JBMO Shortlist 2022

A1 Find all pairs of positive integers (a, b) such that

$$11ab \leq a^3 - b^3 \leq 12ab.$$

A2 Let x, y, and z be positive real numbers such that $xy + yz + zx = 3$. Prove that

$$\frac{x+3}{y+z} + \frac{y+3}{z+x} + \frac{z+3}{x+y} + 3 \geq 27 \cdot \frac{(\sqrt{x} + \sqrt{y} + \sqrt{z})^2}{(x+y+z)^3}.$$

A3 Let a, b, and c be positive real numbers such that $a + b + c = 1$. Prove the following inequality

$$a\sqrt[3]{\frac{b}{a}} + b\sqrt[3]{\frac{c}{b}} + c\sqrt[3]{\frac{a}{c}} \leq ab + bc + ca + \frac{2}{3}.$$

A4 Suppose that a, b, and c are positive real numbers such that

$$a + b + c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Find the largest possible value of the expression

$$\frac{a+b-c}{a^3+b^3+abc} + \frac{b+c-a}{b^3+c^3+abc} + \frac{c+a-b}{c^3+a^3+abc}.$$

A5 The numbers $2, 2, ..., 2$ are written on a blackboard (the number 2 is repeated n times). One step consists of choosing two numbers from the blackboard, denoting them as a and b, and replacing them with $\sqrt{\frac{ab+1}{2}}$. (a) If x is the number left on the blackboard after $n-1$ applications of the above operation, prove that $x \geq \sqrt{\frac{n+3}{n}}$. (b) Prove that there are infinitely many numbers for which the equality holds and infinitely many for which the inequality is strict.

A6 Let a, b, and c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{a^2+b^2}{2ab}+\frac{b^2+c^2}{2bc}+\frac{c^2+a^2}{2ca}+\frac{2(ab+bc+ca)}{3}\geq 5+|(a-b)(b-c)(c-a)|.$$

C1 Anna and Bob, with Anna starting first, alternately color the integers of the set $S=\{1,2,...,2022\}$ red or blue. At their turn each one can color any uncolored number of S they wish with any color they wish. The game ends when all numbers of S get colored. Let N be the number of pairs (a,b), where a and b are elements of S, such that a, b have the same color, and $b-a=3$.
Anna wishes to maximize N. What is the maximum value of N that she can achieve regardless of how Bob plays?

C2 Let $n\geq 2$ be an integer. Alex writes the numbers $1,2,...,n$ in some order on a circle such that any two neighbours are coprime. Then, for any two numbers that are not comprime, Alex draws a line segment between them. For each such segment s we denote by d_s the difference of the numbers written in its extremities and by p_s the number of all other drawn segments which intersect s in its interior.
Find the greatest n for which Alex can write the numbers on the circle such that $p_s\leq|d_s|$, for each drawn segment s.

C3 There are 200 boxes on the table. In the beginning, each of the boxes contains a positive integer (the integers are not necessarily distinct). Every minute, Alice makes one move. A move consists of the following. First, she picks a box X which contains a number c such that $c=a+b$ for some numbers a and b which are contained in some other boxes. Then she picks a positive integer $k>1$. Finally, she removes c from X and replaces it with kc. If she cannot make any mobes, she stops. Prove that no matter how Alice makes her moves, she won't be able to make infinitely many moves.

C4 We call an even positive integer n *nice* if the set $\{1,2,\ldots,n\}$ can be partitioned into $\frac{n}{2}$ two-element subsets, such that the sum of the elements in each subset is a power of 3. For example, 6 is nice, because the set $\{1,2,3,4,5,6\}$ can be partitioned into subsets $\{1,2\}$, $\{3,6\}$, $\{4,5\}$. Find the number of nice positive integers which are smaller than 3^{2022}.

C5 Let S be a finite set of points in the plane, such that for each 2 points A and B in S, the segment AB is a side of a regular polygon all of whose vertices are contained in S. Find all possible values for the number of elements of S.

C6 Let $n\geq 2$ be an integer. In each cell of a $4n\times 4n$ table we write the sum of the cell row index and the cell column index. Initially, no cell is colored. A move consists of choosing two cells which are not colored and coloring one of them in red and one of them in blue.
Show that, however Alex perfors n^2 moves, Jane can afterwards perform a number of moves

(eventually none) after which the sum of the numbers written in the red cells is the same as the sum of the numbers written in the blue ones.

G1 Let $ABCDE$ be a cyclic pentagon such that $BC = DE$ and AB is parallel to DE. Let X, Y, and Z be the midpoints of BD, CE, and AE respectively. Show that AE is tangent to the circumcircle of the triangle XYZ.

G2 Let ABC be a triangle with circumcircle k. The points A_1, B_1, and C_1 on k are the midpoints of arcs $\widehat{BC}$ (not containing A), $\widehat{AC}$ (not containing B), and $\widehat{AB}$ (not containing C), respectively. The pairwise distinct points A_2, B_2, and C_2 are chosen such that the quadrilaterals $AB_1A_2C_1, BA_1B_2C_1$, and $CA_1C_2B_1$ are parallelograms. Prove that k and the circumcircle of triangle $A_2B_2C_2$ have a common center.

Comment. Point A_2 can also be defined as the reflection of A with respect to the midpoint of B_1C_1, and analogous definitions can be used for B_2 and C_2.

G3 Let ABC be an acute triangle such that $AH = HD$, where H is the orthocenter of ABC and $D \in BC$ is the foot of the altitude from the vertex A. Let ℓ denote the line through H which is tangent to the circumcircle of the triangle BHC. Let S and T be the intersection points of ℓ with AB and AC, respectively. Denote the midpoints of BH and CH by M and N, respectively. Prove that the lines SM and TN are parallel.

G4 Given is an equilateral triangle ABC and an arbitrary point, denoted by E, on the line segment BC. Let l be the line through A parallel to BC and let K be the point on l such that KE is perpendicular to BC. The circle with centre K and radius KE intersects the sides AB and AC at M and N, respectively. The line perpendicular to AB at M intersects l at D, and the line perpendicular to AC at N intersects l at F. Show that the point of intersection of the angle bisectors of angles MDA and NFA belongs to the line KE.

G5 Given is an acute angled triangle ABC with orthocenter H and circumcircle k. Let ω be the circle with diameter AH and P be the point of intersection of ω and k other than A. Assume that BP and CP intersect ω for the second time at points Q and R, respectively. If D is the foot of the altitude from A to BC and S is the point of the intersection of ω and QD, prove that $HR = HS$.

G6 Let ABC be a right triangle with hypotenuse BC. The tangent to the circumcircle of triangle ABC at A intersects the line BC at T. The points D and E are chosen so that $AD = BD$, $AE = CE$, and $\angle CBD = \angle BCE < 90^\circ$. Prove that D, E, and T are collinear.

N1 Determine all pairs (k, n) of positive integers that satisfy

$$1! + 2! + ... + k! = 1 + 2 + ... + n.$$

N2	Let $a < b < c < d < e$ be positive integers. Prove that $$\frac{1}{[a,b]} + \frac{1}{[b,c]} + \frac{1}{[c,d]} + \frac{2}{[d,e]} \leq 1$$ where $[x, y]$ is the least common multiple of x and y (e.g., $[6, 10] = 30$). When does equality hold?
N3	Find all quadruples of positive integers (p, q, a, b), where p and q are prime numbers and $a > 1$, such that $$p^a = 1 + 5q^b.$$
N4	Consider the sequence $u_0, u_1, u_2, ...$ defined by $u_0 = 0, u_1 = 1,$ and $u_n = 6u_{n-1} + 7u_{n-2}$ for $n \geq 2$. Show that there are no non-negative integers a, b, c, n such that $$ab(a + b)(a^2 + ab + b^2) = c^{2022} + 42 = u_n.$$
N5	Find all pairs (a, p) of positive integers, where p is a prime, such that for any pair of positive integers m and n the remainder obtained when a^{2^n} is divided by p^n is non-zero and equals the remainder obtained when a^{2^m} is divided by p^m.
N6	Find all positive integers n for which there exists an integer multiple of 2022 such that the sum of the squares of its digits is equal to n.