anna sees it and plays a card of hers. The player whose card has the larger number wins the round, and both cards are discarded. An even-numbered round is played in the same manner except that Anna plays first. At the end of the game, Anna discards her unused card. What is the maximal number of rounds each player can guarantee to win, regardless of how the opponent plays?

6. Let O be the incentre of a tetrahedron $ABCD$ in which the sum of areas of the faces ABC and ABD is equal to the sum of areas of the faces CDA and CDB. Prove that midpoints of BC, AD, AC and BD lie on a plane passing through O.

7. Each cell of an $m \times n$ table is filled with a $+$ sign or a $-$ sign. Such a table is said to be *irreducible* if one cannot change all $-$ signs to $+$ signs by applying, as many times as desired, some permissible operation.

   (a) Suppose the permissible operation is to change the signs of all cells in a row or a column to the opposite signs. Prove that an irreducible table contains an irreducible 2×2 sub-table.

   (b) Suppose the permissible operation is to change the signs of all cells in a row, a column or a diagonal (which may be of any length, including those of length 1, consisting of a corner cell). Prove that an irreducible table contains an irreducible 4×4 sub-table.

Note: The problems are worth 4, 4, 6, 7, 7, 7 and 3+6 points respectively.

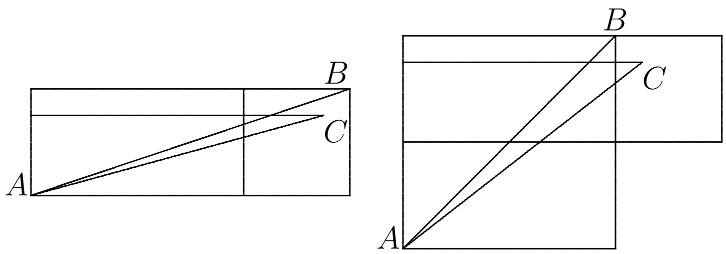
[1] Courtesy of Andy Liu

Solution to Senior A-Level Fall 2003

1. Construct a graph as follows. One group of vertices $X_1, X_2, \ldots, X_n$ represent the brown haired men. Another group of vertices $Y_1, Y_2, \ldots, Y_m$ represent the blond haired women. From each of these vertices, draw an arc pointing to the vertex representing the mate promised by the appropriate matchmaker. If a man represented by X_k is promised a woman who is not blond haired, a new vertex W_k is introduced to represent the mate. Similarly, if a woman represented by Y_k is promised a man who is not brown haired, a new vertex Z_k is introduced to represent the mate. Now some of these arcs may form a cycle. All vertices on such a cycle are X-vertices or Y-vertices, as W-vertices and Z-vertices have no out-going arcs. Moreover, since the arcs point alternately at the two groups, the cycle must be of even length. Hence the amateur mathematician can prescribe marriages, according to alternate arcs along the cycle, between brown haired men and blond haired women. There are also arcs which form a path. Such a path must consist of X-vertices and Y-vertices, ecept that it terminates at a W-vertex or an Z-vertex. The amateur mathematician can prescribe marriages according to alternate arcs along the path, starting with the initial arc. If the path is of even length, all marriages here are between brown haired men and blond haired women. If the path is of odd length, this is still the case except for the marriage corresponding to the terminating arc. In any cases, all brown haired men and all blond haired women are married off with mutual acquaintants.

2. We use induction, the basis being $1 = 3^0 2^0$. For any even integer $n \geq 2$, $\frac{n}{2}$ has a desirable expression $3^{u_1}2^{v_1} + 3^{u_2}2^{v_2} + \cdots + 3^{u_k}2^{v_k}$. Hence $n = 3^{u_1}2^{v_1+1} + 3^{u_2}2^{v_2+1} + \cdots + 3^{u_k}2^{v_k+1}$ is also a desirable expression. For any odd integer $n \geq 3$, let t be the unique integer such that $3^{t+1} > n \geq 3^t$. If $n = 3^t$, then it has a desirable expression $3^t 2^0$. Otherwise, $n - 3^t$ has a desirable expression $3^{u_1}2^{v_1} + 3^{u_2}2^{v_2} + \cdots + 3^{u_k}2^{v_k}$. Then $n = 3^t 2^0 + 3^{u_1}2^{v_1} + 3^{u_2}2^{v_2} + \cdots + 3^{u_k}2^{v_k}$. Since $n - 3^t$ is even, $v_1 > 0$. Moreover, $3^{t+1} > n \geq 3^t + 3^{u_1}2^{v_1} \geq 2 \cdot 3^{u_1}$. Hence $2 \cdot 3^t > 2 \cdot 3^{u_1}$ so that $t > u_1$. It follows that our expression for n is a desirable one.

3. Let A and B be opposite vertices of a $4 \times 4 \times 8$ box and let C be a point on the 4×4 face with B as a vertex, such that the distance from C to each side containing B is 1. To go from A to B or C efficiently, we unfold the box in one of two ways, both shown in the diagram below, and travel in a straight line. In the diagram on the left, $AB = \sqrt{12^2 + 4^2} = \sqrt{160}$ while $AC = \sqrt{11^2 + 3^2} = \sqrt{130}$. In the diagram on the right, $AB = \sqrt{8^2 + 8^2} = \sqrt{128}$ while $AC = \sqrt{9^2 + 7^2} = \sqrt{130}$. All other ways of unfolding the box yield longer distances for AB and AC. Hence the minimum distance from A to B is $\sqrt{128}$ and that from A to C is $\sqrt{130}$, showing that C is further from A than B.



4. We first establish two well-known results.

Lemma 1.

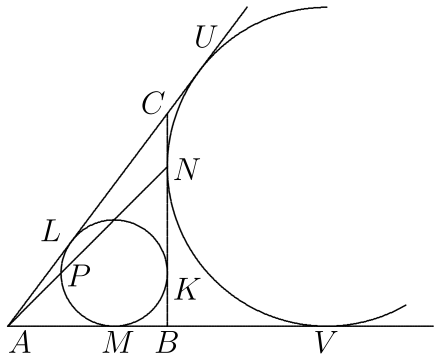
Let O and H be the circumcentre and the orthocentre of triangle ABC respectively, and let D be the midpoint of BC. Then $AH = 2DO$.

Proof:

Let G be the centroid of triangle ABC. Perform a central projection about G carrying A into D. This carries B into the midpoint E of CA and C into the midpoint F of AB. Hence it carries H into the orthocentre of triangle DEF, which is O. It follows that O, G and H lie on a line (called the *Euler line* of ABC). Since OD and AH are both perpendicular to BC, they are parallel. Hence triangles DOG and AHG are similar. Since $AG = 2GD$, we have $AH = 2DO$.

Lemma 2.

PK is a diameter of the incircle of triangle ABC where K is its point of tangency with BC. The extension of AP cuts BC at N. Then $BN = CK$.



Proof:

Perform a central projection about A carrying P into N. This carries the incircle of ABC into its excircle opposite A, N being the point of tangency with BC. Let the incircle touch CA at L and AB at M, and let the excircle touch CA at U and AB at V. Then

$$\begin{aligned}
2BN &= BN + BK - NK \\
&= BU + BM - NK \\
&= AV - AM - NK \\
&= AU - AL - NK \\
&= CV + CL - NK \\
&= CN + CK - NK \\
&= 2CK.
\end{aligned}$$

Dividing by 2 yields the desired result.

We now return to the original problem. Let D and N be as in the Lemmas. That OI is parallel to BC means that $DO = IK$. By Lemma 1, $AH = 2DO = 2IK = PK$. Since AH is parallel to PK, $AHKP$ is a parallelogram so that AP is parallel to HK. By Lemma 2, $ND = BD - BN = CD - CK = DK$. Let the line through D perpendicular to BC cut AN at Q. Then $DQ = \frac{1}{2}PK = DO$. Since O also lies on this line and on the same side of BC as Q, Q and O coincide. Hence O lies on AN, and AO is parallel to HK.

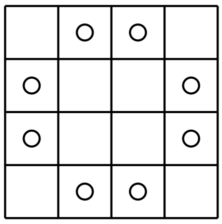
5. Let us first play the game with 5 cards, with Boris holding 2 and 4 and Anna holding 1, 3 and 5. If Boris leads 2, Anna wins both rounds by playing 3 and then leading 5. If Boris leads 4, Anna wins both rounds by playing 5 and then leading 3. We shall prove by induction on n that if there are $4n + 1$ cards, than Boris can guarantee winning $n - 1$ rounds while Anna can guarantee winning $n+1$ rounds. The best strategy for Boris is to lead 2 in the first round. If Anna is going to win this round, she should definitely play 3. Moreover, there is no reason why she should lose this round by playing 1, because 1 and 3 are equivalent after 2 has been discarded. Hence we may assume that Anna wins the first round with 3. If in the second round Anna does not lead $4n + 1$, Boris can win that round by playing the lowest card above Anna's. At this point, although there are now some gaps among the $4n - 3$ cards still in play, the holdings of Boris and Anna are in the alternating pattern. Hence we may play the balance of the game as though the numbers on the cards have been adjusted to run from 1 to $4n - 3$, and reach the desired conclusion by induction. Suppose Anna leads $4n + 1$ in the second round. Clearly, Boris should concede by playing 4, but he can win the next round by leading $4n$, forcing Anna to concded with 1 or 5 (now equivalent) and restoring the alternating pattern. If in any subsequent round, Anna does not lead her highest card, Boris can win that round and at the same time restore the alternating pattern. If Anna continues to lead her highest card, Boris can do likewise. He will then lose the first and the last rounds, and wins every second round in between, winning altogether $n - 1$ rounds. Anna's simplest strategy is to win the first round by playing the lowest card above Boris's, and leading 1 in the second round. The desired conclusion then follows from the induction hypothesis.

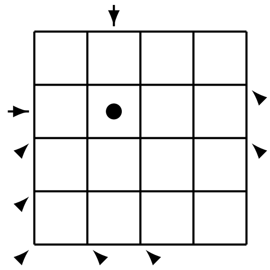
6. Denote by $[P]$ the area of the polygon P. We are given that $[ABC]+[ABD] = [ACD]+[BCD]$. Denote the common value by S. Let V be the volume of $ABCD$ and r be its inradius. Then $V = \frac{r}{3}([ABC]+[ABD]+[ACD]+[BCD])$ so that $r = \frac{3V}{2S}$. Let $E,\ F,\ G$ and H be the respective midpoints of $BC,\ AC,\ AD$ and BD. Let $a,\ b,\ c$ and d be the altitudes of $ACD,\ BCD, ABC$ and ABD from $A,\ B,\ C$ and D respectively. Then $S = [ABC] + [ABD] = \frac{1}{2}AB(c + d)$ so that $2S = AB(c + d)$. Similarly, $2S = CD(a + b)$. Since EF and GH are both parallel to AB while EH and FG are both parallel to CD, $EFGH$ is a parallelogram. Draw a line between EF and GH and parallel to both, such that its distances from EF and GH are in the ratio $c : d$. Draw another line between EH and FG and parallel to both, such that its distance from EH and FG are in the ratio $b : a$. Now the altitude of $ABCD$ from D is equal to $\frac{3V}{[ABC]}$. Hence the distance from G or H to ABC is $\frac{3V}{cAB}$ and the distance from I to ABC is $\frac{3V}{cAB} \cdot \frac{c}{c+d} = \frac{3V}{2S}$. Similarly, the distances from I to $ABD,\ AACD$ and BCD are all equal to $\frac{3V}{2S}$. It follows that I coincides with O, so that O is indeed coplanar with $E,\ F,\ G$ and H.

7. (a) We first note that a 2×2 table is irreducible if and only if it contains an odd number of $-$ signs. It is easy to verify that if the number of $-$ signs is even, all of them can be changed to $+$ signs by the permissible operation. However, since this operation cannot change the parity of the number of $-$ signs, a 2×2 table with an odd number of $-$ signs is indeed irreducible. If an $m \times n$ table contains an irreducible 2×2 sub-table, it is clearly irreducible since we cannot even change all $-$ signs in the sub-table into $+$ signs. Suppose there are no irreducible sub-tables. By applying the permissible operation to columns headed by a $-$ sign, we can make all signs in the first row $+$ signs. We claim that the second row consists only of $-$ signs or only of $+$ signs. Otherwise, there will

be a $-$ sign next to a $+$ sign. Together with the $+$ signs in the first row, these four will constitute an irreducible 2×2 sub-table. By applying the permissible operation to the second row if necessary, we can make it a row of $+$ signs. In the same way, we can fix the remaning rows, showing that a table without an irreducible sub-table is reducible.

(b) We claim that a 4×4 table is irreducible if and only if it contains an odd number of $-$ signs in its eight edge squares, which are marked with circles in the diagram on the left. Since the permissible operation always affects an even number of these squares, it cannot change the parity of the number of $-$ signs in them. It follows that such a 4×4 table is indeed irreducible. If an $m \times n$ table contains an irreducible 4×4 sub-table, it is clearly irreducible since we cannot even change all $-$ signs in the edge squares of the sub-table into $+$ signs.





Suppose a 4×4 table has an even number of $+$ signs in its eight edge squares. By applying the permissible operation one column at a time if necessary, we can change the four signs in the edge squares on the top two rows into $+$ signs. If the two signs in the edge squares on the third row are the same, we can change both into $+$ signs by applying the permissible operation along the that row. Then the two signs in the edge squares on the bottom row must be the same, and can be changed into $+$ signs. If the two signs in the edge squares on the third row are different, then the two signs in the edge squares on the bottom row must also be different. They can be changed into $+$ signs by applying the permissible operation along the short diagonals involving only these squares, preceded by an application along the third row if necessary. The diagram on the right shows how one can apply the permissible operation nine times along the row, columnn and diagonals as indicated, so that a $-$ sign in the central square marked with a black circle can be changed into a $+$ sign without affecting any other squares. Finally, the corner squares can be fixed individually as diagonals of length 1. This justifies our earlier claim that a 4×4 table i irreducible if and only if the number of $-$ signs in its eight edge squares is even. Suppose there are no irreducible sub-tables in an $m \times n$ table. We first change all signs in the top three rows into $+$ signs as follows. We proceed column by column from left to right. By applying the permissible operation along the column if necessary, we can make the sign in the second row a $+$ sign. By applying the permissible operation along a down-diagonal passing through the square in the top row and along an up-diagonal passing through the square in the third row, we can make those $+$ signs too. In the fourth row, apart from the two end squares, the signs in all the others must be the same, as otherwise we would have a irreducible 4×4 sub-table. By applying the permissible operation along this row if necessary., we can make them all $+$ signs. The end squares can be dealt with by applying the permissible operation in diagonals which do not affect any other squares in the top four rows. In the same way, we can fix the table one row at a time and change all the signs into $+$ signs.

Juniors
(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Spring 2004.**

1 [**3**] In triangle ABC the bisector of angle A, the perpendicular to side AB from its midpoint, and the altitude from vertex B, intersect in the same point. Prove that the bisector of angle A, the perpendicular to side AC from its midpoint, and the altitude from vertex C also intersect in the same point.

2 [**3**] Find all possible values of $n \geq 1$ for which there exist n consecutive positive integers whose sum is a prime number.

3 Bucket A contains 3 litres of syrup. Bucket B contains n litres of water. Bucket C is empty. We can perform any combination of the following operations:

- Pour away the entire amount in bucket X,
- Pour the entire amount in bucket X into bucket Y,
- Pour from bucket X into bucket Y until buckets Y and Z contain the same amount.

(a) [**3**] How can we obtain 10 litres of 30% syrup if $n = 20$?

(b) [**2**] Determine all possible values of n for which the task in (a) is possible.

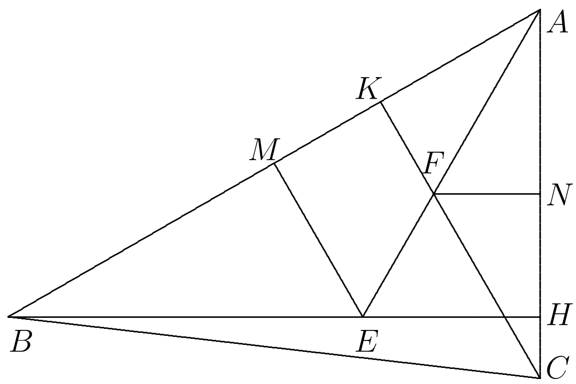
4 [**5**] A positive integer $a > 1$ is given (in decimal notation). We copy it twice and obtain a number $b = \overline{aa}$ which happened to be a multiple of a^2. Find all possible values of b/a^2.

5 [**6**] Two 10-digit integers are called neighbours if they differ in exactly one digit (for example, integers 1234567890 and 1234507890 are neighbours). Find the maximal number of elements in the set of 10-digit integers with no two integers being neighbours.

International Mathematics
TOURNAMENT OF THE TOWNS

Solution to Junior O-Level Spring 2004[1]

1. Let M and N be the respective midpoints of AB and AC. Let the extension of BE cut AC at H, and the extension of CF cut AB at K. Note that triangles AEH, AEM and BEM are congruent to one another. Hence $\angle BEM = \angle MEA = \angle AEH = 60°$. It follows that $\angle MAE = \angle EAH = 30°$. Since triangles AFN and CFN are congruent to each other, $\angle FCN = 30°$, so that $\angle CKA = 90°$. Thus CF is indeed perpendicular to AB.



2. Clearly, we can have $n = 1$ by taking any prime number. We can also have $n = 2$ since each odd prime is the sum of two consecutive numbers. Suppose $p = a + (a+1) + \cdots + (a+k)$ for some prime number p and positive integers a and $k \geq 2$. Then $2p = (k+1)(2a+k)$. Each of $k+1$ and $2a+k$ is greater than 2. This is a contradiciton since p is a prime number. Hence $n = 1$ or 2.

3. (a) We describe the process in the following chart.

Action Taken	Amount in Bucket A	Bucket B	Bucket C
Initial State	3	20	0
Pour from B into C until C=A	3	17	3
Pour away C	3	17	0
Pour from B into C until C=A	3	14	3
Pour away C	3	14	0
Pour from B into C until C=A	3	11	3
Pour away C	3	11	0
Pour from B into C until C=A	3	8	3
Pour away C	3	8	0
Pour from B into C until C=A	3	5	3
Pour from A into C until C=B	1	5	5
Pour from B into A until A=C	5	1	5
Pour from C into A	10	1	0

[1]Courtesy of Andy Liu.

(b) If $n \equiv 0 \pmod 3$, the task is impossible, because the amount of liquid in any bucket at any time will be a multiple of 3, but our target 10 is not. Suppose $n \equiv 2 \pmod 3$. If $n = 2$ or 5, we do not have enough water. If $n = 8$, we can proceed as in (a) from the partition line in the chart. If $n \geq 11$, we can reduce the amount 3 litres at a time. Finally, suppose $n \equiv 1 \pmod 3$. If $n = 1$ or 4, we do not have enough water. If $n = 7$, we can simply pour everything from bucket B into bucket A. If $n \geq 10$, we can reduce the amount 3 litres at a time. In summary, the task is possible except for $n = 1, 2, 4, 5$ and $n \equiv 0 \pmod 3$.

4. Note that $b = a(10^n + 1)$ so that $\frac{b}{a^2} = \frac{10^n+1}{a}$. Suppose it is an integer d. Since a is an n-digit number, $1 < d < 11$. Since $10^n + 1$ is not divisible by 2, 3 or 5, the only possible value for d is 7. The example $a = 143$ and $b = 143143$ shows that we can indeed have $d = 7$.

5. There are 9×10^9 10-digit numbers. If two of them are non-neighbours, they cannot have the same digits in each of the first nine places. Thus the number of 10-digit numbers we can choose is no more than the number of 9-digit numbers, which is 9×10^8. On the other hand, for each 9-digit number, we can add a unique tenth digit so that the sum of all 10 digits is a multiple of 10. If two of the 10-digit numbers obtained this way differ in only one digit, not both digit sums can be multiples of 10. Hence no two are neighbours among these 9×10^8 10-digit numbers.

Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Spring 2004.**

1 [4] Segments AB, BC and CD of the broken line $ABCD$ are equal and are tangent to a circle with centre at the point O. Prove that the point of contact of this circle with BC, the point O and the intersection point of AC and BD are collinear.

2 [4] A positive integer $a > 1$ is given (in decimal notation). We copy it twice and obtain a number $b = \overline{aa}$ which happened to be a multiple of a^2. Find all possible values of $\dfrac{b}{a^2}$.

3 [4] Perimeter of a convex quadrilateral is 2004 and one of its diagonals is 1001. Can another diagonal be 1? 2? 1001?

4 [5] Arithmetical progression $a_1, a_2, a_3, a_4, \ldots$ contains a_1^2, a_2^2 and a_3^2 at some positions. Prove that all terms of this progression are integers.

5 [5] Two 10-digit integers are called neighbours if they differ in exactly one digit (for example, integers 1234567890 and 1234507890 are neighbours). Find the maximal number of elements in the set of 10-digit integers with no two integers being neighbours.

International Mathematics
TOURNAMENT OF THE TOWNS

Solution to Senior O-Level Spring 2004[1]

1. Let O be the centre of the circle, K be the point of tangency with BC and H be the point of intersection of AC and BD. Since $AB = BC$, AC is perpendicular to OB by symmetry. Similarly, BD is perpendicular to OC. Since AC intersects BD at H, H is the orthocentre of triangle OBC. Now the radius OK is perpendicular to the tangent BC. Hence the third altitude OK of triangle OBC passes through H.

2. Note that $b = a(10^n + 1)$ so that $\frac{b}{a^2} = \frac{10^n+1}{a}$. Suppose it is an integer d. Since a is an n-digit number, $1 < d < 11$. Since $10^n + 1$ is not divisible by 2, 3 or 5, the only possible value for d is 7. The example $a = 143$ and $b = 143143$ shows that we can indeed have $d = 7$.

3. Let the quadrilateral be $ABCD$ with $AC = 1001$ and $BD = n$. Note that $1002^2 - 1001^2 = 2003$ lies between 44^2 and 45^2. For $45 \le n \le 1001$, let M be the common midpoint of AC and BD. Initially, let B lie on AM, so that the degenerate quadrilateral $ABCD$ has perimeter 2002. Now rotate BD about M. When BD is perpendicular to AC, the perimeter of $ABCD$ will exceed 2004. Hence at some point during the rotation, the perimeter of $ABCD$ is exactly 2004. It follows that all values of n between 45 and 1001 inclusive are possible. For $2 \le n \le 44$, start with the rhombus $ABCD$ whose perimeter is less than 2004. Translate BD in the direction AC. When C is the midpoint of BD, both AB and AD are longer than 1001, so that the degenerate quadrilateral $ABCD$ has perimeter exceeding $2002 + n \ge 2004$. Hence at some point during the translation, the perimeter of $ABCD$ is exactly 2004. It follows that all values of n between 2 and 44 inclusive are possible. Finally, consider the case $n = 1$. Let M be the point of intersection of AC and BD. Then

$$\begin{aligned} 2004 &= AB + BC + CD + DA \\ &< MA + MB + MB + MC + MC + MD + MD + MA \\ &= 2(AC + BD) \\ &= 2004, \end{aligned}$$

which is a contradiction. It follows that we cannot have $n = 1$.

4. Let the first three terms be $a_1 = a$, $a_2 = a + d$ and $a_3 = a + 2d$, where d is the common difference. Let $a_1^2 = a + kd$, $a_2^2 = a + md$ and $a_3^2 = a + nd$ for some positive integers k, m and n. Then $a^2 = a + kd$, $a^2 + 2ad + d^2 = a + md$ and $a^2 + 4ad + 4d^2 = a + nd$. It follows that $2ad + d^2 = nd - kd$ or $2a + d = m - k$, and $4ad + 4d^2 = nd - kd$ or $4a + 4d = n - k$. Eliminating d, we have $a = \frac{4m-n-3k}{4}$. Hence a is an integral multiple of $\frac{1}{4}$. Eliminating a, we have $d = \frac{n+k-2m}{2}$. Hence d is an integral multiple of $\frac{1}{2}$. Denote by $\{x\}$ the fractional part of x. We consider the following cases.
(1) Let $\{a\} = 0$ and $\{d\} = \frac{1}{2}$. Every term of the progression is an integral multiple of $\frac{1}{2}$ but a_2^2 is not, a contradiction.

[1]Courtesy of Andy Liu.

(2) Let $\{a\} = \frac{1}{2}$. Every term of the progression is an integral multiple of $\frac{1}{2}$ but a_1^2 is not, a contradiction.

(3) Let $\{a\} = \frac{1}{4}$ or $\frac{3}{4}$. Every term of the progression is an integral multiple of $\frac{1}{4}$ but a_1^2 is not, a contradiction.

Thus both a and d are integers, so that every term of the progression is an integer.

5. There are 9×10^9 10-digit numbers. If two of them are non-neighbours, they cannot have the same digits in each of the first nine places. Thus the number of 10-digit numbers we can choose is no more than the number of 9-digit numbers, which is 9×10^8. On the other hand, for each 9-digit number, we can add a unique tenth digit so that the sum of all 10 digits is a multiple of 10. If two of the 10-digit numbers obtained this way differ in only one digit, not both digit sums can be multiples of 10. Hence no two are neighbours among these 9×10^8 10-digit numbers.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2004.**

1. The sum of all terms of a finite arithmetical progression of integers is a power of two. Prove that the number of terms is also a power of two.

2. What is the maximal number of checkers that can be placed on an 8×8 checkerboard so that each checker stands on the middle one of three squares in a row diagonally, with exactly one of the other two squares occupied by another checker?

3. Each day, the price of the shares of the corporation "Soap Bubble, Limited" either increases or decreases by n percent, where n is an integer such that $0 < n < 100$. The price is calculated with unlimited precision. Does there exist an n for which the price can take the same value twice?

4. Two circles intersect in points A and B. Their common tangent nearer B touches the circles at points E and F, and intersects the extension of AB at the point M. The point K is chosen on the extention of AM so that $KM = MA$. The line KE intersects the circle containing E again at the point C. The line KF intersects the circle containing F again at the point D. Prove that the points A, C and D are collinear.

5. All sides of a polygonal billiard table are in one of two perpendicular directions. A tiny billiard ball rolls out of the vertex A of an inner $90°$ angle and moves inside the billiard table, bouncing off its sides according to the law "angle of reflection equals angle of incidence". If the ball passes a vertex, it will drop in and srays there. Prove that the ball will never return to A.

6. At the beginning of a two-player game, the number 2004! is written on the blackboard. The players move alternately. In each move, a positive integer smaller than the number on the blackboard and divisible by at most 20 different prime numbers is chosen. This is subtracted from the number on the blackboard, which is erased and replaced by the difference. The winner is the player who obtains 0. Does the player who goes first or the one who goes second have a guaranteed win, and how should that be achieved?

Note: The problems are worth 4, 5, 5, 6, 6 and 7 points respectively.

Solution to Junior A-Level Spring 2004

1. Let the first term of the arithmetic progression be a, the common difference be d and the number of terms be n. Then the sum of all terms is equal to $\frac{1}{2}n(a+a+(n-1)d) = 2^k$ for some positive integer k. Hence $n(2a+(n-1)d) = 2^{k+1}$, and each factor on the left side must be a power of 2.

2. Clearly, no checkers can be placed on any of the 28 outside squares. Moreover, at least one of each set of three squares with the same label must be left vacant. Thus the number of checkers that can be placed is at most 32. If we leave vacant the 28 outside squares and the 4 central squares, it is easy to verify that all conditions are satisfied. Thus the maximum is 32.

			2	1			
		3	1	2	4		
		1	3	4	2		
			4	3			

3. Suppose the price is repeated after a incrreases and b decreases. Then

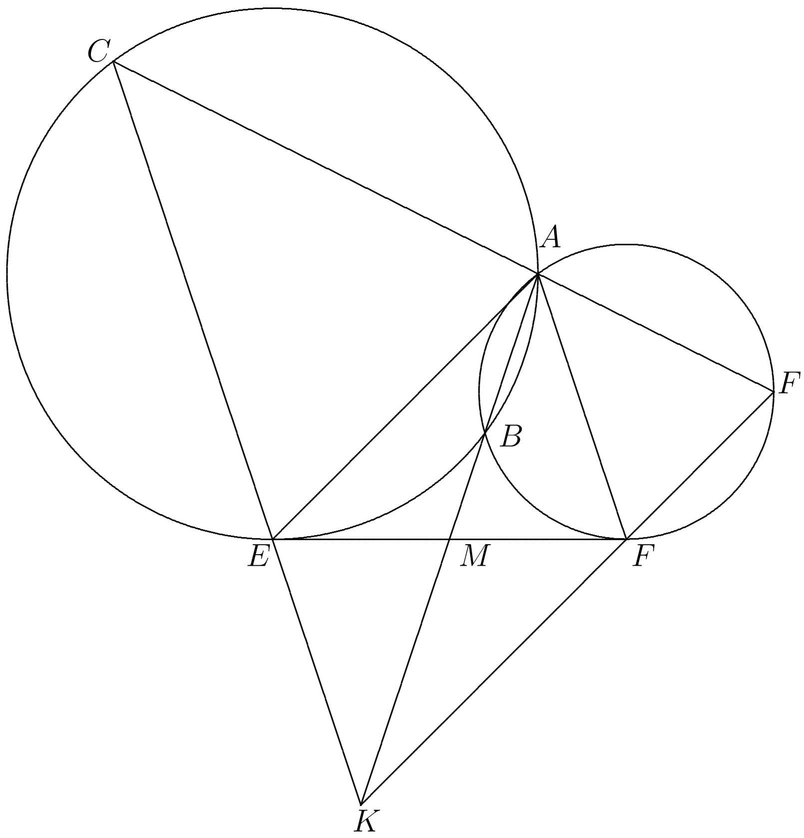
$$(100+n)^a(100-n)^b = 100^{a+b}.$$

Now the right side has only 2 and 5 as its prime divisors, so that the same must be true for the left side. The only power of 2 between 101 and 199 is 128. The only number in this range that is 5 times a power of 2 is 160. There are no numbers in this range that are 25 times a power of 2. The only number in this range that is 125 times a power of 2 is 125 itself. Hence $n = 25$, 28 or 60. Then $100-n = 75$, 72 or 40, so that $n = 60$. However, we cannot have $160^a 40^b = 100^{a+b}$ because the prime factorization of the left side has more 2s than 5s, while that of the right side has an equal number of 2s and 5s.

4. Since $ME^2 = MA \cdot MB = MF^2$, AK and EF bisect each other, so that $AEKF$ is a parallelogram. Moreover, since EF is tangent to the circles,

$$\angle KCA + \angle KDA + \angle CKD = \angle AEF + \angle AFE + \angle EAF = 180°.$$

It follows that C, A and D rae collinear.



5. Let the sides of the billiard table be parallel to the coordinate axes. If the ball starts along a side, it will fall into the vertex at the other end of the side. Otherwise, the absolute value of the slope of each segment of the ball's path is a positive constant. Hence the ball can only return to A by doubling back along the original path. Thus there must be a point of reversal where the path hits a side in a perpendicular direction. Since the sides are all horizontal or vertical while the path never is, the ball can never return to A.

6. Let P be the product of the first 21 primes. The player who goes second has a winning strategy, by always leaving a multiple of P for the opponent. This guarantees a win since both 2004! and 0 are multiples of P. We first show that the opponent cannot turn the table around. Suppose the first player is left with a multiple n of P. Suppose m is subtracted, leaving a difference d. The only way for d to be a multiple of P is for m to be one also, but this is impossible since m is divisible by at most 20 distinct primes. So the first player must leave behind a difference d which is not a multiple of P. Divide d by P and let the remainder be $r < P$. Since P is the smallest number that is divisible by at least 21 distinct primes, r is divisible by at most 20 distinct primes. Hence the second player can subtract r from d and leave behind another multiple of P.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2004.**

1. Each day, the price of the shares of the corporation "Soap Bubble, Limited" either increases or decreases by n percent, where n is an integer such that $0 < n < 100$. The price is calculated with unlimited precision. Does there exist an n for which the price can take the same value twice?

2. All angles of a polygonal billiard table have measures in integral numbers of degrees. A tiny billiard ball rolls out of the vertex A of an interior $1°$ angle and moves inside the billiard table, bouncing off its sides according to the law "angle of reflection equals angle of incidence". If the ball passes throigh a vertex, it will drop in and stays there. Prove that the ball will never return to A.

3. The perpendicular projection of a triangular pyramid on some plane has the largest possible area. Prove that this plane is parallel to either a face or two opposite edges of the pyramid.

4. At the beginning of a two-player game, the number 2004! is written on the blackboard. The players move alternately. In each move, a positive integer smaller than the number on the blackboard and divisible by at most 20 different prime numbers is chosen. This is subtracted from the number on the blackboard, which is erased and replaced by the difference. The winner is the player who obtains 0. Does the player who goes first or the one who goes second have a guaranteed win, and how should that be achieved?

5. The parabola $y = x^2$ intersects a circle at exactly two points A and B. If their tangents at A coincide, must their tangents at B also coincide?

6. The audience shuffles a deck of 36 cards, containing 9 cards in each of the suits spades, hearts, diamonds and clubs. A magician predicts the suit of the cards, one at a time, starting with the uppermost one in the face-down deck. The design on the back of each card is an arrow. An assistant examines the deck without changing the order of the cards, and points the arrow on the back each card either towards or away from the magician, according to some system agreed upon in advance with the magician. Is there such a system which enables the magician to guarantee the correct prediction of the suit of at least

   (a) 19 cards;

   (b) 20 cards?

Note: The problems are worth 4, 6, 6, 6, 7 and 3+5points respectively.

Solution to Senior A-Level Spring 2004

1. Suppose the price is repeated after a incrreases and b decreases. Then

$$(100+n)^a(100-n)^b = 100^{a+b}.$$

Now the right side has only 2 and 5 as its prime divisors, so that the same must be true for the left side. The only power of 2 between 101 and 199 is 128. The only number in this range that is 5 times a power of 2 is 160. There are no numbers in this range that are 25 times a power of 2. The only number in this range that is 125 times a power of 2 is 125 itself. Hence $n = 25$, 28 or 60. Then $100 - n = 75$, 72 or 40, so that $n = 60$. However, we cannot have $160^a 40^b = 100^{a+b}$ because the prime factorization of the left side has more 2s than 5s, while that of the right side has an equal number of 2s and 5s.

2. Let a side of the billiard table be parallel to the x-axes and let $\angle ZAB$ be the 1° interior angle. If the ball starts along AB or AZ, it will fall into B or Z respectively. Suppose the ball follows a path $A_0A_1 \ldots A_n$ where $A_0 = A = A_n$, with $\angle A_1AB$ and $\angle ZAA_{n-1}$ both strictly between 0° and 1°. Then we cannot have $A_1 = A_{n-1}$ since this can only happen if a segment of the path is parpendicular to a side of the billiard table. Since the former must make an angle of a non-integral number of degrees with the x-axis while the latter makes an angle of an integral number of degrees, the situation is impossible. Now each segment of the path divides the billiard table into two polygons. For the $k-th$ segment $A_{k-1}A_k$, let S_k denote the sum of the interior angles of the polygon to the right side of the directed segment. Consider now the sum $S = S_0 - S_1 + S_2 - \cdots + (-1)^n S_n$. Each term is an integral multiple of 180°. Hence S is an even number of degrees. All angles of incidence and angles of reflection cancel in this alternate sum. The interior angles of the billiard table appear in turn, starting with B and ending with Z, and possibly going around several times. Hence each appears the same number of times except for A, which appears one time less. Since we are only interested in the parity of the degrees in S, we can ignore the signs for angles with integral number of degrees. Hence $\angle A_1AB + (-1)^n \angle ZAA_{n-1} + 1^\circ$ is an even number of degrees. This is only possible if n is even and $A_1 = A_{n-1}$. However, this has already been ruled out. Thus the ball can never return to A.

3. If a polygon is orthogonally projected onto a plane, the projection has the largest area when the plane is parallel to the polygon. Let the tetrahedron be $ABCD$. Suppose its orthogonal projection to the plane Π has the largest area. Now the porjection is either a triangle or a convex quadrilateral. In the former case, Π is clearly parallel to a face of the tetrahedron. In the latter case, let the diagonals of the quadrilateral be projected from the sides AC and BD. Perform translations in the direction perpendicular to Π, taking AC into EG and BD into FH, where $EFGH$ is a convex quadrilateral. If Π is parallel to $EFGH$, then it is parallel to both AC and BD. If not, let Ψ be a plane parallel to both AC and BD. As before, perform translations in the direction perpendicular to Ψ, taking AC and BD into coplanar segments. If necessary, perform translations in this plane so that the seqments are the diagonals of a convex quadrilateral $KLMN$. Now the area of the orthogonal projection of $ABCD$ onto Ψ is at least that of $KLMN$, which is equal to the area of $EFGH$, which is in turn greater than that of the orthogonal projection of $ABCD$ onto Π. This is a contradiction.

4. Let P be the product of the first 21 primes. The player who goes second has a winning strategy, by always leaving a multiple of P for the opponent. This guarantees a win since both 2004! and 0 are multiples of P. We first show that the opponent cannot turn the table around. Suppose the first player is left with a multiple n of P. Suppose m is subtracted, leaving a difference d. The only way for d to be a multiple of P is for m to be one also, but this is impossible since m is divisible by at most 20 distinct primes. So the first player must leave behind a difference d which is not a multiple of P. Divide d by P and let the remainder be $r < P$. Since P is the smallest number that is divisible by at least 21 distinct primes, r is divisible by at most 20 distinct primes. Hence the second player can subtract r from d and leave behind another multiple of P.

5. Let the circle be centred at (u, v) with radius e, so that its equation is $(x-u)^2+(y-v)^2 = r^2$. The given geometric information means that the equation $(x-u)^2+(x^2-v)^2 = r^2$ has exactly two real roots, one of which is repeated. Since at least three of the roots are real, all four must be real. If the other root is also repeated, we have tangency at B as well. We must investigate whether the quartic equation could have a triple root and a single root. We may take A to be the point (1,1). Let the x-coordinate of B be $b \neq 1$. Then

$$(x-u)^2 + (x^2-v)^2 - r*2 = (x-1)^3(x-b).$$

Comparing the coefficients of the cubic terms, we have $3+b = 0$ so that $b = -3$. Compaaring the coefficients of the linear terms, we have $-2u = 8$ so that $u = -4$. Comparing the coefficients of the quadratic terms, we have $1-2v = -6$ so that $v = \frac{7}{2}$. Finally, comparing the constant terms, we have $16 + \frac{49}{4} - r^2 = -3$, so that $r = \frac{5\sqrt{5}}{2}$. In summary, the circle $(x+4)^2+(y-\frac{7}{2})^2 = (\frac{5\sqrt{5}}{2})^2$ intersects the parabola $y = x^2$ only at $A(1,1)$ and $B(-3,9)$. with common tangents only at A.

6. **First Solution by Che-Yu Liu, Grade 10 student, Taiwan.**

   (a) For each two cards, the assistant can use the backs to give four different signals, corresponding to the suit of the second of the two cards. Thus the magician can "predict" with accuracy the suits of all even-numbered cards, up to card number 34. Now the magician knows the overall composition of the deck, If the last two cards are of the same suit, no further assistance is required. If they are not, the assistant can use the back of card number 35 to indicate whether it is of the higher ranking or lower ranking suit. It follows that the magician can guarantee at least 19 correct answers.

   (b) The assistant examines the odd-numbered cards except card number 1. By the generalized Pigeomhole Principle, at least 5 of them are of the same suit. The assistant uses the backs of cards number 1 and 2 to signal this suit, and the magician always guesses this suit for all odd-numbered cards. The suits of even-numbered cards except card number 2 can be correctly as in (a). Thus the magician can guarantee at least 5+17=22 correct answers even with an arbitrary distribution among the suits.

Second Solution by Andy Lai, Grade 11 student, Taiwan.

(a) The assistant use the back of cards number 1 and 2 to signal the suit of card number 2 or 3, and the magician guesses that suit for both cards number 2 and 3. This is repeated until the magician makes his guesses on cards number 32 and 33. This guarantees 16 correct answers so far since the magician is right at least once on every two consecutive guesses of the same suit. The assistant then uses the backs of cards number 33 and 34 to signal the suit of card number 34. Now the magician knows the overall composition of the deck, If the last two cards are of the same suit, no further assistance is required. If they are not, the assistant can use the back of card number 35 to indicate whether it is of the higher ranking or lower ranking suit. It follows that the magician can guarantee at least 19 correct answers.

(b) Start off as in (a) until the magician guesses the suit of card number 21. At least 10 correct answers have been obtained. Now the assistant uses the backs of cards number 21 and 22 to signal the suit of card 22, and so on until the magician correctly guesses the suit of card number 34. Cards number 35 and 26 are handled as in (a). Thus far, we can again guarantee 19 corrct answers. If either cards number 2 and 3 or cards number 4 and 5 are of the same suit, we have an additional correct answer. Suppose this is not the case. Now the assistant can arrange for the magician to guess correctly card number 2 or 3, and card number 4 or 5. This flexibility is now used to signal the suit of card number 25. For instance, if card number 25 is Spades, then the correct guesses will occur on cards number 2 and 4. If it is Hearts, they occue on cards number 2 and 5; if Diamonds, 3 and 4; and if Clubs, 3 and 5. Similarly, an extra correct answer can come out of cards number 6 to 9 plus 27, of cards number 10 to 13 plus 29, of cards number 14 to 17 plus 31, and of cards number 18 to 21 plus 33. This guarantees at least 24 correct answers.