26. ТУРНИР ГРАДОВА

Јесење коло.

Припремна варијанта, 17. октобар 2004. год.

8–9 разред (млађи узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

1. (*3 поена*). Могу ли се цели бројеви од 1 до 2004 распоредити у неки низ, тако да збир ма којих 10 узастопних буде дељив са 10?

2. (*4 поена*). У кутији се налази 111 куглица црвене, плаве, зелене и беле боје. Ако се, не гледајући у кутију, извади из ње 100 куглица, онда ће међу њима обавезно бити 4 куглице различитих боја. Колико најмање куглица морамо извући, не гледајући у кутију, да би се међу њима сигурно нашле 3 куглице различитих боја?

3. (*4 поена*). Имамо неколико градова, од којих су неки повезани аутобуским линијама (без међустаница). Из ма којег града може се стићи у ма који други (уз могуће преседање). Ивановић је купио по једну возну карту за сваку маршруту (тј. може проћи по њој само једном на било коју страну). Петровић је купио *n* возних карата за сваку маршруту. Ивановић и Петровић су пошли из града А. Ивановић је искористио све своје карте, нове карте није куповао и стигао је у град В. Петровић је неко време путовао са купљеним картама, нашао се у граду Х и не може из њега поћи док не купи нову карту. Доказати да је Х или град А, или град В.

4. (*5 поена*). Дати су кружница и права која је не сече. Како помоћу шестара и лењира конструисати квадрат чија ће два суседна темена бити на кружници, а друга два темена - на датој правој (ако се зна да такав квадрат постоји)?

5. (*5 поена*). Колико има различитих начина да се број 2004 разложи на целе позитивне сабирке, који су приближно једнаки? Сабирака може бити један или неколико. Бројеве називамо приближно једнаким ако њихова разлика није већа од 1. Начини, који се разликују само по *редоследу сабирака, сматрају се једнаким.*

26. ТУРНИР ГРАДОВА

Јесење коло

Припремна варијанта, 17. октобар 2004. год.

10–11 разред (старији узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

1. (*3 поена*). Три кружнице пролазе кроз тачку X. Тачке A, B, C - су тачке њиховог пресека, различите од X. Нека је A' друга пресечна тачка праве AX и кружнице описане око троугла BCX. Тачке B' и C' одређене су аналогно. Докажите да су троуглови ABC', AB'C и A'BC слични.

2. (*3 поена*). У кутији се налази 100 куглица беле, плаве и црвене боје. Ако се, не гледајући у кутију, извади из ње 26 куглица, онда ће међу њима обавезно бити 10 куглица исте боје. Колико најмање куглица морамо извући, не гледајући у кутију, да би се међу њима сигурно нашло 30 куглица исте боје?

3. (*4 поена*). Дата су два полинома позитивног степена, $P(x)$ и $Q(x)$, при чему важе идентитети $P(P(x))=Q(Q(x))$ и $P(P(P(x)))=Q(Q(Q(x)))$. Да ли тада обавезно важи идентитет $P(x)=Q(x)$?

4. (*4 поена*). Колико има различитих начина да се број 2004 разложи на целе позитивне сабирке, који су приближно једнаки? Сабирака може бити један или неколико. Бројеве називамо приближно једнаким ако њихова разлика није већа од 1. Начини, који се разликују само по редоследу сабирака, сматрају се једнаким.

5. (5 поена). За које N се бројеви од 1 до N могу поређати у другачијем редоследу, тако да аритметичка средина ма које групе од два или више узастопних бројева не буде цео број?

26. ТУРНИР ГРАДОВА

Јесење коло.

Основна варијанта, 24. октобар 2004. год.

8–9 разред (млађи узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

1. (*4 поена*). Троугао ћемо звати рационалним ако су мерни бројеви свих његових углова рационални бројеви. Тачку у троуглу зваћемо рационалном ако, спојивши је са теменима, добијемо три рационална троугла. Докажите да у сваком оштроуглом рационалном троуглу постоје бар три рационалне тачке.

2. (*5 поена*). Кружница, уписана у троугао ABC, додирује странице BC, CA и AB редом у тачкама A', B' и C'. Знамо да је AA'=BB'=CC'. Да ли је тада обавезно троугао ABC једнакостраничан?

3. (*6 поена*). *Колико се највише коња може поставити на шаховску таблу 8x8*, тако да сваки туче не више од 7 осталих.

4. (*6 поена*). Васа је замислио два позитивна броја x и y. Записао је бројеве $x+y$, $x-y$, xy и x/y и показао Петру, али му није рекао, који број је у којој операцији добијен. Докажите да Петар може једнозначно да одреди x и y.

5. (*7 поена*). У троуглу ABC на страници BC означена је тачка K. У троуглове ABK и ACK уписане су кружнице, при чему прва додирује страницу BC у тачки M, а друга у тачки N. Докажите да је тада $BM \cdot CN > KM \cdot KN$.

6. (*8 поена*). Двојица деле комад (парче) сира. На почетку први дели сир на два комада, затим други сваки од тих комада дели на два дела и тако даље, док се не добије 5 комада. Затим први за себе узима један комад, онда други узима један од преосталих комада, затим опет први и тако све док има сира. За сваког играча утврдите колико највише сира он може осигурати за себе, без обзира како игра његов противник.

7. (*8 поена*). Нека су A и B два правоугаоника. Од правоугаоника који су подударни са A, састављен је правоугаоник сличан са B. Доказати да се од правоугаоника подударних са B може саставити правоугаоник сличан са A.

26. ТУРНИР ГРАДОВА

Јесење коло

Основна варијанта, 24. октобар 2004. год.

10–11 разред (старији узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

1. (*5 йоена*). Функције f и g дефинисане су на целој бројевној правој и узајамно су инверзне, тј. $g(f(x))=f(g(x))=y$ за свако x и y. Познато је да се f може приказати у облику збира линеарне и периодичне функције (тј. $f(x)=kx+h(x)$, где је k број, а h - периодична функција). Докажите да се g такође може представити у таквом облику. (Функција h назива се периодичном ако се може наћи такав број $d\neq0$, тако да је $h(x+d)=h(x)$ за сваки број x).

2. (*5 йоена*). Двојица играју следећу игру. Имамо гомилу каменчића. Први играч, кад је на потезу, узима 1 или 10 каменчића. Други играч, при сваком свом потезу, узима m или n каменчића. Предмете узимају наизменично, а почиње први. Губи онај који не може да начини потез (да узме каменчић). Зна се да при ма којој почетној количини каменчића, први играч увек може да игра тако да победи. Какви могу бити бројеви m и n?

3. (*5 йоена*). Васа је замислио два позитивна броја x и y. Затим је записао бројеве $x+y$, $x-y$, xy и x/y и показао Петру, али му није рекао који број је у којој операцији добијен. Докажите да Петар може једнозначно да одреди x и y.

4. (*6 йоена*). Кружница с центром I лежи у кружници са центром O. Нађите геометријско место центара кружница описаних око троуглова IAB, где је AB тетива веће кружнице која додирује мању кружницу.

5. (*7 йоена*). Нека су A и B два правоугаоника. Од правоугаоника који су подударни са A, састављен је правоугаоник сличан са B. Доказати да се од правоугаоника подударних са B може саставити правоугаоник сличан са A.

6. (*8 йоена*). Дат је цео број n, који није дељив ни са 2 ни са 3. Троугао ћемо звати разрешивим ако мера сваког његовог угла има облик $\frac{m\cdot180^\circ}{n}$ (m помножити са 180 степени, па поделити са n), где је m - цео број. Једнаким ћемо сматрати разрешиве троугле са једнаким угловим (тј. сличне троуглове). У почетку је дат један разрешиви троугао. Сваког минута један од постојећих троуглова разрезује се на два разрешива тако да су после резања сви добијени троуглови неједнаки (тј. нису слични). После неког времена показало се да ниједан од троуглова није могуће тако разрезати. Докажите да су до тог момента међу добијаним деловима постојали сви могући разрешиви троуглови.

7. (*8 йоена*). Углови AOB и COD доводе се до поклапања ротацијом. У њих су уписане кружнице које се секу у тачкама E и F. Докажите да су углови AOE и DOF једнаки.

26. ТУРНИР ГРАДОВА

Пролећно коло.

Припремна варијанта. 20. фебруар 2005. год.

8—9. разред (млађи узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

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1. (3 поена) Истовремено из села А и Б су кренули Ана и Бора у сусрет једно другоме (њихове брзине су константне, али не и нужно међусобно једнаке). Да је Ана кренула 30 минута раније, они би се срели 2 km ближе селу Б. Да је Бора кренуо 30 минута раније, сусрет би се десио ближе селу А. За колико ближе?

2. (4 поена) Нека је N било који природан број. Доказати да се у десетичном запису било броја N било броја 3N, налази једна од цифара 1, 2, 9.

3. (5 поена) У првом реду шаховске табле се налази 8 истоветних црних дама, а у последњем реду 8 истоветних белих дама. За који минималан број потеза беле даме могу заменити места са црним дамама? Бели и црни потезе вуку наизменично, по једна дама за један потез. Дама се креће по хоризонтали, по вертикали или по дијагонали за ма који број поља, ако се на путу не налазе друге даме.

4. (5 поена) Дат је квадрат ABCD, а M и N су средишта страница BC и AD респективно. На продужетку дијагонале AC, преко тачке A, одабрана је тачка K. Дуж KM сече страницу AB у тачки L. Докажите да су углови KNA и LNA једнаки.

5. (5 поена) У неком граду све улице иду или правцем север - југ или правцем исток-запад. Возач аутомобила се провозао тим градом, направивши тачно сто скретања на лево. Колико скретања на десно је при томе могао направити, ако ни једно место није прошао два пута и на крају се вратио на место поласка? (Све улице су двосмерне.)

26. ТУРНИР ГРАДОВА

Пролећно коло.
Припремна варијанта, 20. фебруар 2005. год.
10—11 разред (старији узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

1. (3 поена) На координатној равни је нацртано четири графика функкције облика $y = x^2 + ax + b$, где су a и b бројевни коефицијенти. Знамо да постоје тачно 4 тачке пресека, при чему се у свакој секу тачно два графика. Доказати да је збир највеће и најмање од апсциса тачака пресека једнака збиру преостале две апсцисе.

2. (4 поена) Сви природни бројеви су записани један за другим без размака на бесконачној траци: 1234567891011121314... Затим су траку разрезали на делове од по седам цифара у сваком делу. Доказати да се ма који седмоцифрени број

   а) налази бар на једном делу (3 поена);

   б) налази на бесконачно много делова (1 поен).

3. (4 поена) Дат је квадрат ABCD, а M и N су средишта страница BC и AD респективно. На продужетку дијагонале AC иза тачке A одабрана је тачка K. Дуж KM сече страницу AB у тачки L. Докажите да су углови KNA и LNA једнаки.

4. (4 поена) У неком граду све улице иду или правцем север - југ или правцем исток-запад. Аутомобилиста се провозао тим градом, направивши тачно сто скретања на лево. Колико је при том могао направити скретања на десно, ако ни једно место није прошао два пута и на крају се вратио на место поласка? (Све улице су двосмерне).

5. (5 поена) Збир неколико позитивних бројева једнак је 10, а збир њихових квадрата је већи од 20. Доказати да је збир кубова тих бројева већи од 40.

26. ТУРНИР ГРАДОВА

Пролећно коло.

Основна варијанта, 27. фебруар 2005. год.

8–9. разред (млађи узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена, поени за делове једног задатка се сабирају)

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1. (4 поена) На графику квадратног тринома са целобројним коефицијентима уочене су две тачке са целобројним координатама. Докажите да, ако је растојање међу њима изражено целим бројем, онда је дуж која их спаја паралелна са апсцисном осом.

2. (5 поена) Висине AA' и BB' троугла ABC секу се у тачки H. Тачке X и Y су редом средишта дужи AB и CH. Докажите да су праве XY и A'B' међусобно нормалне.

3. На бројчанику исправног сата барона Минхаузена постоје само велика казаљка, мала казаљка и секундара, а све цифре и подеоци су избрисани. Барон тврди да он по том сату може да одређује тачно време, јер, према његовом посматрању, на њему се током дана (од 8.00 до 19.59) не понавља два пута исти распоред казаљки. Да ли је тврђење барона тачно? (Казаљке имају различиту дужину, а крећу се равномерно).

4. Папирни правоугаоник са квадратном мрежом, величине 10×12, неколико пута је пресавијан по линијаама мреже тако да је добијен квадратић 1×1. Колико се парчића може добити пошто се тај квадратић расече по дужи која спаја

а) средишта две његове наспрамне ивице (2 поена);

б) средишта две његове суседне ивице (4 поена)?

(Нађите сва решења и докажите да других нема).

5. (6 поена) Конструктор се састоји из гарнитуре (комплета) правоуглих паралелепипеда. Сви се они могу сместити (сложити) у једну кутију такође облика правоуглог паралелепипеда. У шкартираном (дефектном) комплету код сваког паралелепипеда једна од ивица била је мања од стандардне. Може ли се тврдити да се код кутије, у коју се ставља комплет, такође може смањити једна од ивица? (Паралелепипеди се слажу у кутију тако да су њихове ивице паралелне ивицама кутије).

6. (6 поена) Тома и Сима деле гомилу од 25 новчића с вредностима 1, 2, 3, ... 25 алтина. При сваком потезу један од њих бира новчић из гомиле, а други говори коме да се да тај новчић. Први бира Тома, а потом онај који има тренутно више алтина, а ако имају једнако, онда онај који је бирао прошли пут. Може ли Тома поступати тако да на крају има више алтина од Симе, или, пак, Сима може увек у томе спречити Тому и на крају имати више алтина од Томе?

7. (8 поена) Поља шаховске табле 8×8 нумерисана су по дијагоналама, које иду с лева наниже, почевши од горњег левог угла: 1; следећа дијагонала 2, 3; следећа 4, 5, 6; и тако даље (претпоследња дијагонала 62, 63; последња 64). Пера је поставио на ту таблу 8 жетона тако да је у сваком реду и сваком ступцу био по један жетон. Затим је он преместио жетоне тако да је сваки жетон дошао на поље са већим бројем. Може ли после тога у сваком реду и сваком ступцу да се нађе по један жетон?

26. ТУРНИР ГРАДОВА

Пролећно коло.

Основна варијанта, 27. фебруар 2005. год.

10—11 разред (старији узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена поени по деловима једног задатка се сабирају)

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1. (4 поена) На графику полинома са целобројним коефицијентима уочене су две тачке са целобројним координатама. Докажите да, ако је растојање међу њима изражено целим бројем, онда је дуж која их спаја паралелна са апсцисном осом.

2. (5 поена) Кружница k_1 пролази кроз центар кружнице k_2. Кроз тачку С на кружници k_1 повучене су тангенте на k_2 и оне секу кружницу k_1 у тачкама А и В. Докажите да је дуж АВ нормална на праву која пролази кроз центре кружница.

3. (5 поена) Миша и Влада деле гомилу од 25 новчића с вредностима 1, 2, 3, ... 25 алтина. При сваком потезу један од њих бира новчић из гомиле, а други говори коме да се да тај новчић. Први бира Миша, а потом онај који има тренутно више алтина; ако имају једнако - онај који је бирао прошли пут. Може ли Миша играти тако да на крају има више алтина од Владе, или, пак, Влада може играти тако да он на крају има више алтина од Мише?

4. (6 поена) Постоји ли квадратни трином $f(x)$, такав да за сваки цео позитиван број n једначина $f(f(...f(x)))) = 0$, где је n слова f, има тачно 2^n различитих реалних решења?

5. (7 поена) Икосаедар и додекаедар уписани су у једну исту сферу. Докажите да су они тада и описани око једне исте сфере. (Напомена: Икосаедар има 20 једнаких страна у виду правилних троуглова, из сваког темена полази по 5 ивица и углови које заклапају суседне стране су једнаки. Додекаедар се састоји из 12 једнаких страна - правилних петоуглова, из сваког темена полазе по 3 ивице и углови међу суседним странама су једнаки.)

6. (7 поена) Нека је a угаоно поље шаховске табле 8×8, b њему суседно поље на дијагонали. Докажите да је број начина да "хроми топ" обиђе целу таблу, полазећи са поља a, већи од броја начина да "хроми топ" обиђе целу таблу, полазећи са поља b. ("Хроми топ" се креће по табли по једно поље водоравно или усправно и мора да стане на свако поље табле тачно једном.)

7. У простору је дато 200 тачака. Сваке две од њих спаја дуж, при чему добијене дужи немају пресечних тачака. Свака дуж обојена је једном од К боја. Пеђа хоће сваку од датих тачака да обоји једном од тих К боја, али тако да се не могу наћи две тачке и дуж која је њима одређена исте боје. Може ли Пеђа то увек да уради, ако је:

а) К = 7 (4 поена);

б) К = 10 (4 поена)?

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper[1] **Fall 2004.**

1. Is it possible to arrange the numbers from 1 to 2004 inclusive in some order such that the sum of any ten adjacent numbers is divisible by 10?

2. A bag contains 111 balls, each of which is green, red, white or blue. If 100 balls are drawn at random, there will always be 4 balls of different colours among them. What is the smallest number of balls that must be drawn, at random, in order to guarantee that there will be 3 balls of different colours among them?

3. Various pairs of towns in Russia were linked by direct bus services with no intermediate stops. Alexei Frugal bought one ticket for each route, which allowed travel in either direction but not returning on the same route. He started from Moscow, used up all his tickets without buying any new ones, and finished at Kaliningrad. Boris Lavish bought n tickets for each route, and started from Moscow. However, after using some of his tickets, he got stuck in some town which he could not leave without buying a new ticket. Prove that he got stuck in either Moscow or Kaliningrad.

4. Given a line and a circle which do not intersect, use straight edge and compass to construct a square with two adjacent vertices on the line and the other two on the circle, assuming that such a square exists.

5. In how many ways can 2004 be expressed as the sum of one or more positive integers in non-decreasing order, such that the difference between the last term and the first term is at most 1?

Note: The problems are worth 3, 4, 4, 5 and 5 points respectively.

[1]Courtesy of Andy Liu.

Solution to Junior O-Level Fall 2004

1. We may replace each number by its units digit since this has no effect on divisibility by 10. Then we have 200 of each of 5, 6, 7, 8, 9 and 0, but 201 of each of 1, 2, 3 and 4. Suppose the desired arrangement is possible. Then the 11-th digit must be identical to the 1-st one, the 12-th to the 2-nd, and so on, forming a sequence of period 10. However, $0+1+2+\cdots+9=45$ is not divisible by 10. Hence the period must contain some repeated digit. This digit would have to appear at least 400 times, but none appears more than 201 times. Thus the task is impossible.

2. We first show that 87 is not enough. We may have in the bag 75 green, 12 red, 12 white and 12 blue balls. The total number of balls of any three colours is at most 99. If 100 are drawn at random, there will be 4 balls of different colours. Hence the requirement is satisfied. Now if we draw only 87 balls, we may end up with 75 green and 12 white balls. We now show that 88 is enough. By symmetry, we may assume that the numbers of green, red, white and blue balls is non-increasing. We must have at least 12 blue balls as otherwise we may not have a blue one when we draw 100 balls. Hence there are at least 24 white and blue balls, meaning that the total number of balls of any two colours is at most $111-24=87$. The desired result follows immediately.

3. Consider any town other than Moscow and Kaliningrad. Suppose Alexei visited it k times. Then he came in using k tickets and went out using another k tickets. Hence the number of his tickets with this town on them was $2k$. The number of Boris' tickets with this town on them was $2kn$. They allowed Boris to enter and depart kn times, after which Boris could not come back and be stuck there.

4. Let the given line be horizontal and the circle be above it. Draw the vertical diameter of the circle, and extend it to cut the line at O. From O, draw two lines of slopes ± 2. Suppose they are tangent to the circle at Q and R respectively. Drop perpendiculars from Q and R onto the given line at P and S respectively. Then $PQRS$ is the desired square. If each of the two lines cut the circle at two points, take either the closer pair or the farther pair as Q and R and repeat as before. If the two lines miss the circle completely, the square will not exist, but this is given not to be the case.

5. Consider any k where $1 \le k \le 2004$. Use the Division Algorithm to determine the unique pair of integers (q, r) such that $2004 = kq + r$ with $0 \le r \le k-1$. Then r copies of $q+1$ and $k-r$ copies of q will add up to 2004. Thus there is one desired expression for each value of k, which is clearly unique. Hence there are 2004 such expressions in all.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper[1] **Fall 2004.**

1. Three circles all passing through X intersect one another again pairwise at A, B and C respectively. The extension of the common chord AX of two of the circles intersects the third circle again at D. Similarly, the extensions of BX and CX yield the points E and F respectively. Prove that triangles BCD, CAE and ABF are similar to one another.

2. A bag contains 100 balls, each of which is red, white or blue. If 26 balls are drawn at random, there will always be 10 balls of the same colour among them. What is the smallest number of balls that must be drawn, at random, in order to guarantee that there will be 30 balls of the same colour among them?

3. $P(x)$ and $Q(x)$ are non-constant polynomials such that for all x, $P(P(x)) = Q(Q(x))$ and $P(P(P(x))) = Q(Q(Q(x)))$. Is it necessarily true that $P(x) = Q(x)$ for all x?

4. In how many ways can 2004 be expressed as the sum of one or more positive integers in non-decreasing order, such that the difference between the last term and the first term is at most 1?

5. For which positive integers n is it possible to arrange the numbers from 1 to n in some order, such that the average of any group of two or more adjacent numbers is not an integer?

Note: The problems are worth 3, 3, 4, 4 and 5 points respectively.

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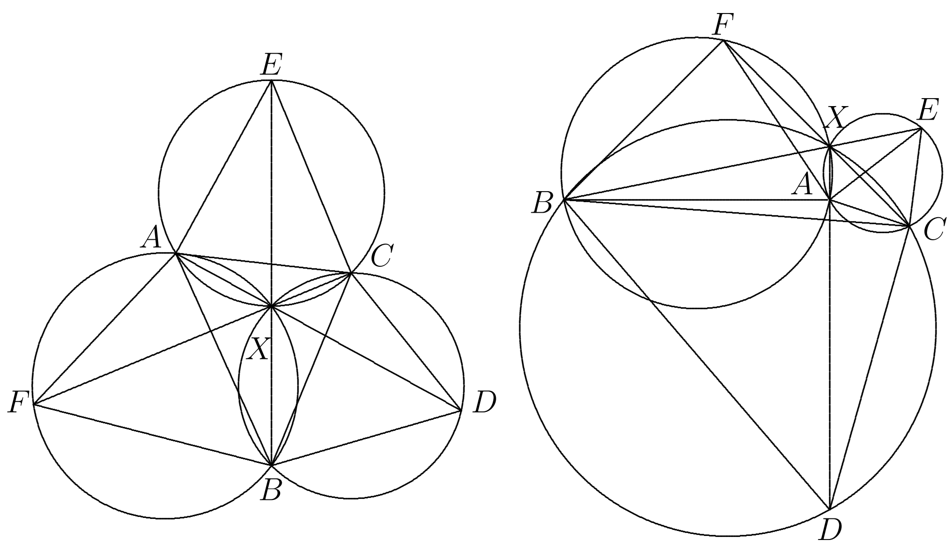
[1]Courtesy of Andy Liu.

Solution to Senior O-Level Fall 2004

1. We have $\angle EAC = \angle EXC = \angle FXB = \angle FAB$. Denote the common value by α. Similarly, we have $\angle FBA = \angle DXC = \angle DBC = \beta$ and $\angle DCB = \angle DXB = \angle ECA = \gamma$. Note that $\alpha + \beta + \gamma = \angle FXB + \angle DXC + \angle DXB = 180^\circ$. Hence $\angle BDC = 180^\circ - \beta - \gamma = \alpha$, $\angle CEA = 180^\circ - \gamma - \alpha = \beta$ and $\angle AFB = 180^\circ - \alpha - \beta = \gamma$. It follows that triangles DBC, AEC and ABF are indeed similar to one another.



Note that X may lie on the extension of one of the common chords, say DA. We have $\angle EAC = \angle EXC = \angle FXB = \angle FAB$. Denote the common value by α. Similarly, we have $\angle CEA = \angle DXC = \angle DBC = \beta$ and $\angle AFB = \angle DXB = \angle DCB = \gamma$. As before, we have $\alpha + \beta + \gamma = \angle FXB + \angle DXC + \angle DXB = 180^\circ$. Hence $\angle BDC = 180^\circ - \beta - \gamma = \alpha$, $\angle ECA = 180^\circ - \gamma - \alpha = \beta$ and $\angle FAB = 180^\circ - \alpha - \beta = \gamma$. It follows that triangles DBC, AEC and ABF are indeed similar to one another.

2. We first show that 65 is not enough. We may have in the bag 47 red, 7 white and 46 blue balls. If 26 are drawn at random, the number of red and blue balls is at least 19. By the Mean Value Principle, there are either at least 10 red balls or at least 10 blue balls, so that the requirement is satisfied. Now if we draw only 65 balls, we may end up with 29 red, 7 white and 29 blue balls. We now show that 66 is enough. We may assume that the number of white balls is not more than the number of red balls and not more than the number of blue balls. If there are at most 7 white balls, then among the 66 balls drawn, the number of red or blue balls is at least 59, so that the desired result follows from the Pigeonhole Principle. If there are at least 9 white balls, then we may draw 9 red, 8 white and 9 blue balls for a total of 26 balls without 10 of the same colour. Hence the number of white balls must be 8. Since we cannot have at least 9 red and at least 9 blue balls in the bag, we may assume that the there are exactly 8 blue balls. When we draw 66 balls, we will get at least 50 red balls.

3. Since $P(x)$ is a polynomial, so is $P(P(x))$, and it takes on infinitely many values. Let x be any of these values. Then $x = P(P(t))$ for some t. Hence $P(x) = P(P(P(t))) = Q(Q(Q(t))) = Q(x)$. Since $Q(x)$ is also a polynomial, and its agrees with $P(x)$ on infinitely many values, we must have $P(x) = Q(x)$ for all x.

4. Consider any k where $1 \leq k \leq 2004$. Use the Division Algorithm to determine the unique pair of integers (q, r) such that $2004 = kq + r$ with $0 \leq r \leq k - 1$. Then r copies of $q + 1$ and $k - r$ copies of q will add up to 2004. Thus there is one desired expression for each value of k, which is clearly unique. Hence there are 2004 such expressions in all.

5. The sum of n consecutive numbers is $\frac{n(2a+n-1)}{2}$ where a is the first of these numbers. Their average is $\frac{2a+n-1}{2}$, which is an integer if and only if n is odd. In our problem, n cannot be odd. We now show that n can be any even number. Arrange the n numbers in their natural order and group them into pairs. Reverse the order within each pair to yield the arrangement 2,1,4,3,6,5,$\ldots, n, n-1$. Consider any k where $2 \leq k \leq n$. Consider first the case where k is odd. Any k adjacent numbers in our arrangement consist of k consecutive integers except that the one which is not in a pair is replaced by its partner, which differs from it by 1. Thus the sum of these k numbers is $mk \pm 1$ for some m, so that their average is not an integer. Finally, consider the case where k is even. Any k adjacent numbers in our arrangment consist of k consecutive integers, possibly with the two at the ends not being in pairs and replaced by their partners. Since one would be increased by 1 while the other would be decreased by 1, the sum is not affected by the replacement. So the average is not an integer.

International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper[1] **Fall 2004.**

1. An angle is said to be rational if its measure in degrees is a rational number. A triangle is said to be rational if all its angles are rational. Prove that there exist at least three different points inside any acute rational triangle such that when each is connected to the three vertices of the original triangle, we obtain three rational triangles.

2. The incircle of triangle ABC touches the sides BC, CA and AB at D, E and F respectively. If $AD = BE = CF$, does it follow that ABC is equilateral?

3. What is the maximum number of knights that can be place on an 8×8 chessboard such that each attacks at most seven other knights?

4. On a blackboard are written four numbers. They are the values, in some order, of $x + y$, $x - y$, xy and $\frac{x}{y}$ where x and y are positive numbers. Prove that x and y are uniquely determined.

5. K is a point on the side BC of triangle ABC. The incircle of triangle BAK touches BC at M. The incircle of triangle CAK touches BC at N. Prove that $BM \cdot CN > KM \cdot KN$.

6. Two persons share a block of cheese as follows. They take turns cutting an existing block of cheese into two, until there are five blocks. Then they take turns choosing one block at a time. The person who makes the first cut also makes the first choice, and gets an extra block. Each wants to get as much cheese as possible. What is the optimal strategy for each, and how much is each guaranteed to get, regardless of the counter measures of the other?

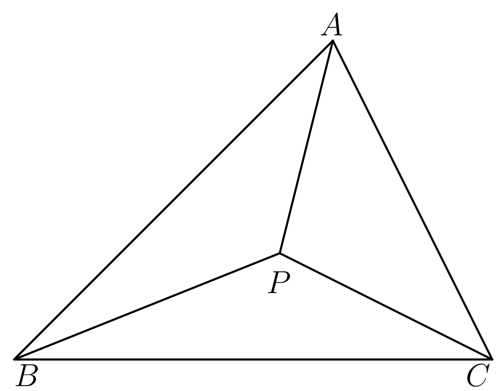
7. We have many copies of each of two rectangles. If a rectangle similar to the first can be made by putting together copies of the second, prove that a rectangle similar to the second can be made by putting together copies of the first, with no overlapping in both instances.

Note: The problems are worth 4, 5, 6, 6, 7, 8 and 8 points respectively.

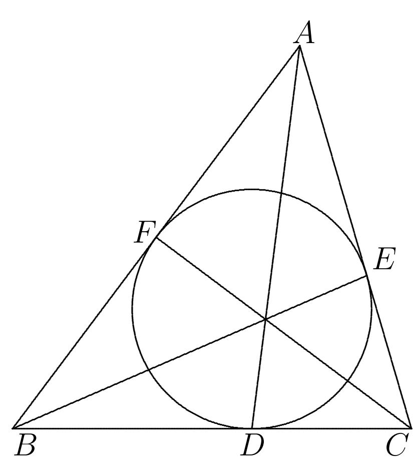
[1]Courtesy of Andy Liu.

Solution to Junior A-Level Fall 2004

1. We remark that if a triangle has two rational angles, then the third angle must also be rational, so that the triangle is rational. Let P be a point inside an acute rational triangle ABC, which is then divided into triangles PBC, PCA and PAB. In each of the following cases, it is sufficient to prove that PBC is a rational triangle, since we can prove that PCA and PAB are also rational in an analogous manner. Suppose P is the incentre of triangle ABC. Since PB bisects the $\angle ABC$ and CP bisects $\angle BCA$, both of which are rational, both $\angle PBC$ and $\angle PCB$ are rational. Hence triangle PBC is rational. Suppose P is the circumcentre of triangle ABC. Since ABC is acute, P is indeed inside it. Now $\angle CPB = 2\angle CAB$ is rational, so that $\angle PCB = \frac{1}{2}(180° - \angle CPB)$ is also rational. Hence triangle PBC is rational. Suppose P is the orthocentre of triangle ABC. Since ABC is acute, P is indeed inside it. Now $\angle PBC = 90° - \angle BCA$ and $\angle PCB = 90° - \angle ABC$ are both rational. Hence triangle PBC is rational. If ABC is not equilateral, then its incentre, circumcentre and orthocentre are disinct points. Thus we have the required three points. If ABC is equilateral, there exist infinitely many points P on the perpendicular bisector of BC such that $\angle PCB$ is rational. Any three such points will meet the requirement of the problem.



2. Assume that $BE = CF$ but $AB \neq AC$. In triangles ABE and ACF, $\angle BAE = \angle CAF$, $AE = AF$ and $BE = CF$. Since $AB \neq AC$, ABE and ACF are not congruent triangles. Hence $\angle ABE \neq \angle ACF$ but we do have $\angle ABE + \angle ACF = 180°$. Hence either $\angle ABE$ or $\angle ACF$ is obtuse, which means that either $AE > AB$ or $AF > AC$. Since $AE = AF$, either $AE > AC$ or $AF > AB$. This is a contradiction. It follows that $AB = AC$, and we can prove in a similar way that $AD = CF$ implies $BC = BA$, so that ABC is indeed equilateral if $AD = BE = CF$.



3. Let us start with a knight on each square of the 8×8 chessboard. If we remove the 4 knights in the central 2×2 subboard, we are left with 60 knights each of which attacks at most 7 others. We now show that 60 is indeed the maximum. Again, we start with a knight on each of the 64 squares. Note that a knight can attack 8 other knights only if it occupies one of the squares in the central 4×4 subboard. We put these 16 knights on a black list. In the following diagram, the number on each square shows the maximum number of knights on the black list that can attack that square. Note that all the numbers are 4 or less. Thus the removal of a knight can take at most 4 other knights off the black list. Even if the removed knight itself is on the black list, we can take at most 5 knights off. Hence removing at most 3 knights will not clear the black list.

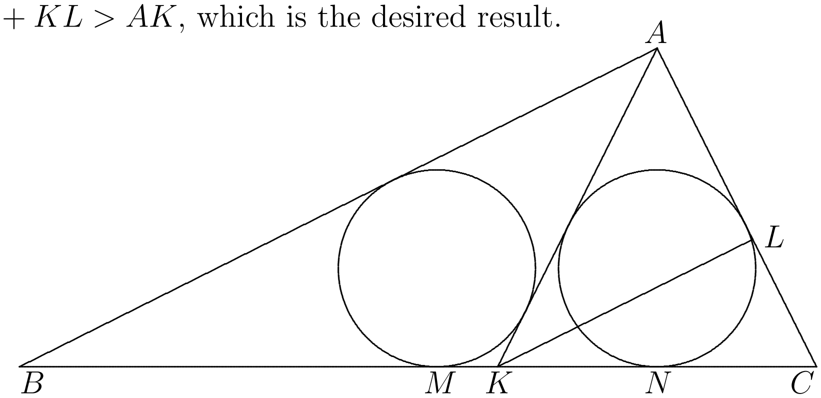
0	1	1	2	2	1	1	0
1	2	2	3	3	2	2	1
1	2	2	3	3	2	2	1
2	3	3	4	4	3	3	2
2	3	3	4	4	3	3	2
1	2	2	3	3	2	2	1
1	2	2	3	3	2	2	1
0	1	1	2	2	1	1	0

4. Note that $(x+y)+(x-y)=2x$ while $(xy)(\frac{x}{y})=x^2$, and that only $x-y$ can be non-positive. We consider three cases.
Case 1. All four numbers are positive.
Let a, b, c and d denote $x+y$, $x-y$, xy and $\frac{x}{y}$ in some order. Choose a pair of them and check if the square of their sum is four times the product of the other two numbers. The pair can be chosen in six ways. There are three subcases.
Subcase 1a. This is satisfied by two disjoint pairs.
We may assume that we have $(a+b)^2=4cd$ and $(c+d)^2=4ab$. Adding these two equations yields $(a-b)^2+(c-d)^2=0$ so that $a=b$ and $c=d$. Substituting back into $(a+b)^2=4cd$, we have $a=\pm c$. Since all four numbers are positive, we must have $a=b=c=d$. This is a contradiction since $x+y \neq x-y$.
Subcase 1b. This is satisfied by two intersecting pairs.
We may assume that we have $(a+b)^2=4cd$ and $(a+c)^2=4bd$ with $b \neq c$. Then we have $b(a+b)^2=4bcd=c(a+c)^2$, or equivalently $(b-c)(a^2+2a(b+c)+(b^2+bc+c^2))=0$. This is a contradiction since $b-c \neq 0$ while $a^2+2a(b+c)_((b^2+bc+c^2)>0$.
Subcase 1c. This is satisfied by only one pair.
We may assume that $(a+b)^2=4cd$. Then we know that the larger one of a and b is $x+y$ and the smaller one $x-y$. We can determine x and y uniquely.
Case 2. One of the numbers is 0. We know that $x=y$ so that $\frac{x}{y}=1$ must also be among the four numbers. The other two are $x+y=2x$ and $xy=x^2$. Since their product is $2x^3$, we can determine $x=y$ uniquely.

Case 3. One of the numbers is negative.
We know that $x < y$ and $\frac{x}{y} < 1$. Check how many numbers in $S = \{x+y, xy, \frac{x}{y}\}$ lie strictly between 0 and 1. There are three subcases.
Subcase 3a. There is exactly one such number.
We know that this number is $\frac{x}{y}$, and we can determine x and y uniquely from $x - y$ and $\frac{x}{y}$.
Subcase 3b. There are exactly two such numbers.
We cannot have $x+y < 1$. Otherwise, we must have $x < 1$ and $y < 1$ so that $xy < 1$, but then all three numbers in S lie strictly between 0 and 1. Hence $x + y > 1$ is the largest number in S, and we can determine x and y uniquely from $x - y$ and $x + y$.
Subcase 3c. There are exactly three such numbers.
From $x + y < 1$, we have $x < 1$ and $y < 1$ so that $xy < x + y$ and $xy < \frac{x}{y}$. Hence the smallest number in S is xy, and we can determine x and y uniquely from $x - y$ and xy.

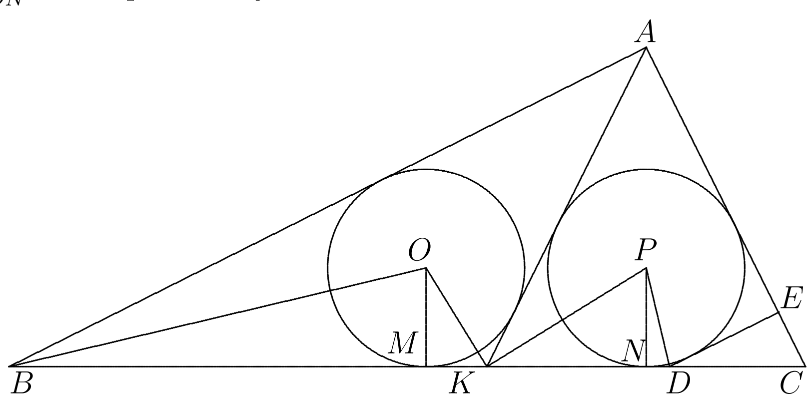
5. **First Solution:**

Note that $BM = \frac{AB+BK-AK}{2}$, $CN = \frac{AC+CK-AK}{2}$, $KM = \frac{AK+BK-AB}{2}$ and $KN = \frac{AK+CK-AC}{2}$. Hence $BM{\cdot}CN > KM{\cdot}KN$ is equivalent to $AC{\cdot}KB+AB{\cdot}KC > AK(KB+KC) = AK{\cdot}BC$, or $\frac{AC\cdot KB}{BC} + \frac{AB\cdot KC}{BC} > AK$. Let L be the point on AC such that KL is parallel to AB. Then triangles ABC and LKC are similar. Hence $AL = \frac{AC\cdot KB}{BC}$ and $KL = \frac{AB\cdot KC}{BC}$. By the Triangle Inequality, $AL + KL > AK$, which is the desired result.



Second Solution:

Construct the tangent to the incircle of triangle CAK parallel to AB and closer to C than to A, cutting BC at D and CA at E. Let O and P be the respective incentres of triangles BAK and CAK. Note that OK is perpendicular to PK since they bisect $\angle BKA$ and $\angle AKC$ respectively. Hence triangles MKO and NPK are similar, so that $\frac{OM}{KM} = \frac{KN}{PN}$. Since AB is parallel to DE, $\angle ABD + \angle BDE = 180°$. They are bisected respectively by OB and PD, which are thus perpendicular to each other. Hence triangle MOB is similar to triangle NDP, so that $\frac{BM}{OM} = \frac{PN}{DN}$. Multiplication yields $KM \cdot KN = BM \cdot DN < BM \cdot CN$.



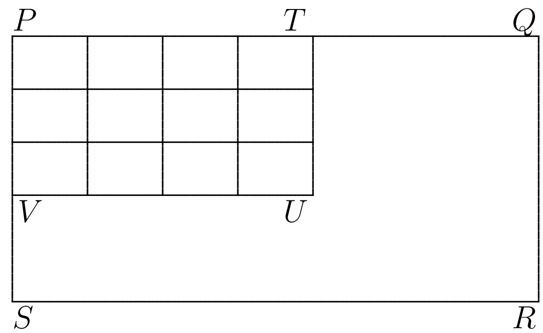
6. Let the total amount of cheese be 1, the first player be Alexei and the second player be Boris. Alexei can be assured of getting at least $\frac{3}{5}$ if he cuts 1 into $\frac{3}{5}$ and $\frac{2}{5}$. We consider two cases.
Case 1. Boris cuts $\frac{2}{5}$ into x and $\frac{2}{5}-x$, where $0 \le x \le \frac{1}{5}$.
Alexei cuts $\frac{3}{5}$ into x and $\frac{3}{5}-x$. Now the four pieces are of sizes $x = x \le \frac{2}{5}-x < \frac{3}{5}-x$. No matter how Boris makes his second cut, the second smallest piece is at most x, and the second largest piece is at most $\frac{2}{5}-x$ since $2(\frac{2}{5}-x) \ge \frac{3}{5}-x$. Hence Boris can get at most $x+(\frac{2}{5}-x)=\frac{2}{5}$.
Case 2. Boris cuts $\frac{3}{5}$ into x and $\frac{3}{5}-x$, where $0 \le x \le \frac{3}{10}$.
If $0 \le x \le \frac{1}{5}$, Alexei cuts $\frac{2}{5}$ into x and $\frac{2}{5}-x$, and this is the same as in Case 1. Hence we may assume that $\frac{1}{5} < x \le \frac{3}{10}$. Alexei cuts $\frac{3}{5}-x$ into $\frac{2}{5}-x$ and $\frac{1}{5}$. Now the four pieces are of sizes $\frac{2}{5}-x < \frac{1}{5} < x < \frac{2}{5}$. There are four subcases.
Subcase 2a. Boris cuts $\frac{2}{5}$ into y and $\frac{2}{5}-y$, where $0 \le y \le \frac{1}{5}$.
We have either $y \le \frac{2}{5}-x < \frac{1}{5} < x \le \frac{2}{5}-y$, in which case Boris gets $(\frac{2}{5}-x)+x=\frac{2}{5}$, or $\frac{2}{5}-x \le y \le \frac{1}{5} \le \frac{2}{5}-y \le x$, in which case Boris still gets $y+(\frac{2}{5}-y)=\frac{2}{5}$.
Subcase 2b. Boris cuts x.
If $\frac{1}{5}$ remains the third largest piece, Alexei gets at least $\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$. If it becomes the second largest piece, Boris gets at most $\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$.
Subcase 2c. Boris cuts $\frac{1}{5}$ into y and $\frac{1}{5}-y$, where $0 \le y \le \frac{1}{10}$.
Since $\frac{2}{5}-x \ge y$, the second smallest piece is at most $\frac{2}{5}-x$. Hence Boris gets at most $(\frac{2}{5}-x)+x=\frac{2}{5}$.
Subcase 2d. Boris cuts $\frac{2}{5}-x$.
Alexei gets at least $\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$.
We now show that Boris can be assured of getting $\frac{2}{5}$. We consider three cases.
Case 1. Alexei cuts 1 into $\frac{3}{5}-x$ and $\frac{2}{5}+x$, where $0 \le x \le \frac{1}{10}$.
Boris cuts $\frac{3}{5}-x$ into $\frac{2}{5}+x$ and $\frac{1}{5}-2x$. If Alexei cuts $\frac{1}{5}-2x$, Boris gets at least $\frac{2}{5}+x \ge \frac{2}{5}$. If Alexei cuts one $\frac{2}{5}+x$, Boris cuts the other $\frac{2}{5}+x$ in the same way and gets at least $\frac{2}{5}+x \ge \frac{2}{5}$.
Case 2. Alexei cuts 1 into $\frac{3}{5}+x$ and $\frac{2}{5}-x$, where $0 \le x \le \frac{1}{5}$.
Boris cuts $\frac{2}{5}-x$ into $\frac{1}{5}$ and $\frac{1}{5}-x$. If Alexei cuts either of these two pieces, Boris cuts $\frac{3}{5}-x$ into halves and gets at least $\frac{3}{10}+\frac{x}{2}+\frac{1}{2}(\frac{1}{5}-x)=\frac{2}{5}$. If Alexei cuts $\frac{3}{5}+x$ into y and $\frac{3}{5}+x-y$ where $0 \le y \le \frac{3}{10}+\frac{x}{2}$, Boris cuts the latter into $\frac{1}{5}+x$ and $\frac{2}{5}-y$. If $\frac{1}{5}-x \le y \le \frac{1}{5} \le \frac{2}{5}-y \le \frac{1}{5}+x$, Boris gets $y+(\frac{2}{5}-y)=\frac{2}{5}$. If $y \le \frac{1}{5}-x \le \frac{1}{5} \le \frac{1}{5}+x \le \frac{2}{5}-y$, Boris still gets $(\frac{1}{5}-x)+(\frac{1}{5}+x)=\frac{2}{5}$.
Case 3. Alexei cuts 1 into $\frac{4}{5}+x$ nd $\frac{1}{5}-x$, where $0 \le x \le \frac{1}{5}$.
Boris cuts $\frac{4}{5}+x$ into $\frac{3}{5}+x$ and $\frac{1}{5}$, and this is the same as Case 2.

7. Suppose we have an $a_1 \times a_2$ rectangle A and a $b_1 \times b_2$ rectangle B. Any rectangle $PQRS$ that can be constructed from copies of A has dimensions $(u_1a_1+u_2a_2) \times (v_1a_1+v_2a_2)$ for some non-negative integers $u_1,\ u_2, v_1$ and v_2. If $PQRS$ is similar to B, then

$$\frac{b_1}{b_2} = \frac{u_1a_1+u_2a_2}{v_1a_1+v_2a_2}.$$

We first consider the case where $\frac{a_1}{a_2}$ is rational, so that it is equal to $\frac{m_1}{m_2}$ for some positive integers m_1 and m_2. Then $\frac{b_1}{b_2} = \frac{u_1m_1+u_2m_2}{v_1m_1+v_2m_2} = \frac{n_1}{n_2}$ for some positive integers n_1 and n_2, so that it is also rational. Using n_1n_2 copies of B, we can construct a square of side $s = n_2b_1+n_1b_2$. Using m_1m_2 copies of this square, we can construct an $sm_1 \times sm_2$ rectangle which is similar to A.

We now consider the case where $\frac{a_1}{a_2}$ is irrational. We claim that in constructing the rectangle $PQRS$ with copies of A, all the copies must be in the same orientation. Let $PTUV$ be the largest subrectangle of $PQRS$ that can be constructed with copies of A all in the same orientation. Suppose U is in the interior of $PQRS$, as illustrated in the diagram below.



If the line TU can be extended without cutting in interior of a copy of A, then the space immediately below UV must be filled with copies of A in the same orientation as those above, as otherwise it contradicts the irrationality of $\frac{a_1}{a_2}$. However, now it contradicts the maximality of $PTUV$. Hence TU cannot be so extended, but this implies that VU can, and we have a contradiction as well. It follows that U must lie on QR or RS. We may assume by symmetry that it lies on QR, so that T coincides with Q. However, the space immediately below UV must be filled with copies of A in the same orientation as those above. This contradicts the maximality of $PTUV$ unless U coincides with R and V with S. Thus our claim is justified. Suppose this construction uses k_1k_2 copies of A in k_1 rows and k_2 columns for some positive integers k_1 and k_2. Then $\frac{k_1a_1}{k_2a_2} = \frac{b_1}{b_2}$ so that $\frac{k_2b_1}{k_1b_2} = \frac{a_1}{a_2}$. Hence we can construct a rectangle similar to A using k_1k_2 copies of B in k_2 rows and k_1 columns.

International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper[1] **Fall 2004.**

1. The functions f and g are such that $g(f(x)) = x$ and $f(g(y)) = y$ for any real numbers x and y. If for all real numbers x, $f(x) = kx + h(x)$ for some constant k and some periodic function $h(x)$, prove that $g(x)$ can similarly be expressed as a sum of a linear function and a periodic function. A function h is said to be periodic if for any real number x, $h(x + p) = h(x)$ for some fixed real number p.

2. Two players alternately remove pebbles from a pile. In each move, the first player must remove either 1 or 10 pebbles, while the second player must remove either m or n pebbles. Whoever cannot make a move loses. If the first player can guarantee a win regardless of the initial number of pebbles in the pile, determine m and n.

3. On a blackboard are written four numbers. They are the values, in some order, of $x + y$, $x - y$, xy and $\frac{x}{y}$ where x and y are positive numbers. Prove that x and y are uniquely determined.

4. A circle with centre I is inside another circle with centre O. AB is a variable chord of the larger circle which is tangent to the smaller circle. Determine the locus of the circumcentre of triangle IAB.

5. We have many copies of each of two rectangles. If a rectangle similar to the first can be made by putting together copies of the second, prove that a rectangle similar to the second can be made by putting together copies of the first, with no overlapping in both instances.

6. Let $n \geq 5$ be a fixed odd prime number. A triangle is said to be admissible if the measure of each of its angles is of the form $\frac{m}{n}180°$ for some positive integer m. Initially, there is one admissible triangle on the table. In each move, one may pick up a triangle from the table and cut it into two admissible ones, neither of which is similar to any other triangle on the table. The two new triangles are put back on the table. After a while, no more moves can be made. Prove that at that point, every admissible triangle is similar to some triangle on the table.

7. From a point O are four rays OA, OC, OB and OD in that order, such that $\angle AOB = \angle COD$. A circle tangent to OA and OB intersects a circle tangent to OC and OD at E and F. Prove that $\angle AOE = \angle DOF$.

Note: The problems are worth 5, 5, 5, 6, 7, 8 and 8 points respectively.

[1] Courtesy of Andy Liu.

Solution to Senior A-Level Fall 2004

1. Let $y = f(x) = kx + h(x)$. Then $y + kp = k(x + p) + h(x + p) = f(x + p)$. It follows that $g(y + kp) = x + p = g(y) + p$. Let $\ell(y) = g(y) - \frac{y}{k}$. Then

$$\ell(y + kp) = g(y + kp) - \frac{y + kp}{k} = g(y) + p - \frac{y}{k} - p = \ell(y).$$

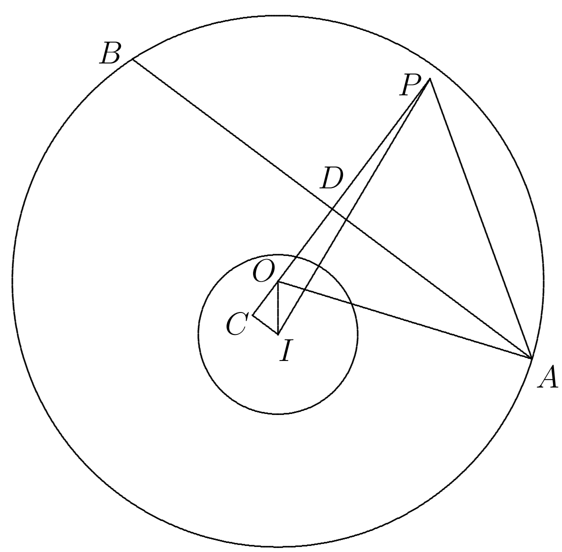
Hence $\ell(y)$ is a periodic function, and $g(y) = \frac{y}{k} + \ell(y)$.

2. Let the first player be Alexei, the second player be Boris, and the total number of pebbles be t. We may assume that $m \leq n$. Suppose $m \leq 8$. If $t = m + 1$, then Alexei can take only 1 pebble and Boris wins by taking the rest. Suppose $n = m + 9$. If $t = m + 10$, then whether Alexei takes 1 or 10 pebbles, Boris can still take the rest and wins. Suppose $m \geq 9$ and $n \neq m+9$. If $t \leq m$, then Alexei wins by taking 1 pebble, leaving Boris with no response. Suppose $t > m$. Alexei has two moves, one of which does not leave behind m pebbles and one of which does not leave behind n pebbles. Suppose taking 1 pebble leaves behind n and taking 10 pebbles leaves behind m. This would mean $n = m+9$, which is not the case. Hence Alexei has a move which leaves behind neither m nor n pebbles, so that the game continues. Since the game cannot continue forever, Boris must eventually lose.

3. Note that $(x+y)+(x-y) = 2x$ while $(xy)(\frac{x}{y}) = x^2$, and that only $x-y$ can be non-positive. We consider three cases.
Case 1. All four numbers are positive.
Let a, b, c and d denote $x + y$, $x - y$, xy and $\frac{x}{y}$ in some order. Choose a pair of them and check if the square of their sum is four times the product of the other two numbers. The pair can be chosen in six ways. There are three subcases.
Subcase 1a. This is satisfied by two disjoint pairs.
We may assume that we have $(a + b)^2 = 4cd$ and $(c + d)^2 = 4ab$. Adding these two equations yields $(a - b)^2 + (c - d)^2 = 0$ so that $a = b$ and $c = d$. Substituting back into $(a + b)^2 = 4cd$, we have $a = \pm c$. Since all four numbers are positive, we must have $a = b = c = d$. This is a contradiction since $x + y \neq x - y$.
Subcase 1b. This is satisfied by two intersecting pairs.
We may assume that we have $(a + b)^2 = 4cd$ and $(a + c)^2 = 4bd$ with $b \neq c$. Then we have $b(a + b)^2 = 4bcd = c(a + c)^2$, or equivalently $(b - c)(a^2 + 2a(b + c) + (b^2 + bc + c^2)) = 0$. This is a contradiction since $b - c \neq 0$ while $a^2 + 2a(b + c)(b^2 + bc + c^2) > 0$.
Subcase 1c. This is satisfied by only one pair.
We may assume that $(a + b)^2 = 4cd$. Then we know that the larger one of a and b is $x + y$ and the smaller one $x - y$. We can determine x and y uniquely.
Case 2. One of the numbers is 0. We know that $x = y$ so that $\frac{x}{y} = 1$ must also be among the four numbers. The other two are $x + y = 2x$ and $xy = x^2$. Since their product is $2x^3$, we can determine $x = y$ uniquely.
Case 3. One of the numbers is negative.
We know that $x < y$ and $\frac{x}{y} < 1$. Check how many numbers in $S = \{x + y, xy, \frac{x}{y}\}$ lie strictly between 0 and 1. There are three subcases.

Subcase 3a. There is exactly one such number.
We know that this number is $\frac{x}{y}$, and we can determine x and y uniquely from $x-y$ and $\frac{x}{y}$.
Subcase 3b. There are exactly two such numbers.
We cannot have $x+y<1$. Otherwise, we must have $x<1$ and $y<1$ so that $xy<1$, but then all three numbers in S lie strictly between 0 and 1. Hence $x+y>1$ is the largest number in S, and we can determine x and y uniquely from $x-y$ and $x+y$.
Subcase 3c. There are exactly three such numbers.
From $x+y<1$, we have $x<1$ and $y<1$ so that $xy<x+y$ and $xy<\frac{x}{y}$. Hence the smallest number in S is xy, and we can determine x and y uniquely from $x-y$ and xy.

4. The circumcentre P of triangle IAB lies on the line through O perpendicular to AB. Let this line cut AB at D, and let C be the point on this line such that CI is perpendicular to it. Let d denote the distance OI, r the radius of the circle with centre I, and R the radius of the circle with centre O. Then $PA^2 = PD^2 + AD^2 = PD^2 + R^2 - OD^2$ and $PI^2 = CI^2 + PC^2 = d^2 - (r - OD)^2 + (PD + r)^2$. These two expressions are equal to each other since $PA = PI$. Simplification yields $R^2 - d^2 = 2r(OD + PD) = 2rPO$. Hence $PO = \frac{R^2-d^2}{2r}$ is a fixed distance, so that the locus of P is a circle with this radius and centre O.

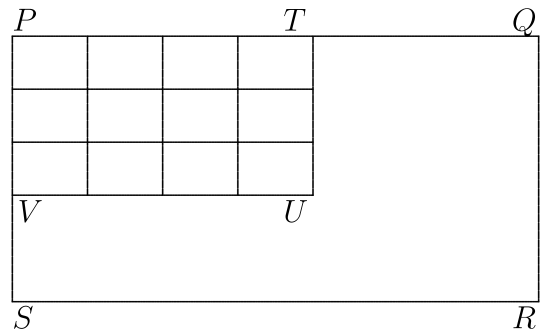


5. Suppose we have an $a_1 \times a_2$ rectangle A and a $b_1 \times b_2$ rectangle B. Any rectangle $PQRS$ that can be constructed from copies of A has dimensions $(u_1a_1 + u_2a_2) \times (v_1a_1 + v_2a_2)$ for some non-negative integers u_1, u_2, v_1 and v_2. If $PQRS$ is similar to B, then

$$\frac{b_1}{b_2} = \frac{u_1a_1 + u_2a_2}{v_1a_1 + v_2a_2}.$$

We first consider the case where $\frac{a_1}{a_2}$ is rational, so that it is equal to $\frac{m_1}{m_2}$ for some positive integers m_1 and m_2. Then $\frac{b_1}{b_2} = \frac{u_1m_1+u_2m_2}{v_1m_1+v_2m_2} = \frac{n_1}{n_2}$ for some positive integers n_1 and n_2, so that it is also rational. Using n_1n_2 copies of B, we can construct a square of side $s = n_2b_1 + n_1b_2$. Using m_1m_2 copies of this square, we can construct an $sm_1 \times sm_2$ rectangle which is similar to A.

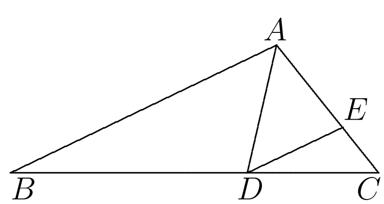
We now consider the case where $\frac{a_1}{a_2}$ is irrational. We claim that in constructing the rectangle $PQRS$ with copies of A, all the copies must be in the same orientation. Let $PTUV$ be the largest subrectangle of $PQRS$ that can be constructed with copies of A all in the same orientation. Suppose U is in the interior of $PQRS$, as illustrated in the diagram below.



If the line TU can be extended without cutting in interior of a copy of A, then the space immediately below UV must be filled with copies of A in the same orientation as those above, as otherwise it contradicts the irrationality of $\frac{a_1}{a_2}$. However, now it contradicts the maximality of $PTUV$. Hence TU cannot be so extended, but this implies that VU can, and we have a contradiction as well. It follows that U must lie on QR or RS. We may assume by symmetry that it lies on QR, so that T coincides with Q. However, the space immediately below UV must be filled with copies of A in the same orientation as those above. This contradicts the maximality of $PTUV$ unless U coincides with R and V with S. Thus our claim is justified. Suppose this construction uses k_1k_2 copies of A in k_1 rows and k_2 columns for some positive integers k_1 and k_2. Then $\frac{k_1a_1}{k_2a_2} = \frac{b_1}{b_2}$ so that $\frac{k_2b_1}{k_1b_2} = \frac{a_1}{a_2}$. Hence we can construct a rectangle similar to A using k_1k_2 copies of B in k_2 rows and k_1 columns.

6. Let the measures of the angles of a resolvable triangle be $\frac{a}{n}$, $\frac{b}{n}$ and $\frac{c}{n}$ times $180°$, where a, b and c are positive integers such that $a+b+c=n$. We label such a triangle (a,b,c). For $n=3$, there is only one resolvable triangle, namely (1,1,1), and the result is trivially true. For $n=5$, we have (3,1,1) and (2,2,1). Each can be cut into two triangles which are similar to itself and to the other. Thus the result is also true. Henceforth, we assume that $n \geq 7$.

   We generalize the case $n=5$ as follows. We claim that whenever a resolvable triangle T can be cut into two resolvable ones, one similar to itself and another similar to a different resolvable triangle S, then S can also be cut into two such triangles.



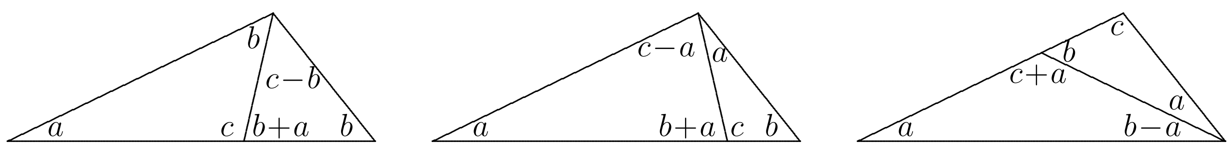
   Let ABC be a resolvable triangle which is cut into two resolvable ones DBA and DCA, with DBA similar to ABC. Then $\angle BAD = \angle BCA$. Now cut DCA along DE parallel to BA. Clearly, EDC is similar to ABC. Since $\angle EDA = \angle BAD = \angle BCA$, EDA is also similar to DCA. This justifies the claim.

   Two such triangles are said to be *compatible* with each other, and the dissection dividing either into triangles similar to both is called their common dissection.

For a fixed n, construct a graph as follows. Each vertex represents a similarity type of resolvable triangles. Two vertices are joined by an edge if and only if the triangles they represent are compatible with each other. Colour red the vertex representing the resolvable triangle given initially, and any other vertices as the triangles they represent appear on the table. We shall only use a common dissection to cut a resolvable triangle into a compatible pair. It follows that once coloured red, a vertex remains red.

Suppose not all vertices are red. If the graph is connected, then there is a pair of adjacent vertices exactly one of which is red. We can make the other vertex red by performing a common dissection. Hence the desired result follows if we can prove that the graph is indeed connected.

The degree of each vertex representing a non-isosceles resolvable triangle is 3. This is because there are common dissections with three other triangles. If the triangle is (a, b, c) where $a < b < c$, then it has a common dissection with each of $(c-b, b, b+a)$, $(b-a, a, c+a)$ and $(c-a, a, b+a)$.



The degree of each vertex representing an isosceles resolable triangle is 1. If it is of the form (a, b, b) where $a < b$, we can only the second or the third common dissection to generate $(b-a, a, b+a)$. If it is of the form (a, c, c) where $a < c$, we can use either the first or the second common dissection to generate $(a, 2a, c-a)$. Moreover, the newly generated triangle can only be isosceles if $n = 5$. Since we are now concerned only with the cases $n \geq 7$, we can safely removal such vertices without affecting the connectivity of the graph. Of course, some of the other vertices will have their degrees reduced from 3 to 2.

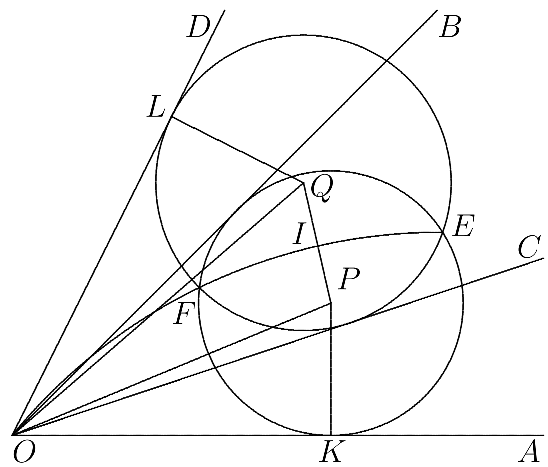
Let (a, b, c) be a resolvable triangle with $a \leq b \leq c$. Put the vertex representing it in level a. The vertices on level 1 form a chain $(1,2,n-3)$ — $(1,3,n-4)$ — $\cdots$ — $(1,\frac{n-3}{2},\frac{n+1}{2})$ by the third common dissection. We claim that each vertex in level $a > 1$ is either joined to some vertex at a lower level, either directly or via a chain in level a. Then we can conclude that the graph is connected.

If $a + b > c > b$, we use the first common dissection to obtain $(b+a, b, c-b)$. Since $c - b < a$, the vertex representing this triangle is in a lower level. If $2a > b > a$, we use the third common dissection to obtain $(b-a, a, c+a)$. Since $b - a < a$, the vertex representing this triangle is in a lower level.

Suppose $c > a + b \geq 3a$. We may use the second common dissection to obtain $(a, b+a, c-a)$. For some positive integer k, we will have $a + (b + ka) > c - ka$. Alternatively, we may use the third common division to obtain $(a, b-a, c+a)$. For some positive integer ℓ, we will have $2a > b - \ell a$. In both cases, we are moving within the same level towards a vertex which allows for descent into a lower level.

We will have a problem in the first approach if $b + ka = c - ka$, and in the second approach if $2a = b - \ell a$. Either may occur, but if they occur simultaneously, we have $b = (\ell + 1)a$ while $c = (2k + \ell + 1)a$. Since n is prime, this is only possible if $a = 1$. However, we have already proved that level 1 is connected.

7. Let the circles inscribed in $\angle AOB$ and $\angle COD$ have centres P and Q, and tangent to OA and OD at K and L, respectively. Then we have $\angle POK = \frac{1}{2}\angle AOB = \frac{1}{2}\angle COD = \angle QOL$ and $\angle PKO = 90° = \angle QLO$. Hence triangles POK and QOL are similar. It follows that $\frac{PO}{QO} = \frac{PK}{QL} = \frac{PE}{QE} = \frac{PF}{QF}$, so that the circumcircle of triangle OEF is the locus of all points M satisfying $\frac{PM}{QM} = \frac{PK}{QL}$. Now PQ will intersect this circle at the midpoint I of the arc EF. Hence $\angle IOE = \angle IOF$. Moreover, since $\frac{PI}{QI} = \frac{PO}{QO}$, we have $\angle POI = \angle QOI$. Hence $\angle AOE = \angle AOP + \angle POI - \angle IOE = \angle DOQ + \angle QOI - \angle IOF = \angle DOF$.



International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper[1] **Spring 2005.**

1. Anna and Boris move simultaneously towards each other, from points A and B respectively. Their speeds are constant, but not necessarily equal. Had Anna started 30 minutes earlier, they would have met 2 kilometers nearer to B. Had Boris started 30 minutes earlier instead, they would have met some distance nearer to A. Can this distance be uniquely determined?

2. Prove that one of the digits 1, 2 and 9 must appear in the base-ten expression of n or $3n$ for any positive integer n.

3. There are eight identical Black Queens in the first row of a chessboard and eight identical White Queens in the last row. The Queens move one at a time, horizontally, vertically or diagonally by any number of squares as long as no other Queens are in the way. Black and White Queens move alternately. What is the minimal number of moves required for interchanging the Black and White Queens?

4. M and N are the midpoints of sides BC and AD, respectively, of a square $ABCD$. K is an arbitrary point on the extension of the diagonal AC beyond A. The segment KM intersects the side AB at some point L. Prove that $\angle KNA = \angle LNA$.

5. In a certain big city, all the streets go in one of two perpendicular directions. During a drive in the city, a car does not pass through any place twice, and returns to the parking place along a street from which it started. If it has made 100 left turns, how many right turns must it have made?

Note: The problems are worth 3, 4, 5, 5 and 5 points respectively.

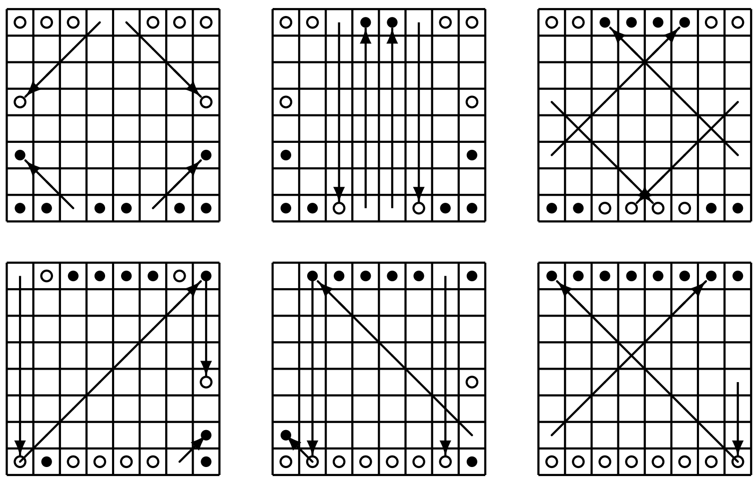
[1] Courtesy of Andy Liu.

Solution to Junior O-Level Spring 2005

1. Let the distance AB be x kilometres. Let the speeds of Anna and Boris be a and b kilometres respectively. Then the distance covered by Anna is $\frac{ax}{a+b}$ and that by Boris $\frac{bx}{a+b}$. When Anna covers 2 more kilometres and Boris 2 less, the difference in time spent is $\frac{1}{2}$ hours. It follows that $\frac{1}{a}(\frac{ax}{a+b}+2)-\frac{1}{b}(\frac{bx}{a+b}-2)=\frac{1}{2}$, which simplifies to $\frac{1}{a}+\frac{1}{b}=\frac{1}{4}$. Since this expression is symmetric, the two of them will meet 2 kilometres closer to A when Boris starts 30 minutes early.

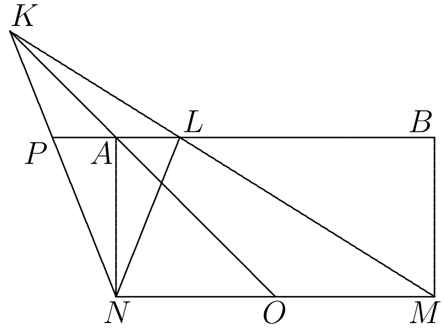
2. If the leading digit of n is 1, 2 or 9, there is nothing to prove. If it is 3, then the leading digit of $3n$ is either 9 or 1. If the leading digit of n is 4 or 5, the leading digit of $3n$ will be 1. If it is 6, then the leading digit of $3n$ is either 1 or 2. If the leading digit of n is 7 or 8, the leading digit of $3n$ will be 2. All cases have been covered, and the desired conclusion follows.

3. We first show that the task can be accomplished in 23 moves.



We now prove that we need at least 23 moves. Each the 16 Queens must move at least once. Of the two Queens on each inside column, at most one can move only once. This means at least 6 extra moves. Of the four Queens at the corners, at most three can move only once. This means at least 1 extra move. Hence the minimum is 23 moves.

4. Let AC cut MN at O, and extend BA to cut KN at P. Since PL is parallel to NM and O is the midpoint of NM, A is the midpoint of AL. Hence triangles PAN and LAN are congruent to each other, so that $\angle KNA=\angle LNA$.



5. In tracing a simple closed curve, the net change in the direction of the car is 360°, clockwise or counterclockwise. Hence it must have made 96 or 104 right turns.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper[1] **Spring 2005.**

1. The graphs of four functions of the form $y = x^2 + ax + b$, where a and b are real coefficients, are plotted on the coordinate plane. These graphs have exactly four points of intersection, and at each one of them, exactly two graphs intersect. Prove that the sum of the largest and the smallest x-coordinates of the points of intersection is equal to the sum of the other two.

2. The base-ten expressions of all the positive integers are written on an infinite ribbon without spacing: 1234567891011.... Then the ribbon is cut up into strips seven digits long. Prove that any seven digit integer will:

   (a) appear on at least one of the strips;

   (b) appear on an infinite number of strips.

3. M and N are the midpoints of sides BC and AD, respectively, of a square $ABCD$. K is an arbitrary point on the extension of the diagonal AC beyond A. The segment KM intersects the side AB at some point L. Prove that $\angle KNA = \angle LNA$.

4. In a certain big city, all the streets go in one of two perpendicular directions. During a drive in the city, a car does not pass through any place twice, and returns to the parking place along a street from which it started. If it has made 100 left turns, how many right turns must it have made?

5. The sum of several positive numbers is equal to 10, and the sum of their squares is greater than 20. Prove that the sum of the cubes of these numbers is greater than 40.

Note: The problems are worth 3, 3+1, 4, 4 and 5 points respectively.

[1] Courtesy of Andy Liu.

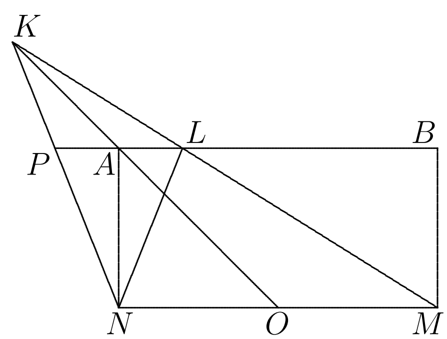
Solution to Senior O-Level Spring 2005

1. Let the parabolas be $y_i = x^2 + a_i x + b_i$, $1 \le i \le 4$. Now y_i and y_j intersect if and only if $a_i \ne a_j$, and if that it the case, they intersect at exactly one point with $x = \frac{b_i - b_j}{a_j - a_i}$. Since we have only four points of intersection, we must have two distinct values of a_i, each appearing twice. Hence we may assume that $a_2 = a_1$ and $a_4 = a_3$. By symmetry, we may assume that $b_1 < b_2$, $b_3 < b_4$ and $a_1 < a_3$. This means that y_1 is below y_2, y_3 is below y_4 and the common axis of y_1 and y_2 is to the right of the common axis of y_3 and y_4. It follows that the rightmost point of intersection is that of y_2 with y_3 while the leftmost point of intersection is that of y_1 with y_4. The sum of their x-coordinates is $\frac{b_1 - b_4}{a_3 - a_1} + \frac{b_2 - b_3}{a_3 - a_1} = \frac{b_1 + b_2 - b_3 - b_4}{a_3 - a_1}$. The sum of the x-coordinates of the other two points of intersections is $\frac{b_1 - b_3}{a_3 - a_1} + \frac{b_2 - b_4}{a_3 - a_1} = \frac{b_1 + b_2 - b_3 - b_4}{a_3 - a_1}$ as well.

2. (a) Suppose n is a seven-digit number. Consider the seven consecutive eight-digit numbers $10n$, $10n + 1$, $\ldots$, , $10n + 6$. Since 7 and 8 are relatively prime, some strip will start with one of these numbers and n appears on it.

   (b) As in (a), we can consider the seven consecutive nine-digit numbers $100n$, $100n + 1$, $\ldots$, $100n + 6$, the seven consecutive ten-digit numbers $1000n$, $1000n + 1$, $\ldots$, $1000n + 6$, and so on. For each number of digits not divisible by 7, we get a strip on which n appears.

3. Let AC cut MN at O, and extend BA to cut KN at P. Since PL is parallel to NM and O is the midpoint of NM, A is the midpoint of AL. Hence triangles PAN and LAN are congruent to each other, so that $\angle KNA = \angle LNA$.



4. In tracing a simple closed curve, the net change in the direction of the car is 360°, clockwise or counterclockwise. Hence it must have made 96 or 104 right turns.

5. Suppose $a_1 + a_2 + \cdots + a_n = 10$ and $a_1^2 + a_2^2 + \cdots + a_n^2 > 20$. By Cauchy's Inequality,

$$\begin{aligned} 10(a_1^3 + a_2^3 + \cdots + a_n^3) &= (a_1 + a_2 + \cdots + a_n)(a_1^3 + a_2^3 + \cdots + a_n^3) \\ &\ge (\sqrt{a_1}\sqrt{a_1^3} + \sqrt{a_2}\sqrt{a_2^3} + \cdots + \sqrt{a_n}\sqrt{a_n^3})^2 \\ &= (a_1^2 + a_2^2 + \cdots + a_n^2)^2 \\ &> 400. \end{aligned}$$

   Hence $a_1^3 + a_2^3 + \cdots + a_n^3 > 40$.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper[1] **Spring 2005.**

1. On the graph of a polynomial with integral coefficients are two points with integral coordinates. Prove that if the distance between these two points is integral, then the segment connecting them is parallel to the x-axis.

2. The altitudes AD and BE of triangle ABC meet at its orthocentre H. The midpoints of AB and CH are X and Y, respectively. Prove that XY is perpendicular to DE.

3. Baron Münchhausen's watch works properly, but has no markings on its face. The hour, minute and second hands have distinct lengths, and they move uniformly. The Baron claims that since none of the mutual positions of the hands is repeats twice in the period between 8:00 and 19:59, he can use his watch to tell the time during the day. Is his assertion true?

4. A 10×12 paper rectangle is folded along the grid lines several times, forming a thick 1×1 square. How many pieces of paper can one possibly get by cutting this square along the segment connecting

   (a) the midpoints of a pair of opposite sides;

   (b) the midpoints of a pair of adjacent sides?

5. In a rectangular box are a number of rectangular blocks, not necessarily identical to one another. Each block has one of its dimensions reduced. Is it always possible to pack these blocks in a smaller rectangular box, with the sides of the blocks parallel to the sides of the box?

6. John and James wish to divide 25 coins, of denominations 1, 2, 3, ..., 25 kopeks. In each move, one of them chooses a coin, and the other player decides who must take this coin. John makes the initial choice of a coin, and in subsequent moves, the choice is made by the player having more kopeks at the time. In the event that there is a tie, the choice is made by the same player in the preceding move. After all the coins have been taken, the player with more kokeps wins. Which player has a winning strategy?

7. The squares of a chessboard are numbered in the following way. The upper left corner is numbered 1. The two squares on the next diagonal from top-right to bottom-left are numbered 2 and 3. The three squares on the next diagonal are numbered 4, 5 and 6, and so on. The two squares on the second-to-last diagonal are numbered 62 and 63, and the lower right corner is numbered 64. Peter puts eight pebbles on the squares of the chessboard in such a way that there is exactly one pebble in each column and each row. Then he moves each pebble to a square with a number greater than that of the original square. Can it happen that there is still exactly one pebble in each column and each row?

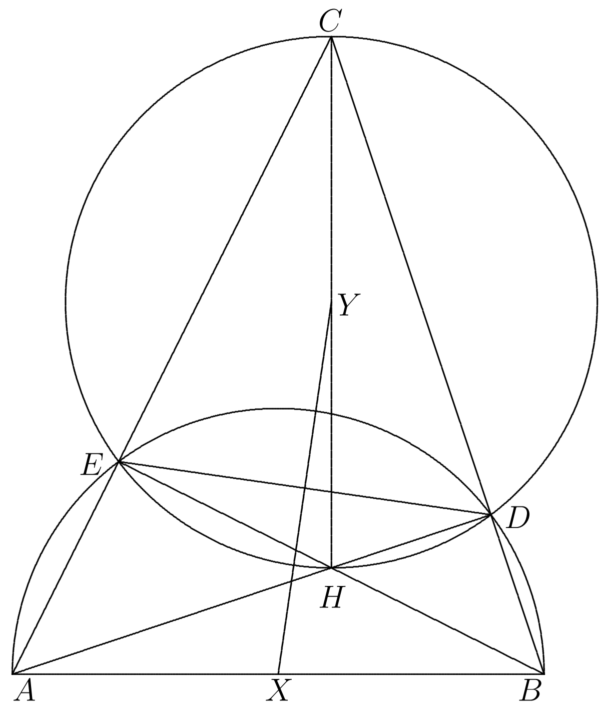
Note: The problems are worth 4, 5, 5, 2+4, 6, 6 and 8 points respectively.

[1] Courtesy of Andy Liu.

Solution to Junior A-Level Spring 2005

1. Let $f(x)$ be a polynomial with integral coefficients such that $f(x_1)$ and $f(x_2)$ are integers for some integers x_1 and x_2. Since $x_1^k - x_2^k$ is divisible by $x_1 - x_2$ for all k, $f(x_1) - f(x_2) = n(x_1 - x_2)$ for some integer n. If in addition the distance between the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is also an integer m, then $(x_1 - x_2)^2 + (f(x_1) - f(x_2))^2 = m^2$. Then $(x_1 - x_2)^2(1 + n^2) = m^2$, so that $1 + n^2$ is also the square of an integer. This is only possible for $n = 0$. Hence $f(x_1) - f(x_2) = 0$, so that $f(x_1) = f(x_2)$, and the line joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is indeed parallel to the x-axis.

2. Since $\angle ADB = 90° = \angle AEB$, D and E lie on a circle with diameter AB, and hence with centre X. Since $\angle CDH = 90° = \angle CEH$, D and E lie on a circle with diameter CH, and hence with centre Y. The common chord DE of the two circles is therefore perpendicular to the line of centres XY.



3. We first show that the three hands coincide only at 12:00 or 24:00. Suppose this occurs again. Consider the angular distance θ covered by the hour hand where $0° < \theta < 360°$. The angular distance covered by the minute hand is $360°n + \theta$, where n is the number of revolutions it has made. Since the minute hand moves at 12 times the speed of the hour hand, $360°n + \theta = 12\theta$, so that $\theta = 360°\frac{n}{11}$. The angular distance covered by the second hand is $360°m + \theta$, where m is the number of revolutions it has made. Since the second hand moves at 720 times the speed of the hour hand, $360°m + \theta = 720\theta$, so that $\theta = 360°\frac{m}{719}$. From $\frac{n}{11} = \frac{m}{719}$, n must be a multiple of 11 and m a mutliple of 719 as 11 and 719 are relatively prime. However, this contradicts $0° < \theta < 360°$. This justifies the Baron's claim. If there are two indistinguishable times within a twelve-hour period, shift the times so that one of them is at 12:00 or 24:00 and the other not. However, since one set of hands coincide, so must the other, and we have already proved that this is not possible.

4. (a) Let the edge of length 12 be horizontal. No matter how the piece of paper is folded into a 1×1 stack, the horizontal edges of each square remains horizontal. Thus if the cut is horizontal, we obtain 10+1=11 strips of paper. If the cut is vertical, we obtain 12+1=13 strips of paper.

   (b) Label the vertices of the 1×1 squares as follows. Along the top row, they are labelled alternately A and B. Along the second row, they are labelled alternately C and D. Thereafter, the rows are labelled alternately as above, so that along the bottom row, the vertices are labelled alternately A and B. There are $6 \times 7 = 42$ A vertices, $6 \times 6 = 36$ B vertices, $5 \times 7 = 35$ C vertices and $5 \times 6 = 30$ D vertices. No matter how the piece of paper is folded into a 1×1 stack, all A vertices will be on top of one another, as will all the B vertices, all the C vertices and all the D vertices. If the cut isolates the A vertices, we have 42+1=43 pieces of paper. If the cut isolates the B vertices, we have 36+1=37 pieces of paper. If the cut isolates the C vertices, we have 35+1=36 pieces of paper. If the cut isolates the D vertices, we have 30+1=31 pieces of paper.

5. Let the box be $10000 \times 1000 \times 100$, the first block 10000×1000, the second block $10000 \times 800 \times 80$ and the third block $10000 \times 200 \times 80$. Let the reduced first block be $10000 \times 999 \times 20$, the reduced second block be $10000 \times 800 \times 79$ and the reduced third block be $9999 \times 200 \times 80$. The dimension of the box which exceeds 1000 must remain at 10000. The dimension of the box which exceeds 100 must remain 1000=800+200, and the third dimension of the box must remain 100=80+20. Hence no smaller box can hold the three reduced blocks.

6. James can always get more kopeks than John. Upon John's initial offer, James can either take it or leave it. If there is a way for him to get more kopeks than John by taking it, there is nothing further to prove. If there are no ways, then he makes John take it, and there are no ways for John to get more kopeks than he.

7. Label the rows from 1 to 8 from top to bottom, and the columns from 1 to 8 from left to right. Note that the sum of the row number and column number of a square is constant along any diagonal from top-right to bottom-left, and this sum increases as the diagonals shift from top-left to bottom-right. For eight pebbles each in a different row and a different column, the sum of their row and column numbers must be 2(1+2+3+4+5+6+7+8). In moving a pebble from a square to another so that the number on the square increases, it must either slide downwards along a diagonal from top-right to bottom-left, or move to a diagonal closer to the bottom-right. Since the sum of all the row and column numbers cannot decrease, every pebble must stay on its original diagonal from top-right to bottom-left. However, this means that every pebble slides downwards, so that there will not be any left in the first row.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper[1] **Spring 2005.**

1. On the graph of a polynomial with integral coefficients are two points with integral coordinates. Prove that if the distance between these two points is integral, then the segment connecting them is parallel to the x-axis.

2. A circle ω_1 with centre O_1 passes through the centre O_2 of a second circle ω_2. The tangent lines to ω_2 from a point C on ω_1 intersect ω_1 again at points A and B respectively. Prove that AB is perpendicular to O_1O_2.

3. John and James wish to divide 25 coins, of denominations 1, 2, 3, ..., 25 kopeks. In each move, one of them chooses a coin, and the other player decides who must take this coin. John makes the initial choice of a coin, and in subsequent moves, the choice is made by the player having more kopeks at the time. In the event that there is a tie, the choice is made by the same player in the preceding move. After all the coins have been taken, the player with more kokeps wins. Which player has a winning strategy?

4. For any function $f(x)$, define $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(x))$ for any integer $n \geq 2$. Does there exist a quadratic polynomial $f(x)$ such that the equation $f^n(x) = 0$ has exactly 2^n distinct real roots for every positive integer n?

5. Prove that if a regular icosahedron and a regular dodecahedron have a common circumsphere, then they have a common insphere.

6. A *lazy* rook can only move from a square to a vertical or a horizontal neighbour. It follows a path which visits each square of an 8×8 chessboard exactly once. Prove that the number of such paths starting at a corner square is greater than the number of such paths starting at a diagonal neighbour of a corner square.

7. Every two of 200 points in space are connected by a segment, no two intersecting each other. Each segment is painted in one colour, and the total number of colours is k. Peter wants to paint each of the 200 points in one of the colours used to paint the segments, so that no segment connects two points both in the same colour as the segment itself. Can Peter always do this if

   (a) $k = 7$;

   (b) $k = 10$?

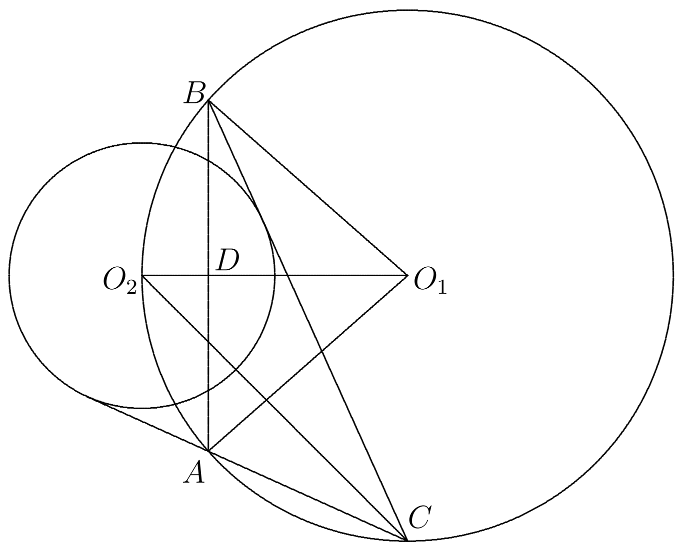
Note: The problems are worth 4, 5, 5, 6, 7, 7 and 4+4 points respectively.

[1] Courtesy of Andy Liu.

Solution to Senior A-Level Spring 2005

1. Let $f(x)$ be a polynomial with integral coefficients such that $f(x_1)$ and $f(x_2)$ are integers for some integers x_1 and x_2. Since $x_1^k - x_2^k$ is divisible by $x_1 - x_2$ for all k, $f(x_1) - f(x_2) = n(x_1 - x_2)$ for some integer n. If in addition the distance between the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is also an integer m, then $(x_1 - x_2)^2 + (f(x_1) - f(x_2))^2 = m^2$. Then $(x_1 - x_2)^2(1 + n^2) = m^2$, so that $1 + n^2$ is also the square of an integer. This is only possible for $n = 0$. Hence $f(x_1) - f(x_2) = 0$, so that $f(x_1) = f(x_2)$, and the line joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is indeed parallel to the x-axis.

2. Since CA and CB are tangents to ω_2, we have $\angle ACO_2 = \angle BCO_2$. It follows that we have $\angle AO_1O_2 = 2\angle ACO_2 = 2\angle BCO_2 = \angle BO_1O_2$. Moreover, $O_1A = O_1B$ and $O_1D = O_1D$, where D is the point of intersection of AB and O_1O_2. It follows that triangles O_1AD and O_1BD are congruent. Hence $\angle ADO_1 = \angle BDO_1$. Since their sum is $180°$, each is $90°$ and O_1O_2 is indeed perpendicular to AB.

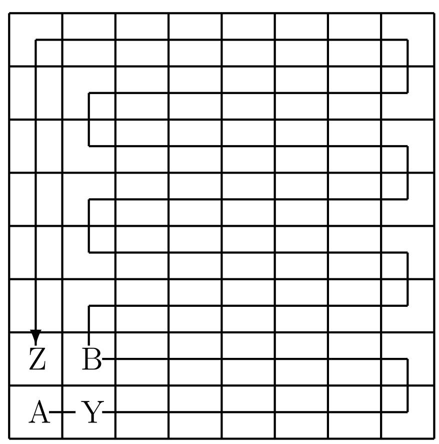


3. James can always get more kopeks than John. Upon John's initial offer, James can either take it or leave it. If there is a way for him to get more kopeks than John by taking it, there is nothing further to prove. If there are no ways, then he makes John take it, and there are no ways for John to get more kopeks than he.

4. Such a function is $f(x) = x^2 - 2$. For $f(x) = 0$, we have $x^2 = 2$, and the roots are $\pm\sqrt{2}$. We claim that every root of $f^{n+1}(x) = 0$ has the form $r_{n+1} = \pm\sqrt{2 \pm r_n}$ for some root $\pm r_n$ of $f^n(x) = 0$. Indeed, $f^{n+1}(r_{n+1}) = f^n((\pm\sqrt{2 \pm r_n})^2 - 2) = f^n(\pm r_n) = 0$. Since the degree of $f^{n+1}(x)$ is double that of $f^n(x)$, these are all the roots. We prove by induction on n that $\pm r_n$ are real and $|r_n| < 2$ for all n. For $n = 1$, this is certainly the case with $\pm\sqrt{2}$. Suppose the result holds for some $n \geq 1$. Since $|r_n| < 2$, $2 \pm r_n > 0$ so that $r_{n+1} = \pm\sqrt{2 \pm r_n}$ are real. Moreover, $|2 \pm r_n| \leq 2 + |r_n| < 4$, so that $|r_{n+1}| < 2$. Finally, observe that $\sqrt{2}$ and $-\sqrt{2}$ are distinct, and that distinct roots of $f^n(x) = 0$ lead to distinct roots of $f^{n+1}(x) = 0$.

5. Let O be the circumcentre of the icosahedron, C the centre of one of its faces and A a vertex of that face. Its circumradius is OA, and its inradius is OC. Construct a dual dodecahedron by joining the centrers of adjacent faces of the icosahedron. Now C is a vertex of three faces of this dodecahedron, and the centre B of one of these faces lies on OA. Its circumradius is OC and its inradius is OB. Note that in triangles OAC and OCB, $\angle AOC = \angle COB$ and $\angle OCA = 90° = \angle OBA$. Hence they are similar to each other, so that $\frac{OA}{OC} = \frac{OC}{OB}$. If we rescale the two solids so that their circumradii are equal, then so are their inradii.

6. The diagram below shows a path from A to Z along which a lazy rook visits every square of the 8×8 chessboard once and only once, where A, B, Y and Z are as labelled. Note that A and B have the same colour in the usual chessboard pattern. Since the squares visited by the lazy rook must alternate in colour, no path can start from A and end at B, or vice versa. We claim that there are more such paths starting from A than those starting from B. For each path starting from B, since the path cannot end at A, the lazy rook must visit A between visits to Y and Z. Suppose the lazy rook visits Y first. Then the path corresponds to the following one starting from A: move to Y, follow the original path in reverse to B, move to Z, and follow the original path to the end. If the lazy rook visits Z first, then start from A, move to Z, follow the original path in reverse to B, move to Y, and follow the original path to the end. The path in the diagram below does not correspond to any path starting from B because no path starting from B can end at Z unless it moves from A to Z. This justifies our claim.



7. (a) Peter cannot always do so when $k = 7$, even when there are only 128 points. We ignore the remaining 72 points and segments joining them to one another or to our 128 points. Divide the 128 points into 64 pairs, and paint the segments joining the two points in each pair red. Combine the 64 pairs into 32 quartets. In each quartet, all segments joining one point from each pair are painted blue. Combine the 32 quartets into 16 octets. In each octet, all segments joining one point from each quartet are painted yellow. Combine the 16 octets into 8 hexidecatets. In each hexidecatets, all segments joining one point from each octet are painted green. Combine the 8 hexidecatets into 4 groups. In each group, all segments joining one point from each hexidecatet are painted orange. Combine the 4 groups into 2 halves. In each half, all segments joining one point from each group are painted violet. Finally, combine the 2 halves into 1 set. In the set, all segments joining one point from each half are painted black. Now Peter cannot have a black point in each half. Hence there is a half with no black points. Discard the other half. Now Peter cannot have a violet point in each group. Hence there is a group with no violet points. Discard the other group. Now Peter cannot have an orange point in each hexidecatet. Hence there is a hexidecatet with no orange points. Discard the other hexidecatet. Now Peter cannot have a green point in each octet. Hence there is an octet with no green points. Discard the other octet. Now Peter cannot have a yellow point in each quartet. Hence there is a quartet with no yellow points. Discard the other quartet. Now Peter cannot have a blue point in each pair. Hence there is a pair with no belue points. Discard the other pair. In the remaining pair, both points are red and they are joined by a red segment.

(b) **Solution by Cheng-Chiang Tasi, Kaohsiung High School, Taiwan.**
Peter still cannot do so when $k = 10$, even when there are only 121 points. We ignore the remaining 79 points and segments joining them to one another or to our 121 points. We construct a finite geometry based on arithmetic modulo 11. Each point is given coordinates (i, j), where each of i and j is an integer between 0 and 10 inclusive. Consider two points (i_1, j_1) and (i_2, j_2). If $i_1 = i_2$, the segment joining them is vertical. If $j_i = j_2$, the segment joining them is horizontal. We either paint such segments arbitrarily or leave them unpainted. In all other cases, the segment joining the two points has slope m, where m is an integer between 1 and 10 inclusive. We paint such a segment in the m-th colour. Now Peter paint the 121 points in 10 colours. By the Pigeonhole Principle, there must be at least 13 points of the same colour, say the m-th one. Now there are 11 lines in this geometry with slope m, each passing through exactly 11 points. By the Pigeonhole Principle again, at least 2 of these 13 points must be on the same line. Then we have 2 points in the m-th colour, joined by a segment also in the m-th colour.